GENERALIZED CESARO MEANS OF ORDER -1

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A series $\sum a_n$ is said to be summable $(C, -1)$ to s if it converges to s and $na_n = o(1)$ [**8**]. It is well known that this definition is equivalent to $t_n \to s$ $(n \to \infty)$, where $t_n = s_n + na_n$, $s_n = a_0 + \ldots + a_n$. The series is summable $| C, -1 |$ to s if the sequence $t = \{t_n\}$ is of bounded variation ($t \in$ B.V.), i.e. $\sum | \Delta t_n | = \sum | t_n - t_{n-1} | < \infty$, and $\sum \Delta t_n = \lim t_n = s$.† An equivalent condition is $\sum | a_n | < \infty$, $\sum a_n = s$ and $\sum | \Delta(na_n) | < \infty$. For, suppose that $\sum a_n = s \mid C, -1 \mid$. Since $\{s_n\}$ is the sequence of $(C, 1)$-means of $\{t_n\}$ and since $| C, 0 | \subset | C, 1 |$, we have $\sum | a_n | < \infty$ and $\sum a_n = s$, whence $\sum | \Delta(na_n) | < \infty$. Conversely, $\sum | a_n | < \infty$, $\sum a_n = s$ and $\sum | \Delta(na_n) | < \infty$ imply $t \in$ B.V. and $\sum \Delta t_n = s + \lim na_n$. But $\lim na_n = 0$, since $\sum | a_n | < \infty$.

Now let $\sum a_n$ be a given series, with $s_n = a_0 + \ldots + a_n$, and define the sequence $\{t_n\}$ so that s_n is the discontinuous Riesz mean of order 1 of t_n:

$$s_n = \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) t_k,$$

where $0 \leqq \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$. Then we have

$$t_n = s_n + \mu_n a_n, \quad \text{with } \mu_n = \frac{\lambda_n}{\lambda_{n+1} - \lambda_n}. \tag{1}$$

We shall say that $\sum a_n = s(C, \lambda_n, -1)$ if and only if $t_n \to s$ $(n \to \infty)$. By the regularity of $(\bar{R}, \lambda_n, 1)$ summability it is easily seen that an equivalent definition is that $\sum a_n$ converges to s and $\mu_n a_n = o(1)$. If $\lambda_n = n$, $(C, \lambda_n, -1)$ reduces to $(C, -1)$, so that the new method generalizes the Cesàro method of order -1.

We have used the notation $(C, \lambda_n, -1)$ rather than $(\bar{R}, \lambda_n, -1)$ since a definition‡ of discontinuous $(\bar{R}, \lambda_n, -1)$ summability is already available. Now it is known [**5**], that (C, k) and $(\bar{R}, n, k)$ are equivalent for $-1 < k < 2$, and Dr Kuttner has shown me a proof, similar to that of [**5**], that $(\bar{R}, n, -1)$ implies $(C, -1)$ but that the converse implication is false. Thus $(C, \lambda_n, -1)$ is not equivalent to $(\bar{R}, \lambda_n, -1)$ even when $\lambda_n = n$.

Using (1) we define $\sum a_n = s \mid C, \lambda_n, -1 \mid$ if and only if $t \in$ B.V. and $t_n \to s$. Thus we have the inclusion $| C, \lambda_n, -1 | \subset (C, \lambda_n, -1)$. We now give an equivalent condition for $| C, \lambda_n, -1 |$ summability.

THEOREM 1. *$\sum a_n = s \mid C, \lambda_n, -1 \mid$ if and only if $\sum | a_n | < \infty$, $\sum a_n = s$ and $\sum | \Delta(\mu_n a_n) | < \infty$.*

† All summations run from 0 to ∞, and we take $t_{-1} = 0$.

‡ $\sum a_n = s(\bar{R}, \lambda_n, -1)$ if $\displaystyle\sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)^{-1} a_k \to s$ $(n \to \infty)$.

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Proof. By the absolute regularity of $(\bar{R}, \lambda_n, 1)$, $\sum a_n = s \mid C, \lambda_n -1 \mid$ implies $\sum \mid a_n \mid < \infty$ and $\sum a_n = s$, whence $\sum \mid \Delta(\mu_n a_n) \mid < \infty$. Conversely, $\sum \mid a_n \mid < \infty$, $\sum a_n = s$ and $\sum \mid \Delta(\mu_n a_n) \mid < \infty$ imply $t \in$ B.V. and $\sum \Delta t_n = s + \lim \mu_n a_n$. Now suppose, if possible, that $\lim \mu_n a_n = l \neq 0$. We first note that

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n} = \infty. \tag{2}$$

For the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{\mu_n}\right) = \prod_{1}^{\infty} \frac{\lambda_{n+1}}{\lambda_n}$$

diverges to $+\infty$, which implies that (2) holds. Since $a_n \sim l/\mu_n$, (2) implies that $\sum a_n$ diverges, a contradiction. Hence $\mu_n a_n = o(1)$, so that $\sum a_n = s \mid C, \lambda_n, -1 \mid$.

Our next theorem gives the class of sequences $\{\lambda_n\}$ for which the generalized methods are equivalent to convergence (or absolute convergence). It is known [3, Theorem 21] that a sufficient condition for (R, λ_n, k) summability $(k > 0)$ to be equivalent to convergence is

$$\Lambda_n = \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_n} = O(1).$$

Since $\Lambda_n = \mu_n + 1$, $\mu_n = O(1)$ is also a sufficient condition. Theorem 2 shows that $\mu_n = O(1)$ is necessary as well as sufficient for $(C, \lambda_n, -1)$ to be equivalent to $(C, 0)$ and for $\mid C, \lambda_n, -1 \mid$ to be equivalent to $\mid C, 0 \mid$, where $(C, 0)(\mid C, 0 \mid)$ denotes convergence (absolute convergence).

THEOREM 2. *$(C, \lambda_n, -1)(\mid C, \lambda_n, -1 \mid)$ is equivalent to $(C, 0)(\mid C, 0 \mid)$ if and only if $\mu_n = O(1)$, or what amounts to the same thing, if and only if $\Lambda_n = O(1)$.*

Proof. The inclusions $(C, \lambda_n, -1) \subset (C, 0)$ and $\mid C, \lambda_n, -1 \mid \subset \mid C, 0 \mid$ follow immediately from the equivalent definitions of $(C, \lambda_n, -1)$ and $\mid C, \lambda_n, -1 \mid$. Suppose then that $\sum a_n$ converges to s and $\mu_n = O(1)$. Then $\mu_n a_n = O(1) . o(1) = o(1)$, whence $(C, 0) \subset (C, \lambda_n, -1)$. Also it is clear that, if $\sum a_n = s$ implies $\mu_n a_n = o(1)$, then $\mu_n = O(1)$. Also it is clear that, if $\sum a_n = s$ implies $\mu_n a_n = o(1)$, then $\mu_n = O(1)$. This gives the first result of the theorem.

Now let $\sum \mid a_n \mid < \infty$ and $\mu_n = O(1)$. Then

$$\sum \mid \Delta(\mu_n a_n) \mid = O(1) \sum \mid a_n \mid < \infty,$$

so that $\mid C, 0 \mid \subset \mid C, \lambda_n, -1 \mid$. Finally, suppose that $\sum \mid a_n \mid < \infty$ implies $\sum \mid \Delta(\mu_n a_n) \mid < \infty$, i.e. that $\sum \mid x_n \mid < \infty$, where

$$x_n = \sum_{k=0}^{\infty} c_{n,k} a_k,$$

and $c_{n,n} = \mu_n$, $c_{n,n-1} = -\mu_{n-1}$, $c_{n,k} = 0 \, (k \neq n-1, n)$. By a theorem of Knopp and Lorentz [4], $\sum \mid x_n \mid < \infty$ if and only if

$$\sup_k \sum_{n=0}^{\infty} \mid c_{n,k} \mid < \infty.$$

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Using the necessity of this condition in our case we see that $\mu_n = O(1)$. This proves the theorem.

The next result involves a change in the type of summability. We take sequences $\{\lambda_n\}$ and $\{\lambda'_n\}$, with the corresponding sequences $\{\mu_n\}$ and $\{\mu'_n\}$.

THEOREM 3. *$(C, \lambda_n, -1) \subset (C, \lambda'_n, -1)$ if and only if $\Lambda'_n = O(\Lambda_n)$.*

Proof. If $\sum a_n = s(C, \lambda_n, -1)$, then $a_n = o(1)$ and $\mu_n a_n = o(1)$; and $\Lambda'_n = O(\Lambda_n)$ then gives

$$\mu'_n a_n = (\Lambda'_n - 1)a_n = O((\Lambda_n + 1) \mid a_n \mid) = O((\mu_n + 2) \mid a_n \mid) = o(1),$$

whence $\sum a_n = s(C, \lambda'_n, -1)$. Now suppose that $(C, \lambda_n, -1) \subset (C, \lambda'_n, -1)$, i.e. that $t_n = s_n + \mu_n a_n \to s$ $(n \to \infty)$ implies $t'_n = s_n + \mu'_n a_n \to s$ $(n \to \infty)$. Then

$$t'_n = \sum_{k=0}^{n} c_{n,k} t_k \quad (c_{n,n} = \Lambda'_n / \Lambda_n)$$

converges to s whenever t_n does. By the Toeplitz theorem it is necessary that $c_{n,n} = O(1)$, i.e. that $\Lambda'_n = O(\Lambda_n)$. This completes the proof.

With the restriction $\lambda_{n+1} = O(\lambda_n)$ we note that $\mu'_n = O(\mu_n)$ is also necessary and sufficient.

Specializing λ_n and λ'_n in Theorem 3 we have the inclusion $(C, \log n, -1) \subset (C, n, -1)$, which may be contrasted with a typical " second theorem of consistency " for Riesz means [3, Theorem 18], in which $(R, n, k) \subset (R, \log n, k)$ for $k > 0$.

In the next theorem we give some results on summability factors for the methods $(C, \lambda_n, -1)$ and $\mid C, \lambda_n, -1 \mid$, which extend and generalize some known theorems on Cesàro summability factors ([2], [7]). If A, B are any summability methods, we use the notation $\{\varepsilon_n\} \in (A, B)$ to mean that the A-summability of $\sum a_n$ implies the B-summability of $\sum a_n \varepsilon_n$.

THEOREM 4. *(a) $\{\varepsilon_n\} \in ((C, k), \mid C, \lambda_n, -1 \mid)$ for $k \geqq -1$, if and only if*

$$\sum_{1}^{\infty} \Lambda_n n^k \mid \varepsilon_n \mid < \infty.$$

(b) $\{\varepsilon_n\} \in ((C, \lambda_n, -1), \mid C, \lambda_n, -1 \mid)$ if and only if

$$\sum \mid \varepsilon_n \mid < \infty.$$

(c) $\{\varepsilon_n\} \in (\mid C, 0 \mid, \mid C, \lambda_n, -1 \mid)$ if and only if

$$\text{(i) } \mu_n \varepsilon_n = O(1), \quad \text{(ii) } \varepsilon_n = O(1).$$

(d) $\{\varepsilon_n\} \in (\mid C, -1 \mid, \mid C, 0 \mid)$ if and only if

$$\sum_{n=k}^{\infty} n^{-2} \mid \varepsilon_n \mid = O(k^{-1}).$$

Proof. (a) For the sufficiency, the well-known limitation theorem for the (C, k)-summability of $\sum a_n$ gives $a_n = o(n^k)$. Hence, since $1 < \Lambda_n$, $\sum_{1}^{\infty} \mid a_n \varepsilon_n \mid = O(1) \sum_{1}^{\infty} n^k \mid \varepsilon_n \mid < \infty$, and since $\mu_n < \Lambda_n$,

$$\sum_{1}^{\infty} \mid \Delta(\mu_n a_n \varepsilon_n) \mid = O(1) \sum_{1}^{\infty} \mu_n n^k \mid \varepsilon_n \mid < \infty.$$

For the necessity, define

$$r_n = \sum_{s=0}^{n} a_s \varepsilon_s + \mu_n a_n \varepsilon_n,$$

$$q_n^k = \begin{cases} C_n^k & (k > -1), \\ s_n + n a_n & (k = -1), \end{cases}$$

where C_n^k is the Cesàro mean of $\sum a_n$. If $\{\varepsilon_n\} \in ((C, k), |C, \lambda_n, -1|)$, then $\sum |\Delta r_n| < \infty$, whenever $\{q_n^k\}$ converges. Now, if $A_0^\alpha = 1$,

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \quad (n \geqq 1),$$

for real α, then

$$\begin{aligned} \Delta r_n &= a_n \varepsilon_n + \Delta(\mu_n a_n \varepsilon_n) \\ &= \Lambda_n a_n \varepsilon_n - \mu_{n-1} a_{n-1} \varepsilon_{n-1} \\ &= \begin{cases} \Lambda_n \varepsilon_n \sum_{s=0}^{n} A_{n-s}^{-k-2} A_s^k C_s^k - \mu_{n-1} a_{n-1} \varepsilon_{n-1} & (k > -1), \\ \dfrac{\Lambda_n \varepsilon_n}{n+1} \sum_{s=0}^{n} q_s^{-1} - \dots & (k = -1), \end{cases} \\ &= \sum_{s=0}^{n} a_{n,s}^k q_s^k, \end{aligned}$$

where

$$a_{n,n}^k = \begin{cases} \Lambda_n \varepsilon_n A_n^k & (k > -1), \\ \dfrac{\Lambda_n \varepsilon_n}{n+1} & (k = -1). \end{cases}$$

Hence by a slight modification of a theorem of Chow [**2**, Lemma 6†], $\sum |a_{n,n}^k| < \infty$, which is equivalent to the condition in (*a*).

If $\lambda_n = n$ we find‡ that $\{\varepsilon_n\} \in ((C, k), |C, -1|)$, $k \geqq -1$, if and only if $\sum n^{k+1} |\varepsilon_n| < \infty$.

(*b*) The necessity of $\sum |\varepsilon_n| < \infty$ follows by the argument used in part (*a*). For the sufficiency, since $\sum a_n$ converges and $\mu_n a_n = o(1)$, we have

$$\sum |a_n \varepsilon_n| = O(1) \sum |\varepsilon_n| < \infty, \quad \text{and} \quad \sum |\Delta(\mu_n a_n \varepsilon_n)| = O(1) \sum |\varepsilon_n| < \infty.$$

(*c*) It is well known that (ii) is necessary and sufficient for $\varepsilon_n \in (|C, 0|, |C, 0|)$. By the theorem of Knopp and Lorentz referred to in Theorem 2, we find that (i) is necessary and sufficient for $\sum |\Delta(\mu_n a_n \varepsilon_n)| < \infty$ whenever $\sum |a_n| < \infty$. Hence (i) and (ii) are necessary and sufficient for (*c*).

† Chow's Lemma 6 still holds with o in place of O.
‡ This extends Chow's Theorem 2 [**2**].

If $\{\mu_n\}$ is bounded away from zero, then (i) implies (ii). In particular, when $\lambda_n = n$, we have $\{\varepsilon_n\} \in (|C, 0|, |C, -1|)$ if and only if $n\varepsilon_n = O(1)$; a result which extends a theorem of Peyerimhoff [7].

(*d*) Noting that

$$a_n = \frac{1}{n(n+1)} \sum_{m=1}^{n} m\Delta t_m \quad (n \geqq 1),$$

and again applying the Knopp-Lorentz theorem, we see that $\{\varepsilon_n\} \in (|C, -1|, |C, 0|)$ if and only if

$$\sup_{k>0} \sum_{n=k}^{\infty} \frac{k\,|\varepsilon_n|}{n(n+1)} < \infty,$$

which is equivalent to the condition in (*d*).

This completes the proof of Theorem 5.

In our last theorem we consider matrix transformations of $(C, \lambda_n, -1)$ summable series $\sum a_n$:

$$A_n(a) = \sum a_{nk} a_k.$$

We give necessary and sufficient conditions for $A = (a_{nk})$ to be regular, i.e. for $A_n \to s$ $(n \to \infty)$ whenever $\sum a_n = s(C, \lambda_n, -1)$. In a recently submitted note [**6**, Theorem 1], I have given conditions for the regularity of A, where $A_n \to s$ $(n \to \infty)$ whenever $\sum a_n = s(C, -1)$. Thus, Theorem 6 generalizes this result. Since the proof of Theorem 6 is essentially the same as that in [**6**], I do no more than indicate the argument.

THEOREM 6. *$A_n(a) \to s(n \to \infty)$ whenever $\sum a_n = s(C, \lambda_n, -1)$, if and only if there is a regular series to sequence matrix $B = (b_{nk})$ such that*

$$a_{nk} = b_{nk} + \mu_k (b_{nk} - b_{n,k+1}). \tag{3}$$

Proof. Let (3) hold and $\sum a_n = s(C, \lambda_n, -1)$. Since $\sum a_k$ converges to s and B is regular, the B-transform of $\sum a_k$ converges to s. Also the matrix $(b_{nk} - b_{n,k+1})$ takes null sequences into null sequences. Hence (3) is sufficient.

Now suppose that $A_n(a)$ exists for each n and $A_n(a) \to s$ $(n \to \infty)$ whenever

$$\sum a_n = s(C, \lambda_n, -1),$$

i.e. by (1), whenever $t \in c$ (c being the space of convergent sequences $t = \{t_n\}$ with norm $\|t\| = \sup |t_n|$). If we express $a = \{a_n\}$ in terms of t we easily find that $a_n = a_n(t)$ is a continuous linear functional on c. Since $\sum a_{nk} a_k$ converges for each n, it follows that $A_n(a)$ is a continuous linear functional on c. Thus, for each n [**1**, p. 65],

$$A_n(a) = d_n \lim t_k + \sum_k d_{nk} t_k, \tag{4}$$

$$\|A_n\| = |d_n| + \sum_k |d_{nk}|, \quad \text{with } \sum_k |d_{nk}| < \infty. \tag{5}$$

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Taking $a = e^{(k)}$, $e_i^{(k)} = 0$, $i \neq k$, $e_k^{(k)} = 1$, we deduce from (4) the existence of a matrix B such that (3) holds and $b_{nk} - b_{n,k+1} = d_{nk}$. Also we see that $a_{nk} \to 1$ ($n \to \infty$, k fixed). On applying the Banach-Steinhaus theorem, (5) yields

$$\sup_n \sum_k | b_{nk} - b_{n,k+1} | < \infty. \tag{6}$$

Taking $t = e^{(k)}$ we have $d_{nk} \to 0$ ($n \to \infty$, k fixed), whence

$$b_{nk} \to 1 \quad (n \to \infty,\ k \text{ fixed}). \tag{7}$$

But (6) and (7) are the conditions for B to be a regular series to sequence matrix, so the result is proved.

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