

COMPARING THE GENERALISED ROUNDNESS OF METRIC SPACES

LUKIEL LEVY-MOORE, MARGARET NICHOLS and ANTHONY WESTON[✉]

(Received 12 August 2016; accepted 27 August 2016; first published online 2 November 2016)

Abstract

Motivated by the local theory of Banach spaces, we introduce a notion of finite representability for metric spaces. This allows us to develop a new technique for comparing the generalised roundness of metric spaces. We illustrate this technique by applying it to Banach spaces and metric trees. In the realm of Banach spaces we obtain results such as the following: (1) if \mathcal{U} is any ultrafilter and X is any Banach space, then the second dual X^{**} and the ultrapower $(X)_{\mathcal{U}}$ have the same generalised roundness as X , and (2) no Banach space of positive generalised roundness is uniformly homeomorphic to c_0 or ℓ_p , $2 < p < \infty$. For metric trees, we give the first examples of metric trees of generalised roundness one that have finite diameter. In addition, we show that metric trees of generalised roundness one possess special Euclidean embedding properties that distinguish them from all other metric trees.

2010 *Mathematics subject classification*: primary 46B07; secondary 05C05, 05C12, 46B80.

Keywords and phrases: generalised roundness, negative type, uniform homeomorphism, scale isomorphism.

1. Introduction

Direct calculation of the generalised roundness of an infinite metric space is, in general, a difficult task. In this paper we develop a versatile technique for comparing the generalised roundness of metric spaces. This leads to substantial new insights into the generalised roundness of Banach spaces and metric trees.

DEFINITION 1.1. The *generalised roundness* of a metric space (X, d) , denoted by $\varphi_{(X,d)}$ or simply φ_X , is the supremum of the set of all $p \geq 0$ that satisfy the following condition: for all integers $k \geq 2$ and all choices of (not necessarily distinct) points $a_1, \dots, a_k, b_1, \dots, b_k \in X$,

$$\sum_{1 \leq i < j \leq k} \{d(a_i, a_j)^p + d(b_i, b_j)^p\} \leq \sum_{1 \leq i, j \leq k} d(a_i, b_j)^p. \quad (1.1)$$

The research in this paper was initiated at the 2011 Cornell University *Summer Mathematics Institute* (NSF grant DMS-0739338) and completed at the University of South Africa (Unisa). The second named author was partially supported by the National Science Foundation *Graduate Research Fellowship Program* (NSF grant DGE-1144082).

© 2016 Australian Mathematical Publishing Association Inc. 0004-9727/2016 \$16.00

The configuration of points $D_k = [a_1, \dots, a_k; b_1, \dots, b_k] \subseteq X$ underlying (1.1) will be called a *simplex* in X . We will say that $p \geq 0$ is a *generalised roundness exponent* for (X, d) if (1.1) holds for every simplex in X .

The notion of generalised roundness was introduced by Enflo [8] to study universal uniform embedding spaces. By showing that such spaces must have generalised roundness zero, Enflo was able to prove that Hilbert spaces are not universal uniform embedding spaces. This resolved a prominent question of Smirnov. Some time later, Lennard *et al.* [25] exhibited an important connection between generalised roundness and the classical isometric embedding notion of negative type. Lafont and Prassidis [23] used this connection to show that if a finitely generated group Γ has a Cayley graph of positive generalised roundness, then Γ must satisfy the coarse Baum–Connes conjecture and hence the strong Novikov conjecture. The interplay between these notions has a very interesting history (see Prassidis and Weston [30]).

The set of generalised roundness exponents of a given metric space (X, d) is always a closed interval of the form $[0, \varphi]$ or $[0, \infty)$, including the possibility that $\varphi = 0$, in which case the interval degenerates to $\{0\}$. This result is a direct consequence of Schoenberg [34, Theorem 2.7] and Lennard *et al.* [25, Theorem 2.4]. Faver *et al.* [10] have shown that the interval $[0, \infty)$ arises if and only if d is an ultrametric. For finite metric spaces it is always the case that $\varphi > 0$. This is the main result in Weston [36].

Enflo [8] constructed a separable metric space that is not uniformly embeddable in any metric space of positive generalised roundness. Dranishnikov *et al.* [7] modified Enflo's example to construct a locally finite metric space that is not coarsely embeddable in any Hilbert space, thereby settling a prominent question of Gromov. Kelleher *et al.* [19] unified these examples to construct a locally finite metric space that is not uniformly or coarsely embeddable in any metric space of positive generalised roundness. One may also use generalised roundness as a highly effective isometric invariant by exploiting the connection between generalised roundness and negative type due to Lennard *et al.* [25]. A general principle for using generalised roundness as an isometric invariant was recently isolated by Kelleher *et al.* [20, Theorem 3.24]. It is therefore a matter of great utility to be able to calculate the generalised roundness of certain metric spaces.

In recent work, Sánchez [33] has provided a method of calculating, at least numerically, the generalised roundness of a given finite metric space (X, d) . However, as the size of the space grows, Sánchez' method rapidly becomes computationally intensive. Nevertheless, the method is an important tool for the analysis of the generalised roundness of finite metric spaces. In [33], the method is used to calculate the generalised roundness of certain finite graphs endowed with the usual combinatorial metric. The metric graphs that we consider in this paper are countable metric trees and so we are unable to use Sánchez' method.

It is prudent at this point to pin down some basic definitions pertaining to metric graphs. A graph G is *connected* if there is a (finite) path between any two vertices of G . A *tree* is an undirected, connected, locally finite graph without cycles. These

definitions imply that the vertex and edge sets of a tree are at most countable. Assigning a positive length to each edge of a given tree T induces a shortest-path metric d on the vertices of the tree. The resulting metric space is denoted by (T, d) and is called a *metric tree*.

Generalised roundness properties of metric trees have been studied by several authors. All additive metric spaces, and hence all metric trees, have generalised roundness at least one. This fact is folklore and it may be derived in several different ways (see Kelly [21, Theorem II] and Faver *et al.* [10, Proposition 4.1]). Examples of Caffarelli *et al.* [2] show that some countable metric trees have generalised roundness exactly one. The situation is different for finite metric trees. Indeed, Hjorth *et al.* [14] have shown that all finite metric trees have strict 1-negative type, which ensures that all finite metric trees have generalised roundness greater than one. (One way to see this is to appeal to Lennard *et al.* [25, Theorem 2.4] and Li and Weston [27, Corollary 4.2].) Hence, metric trees of generalised roundness one are necessarily countable. Simple examples show that the converse of this statement is not true in general.

We conclude this introduction with some comments about the structure and main results of this paper. In Section 2, motivated by the local theory of Banach spaces, we introduce a notion of finite representability for metric spaces. Our purpose in introducing such a notion is to provide a new technique for comparing the generalised roundness of metric spaces. The remainder of Section 2 is then devoted to a preliminary investigation of this technique in the context of infinite-dimensional Banach spaces. We prove, for example, that if \mathcal{U} is any ultrafilter and X is any Banach space, then the second dual X^{**} and the ultrapower $(X)_{\mathcal{U}}$ have the same generalised roundness as X . In other words, $\phi_X = \phi_{X^{**}} = \phi_{(X)_{\mathcal{U}}}$. It is also noted that no Banach space of positive generalised roundness is uniformly homeomorphic to c_0 or ℓ_p , $2 < p < \infty$.

Caffarelli *et al.* [2] identified several classes of metric trees of generalised roundness one. The types of trees studied in [2] were spherically symmetric, infinitely bifurcating or comb-like trees endowed with the usual combinatorial path metric. In other words, all edges in the trees were assumed to have length one and all other distances were determined geodesically. In Sections 3 and 4 we relax this condition by considering trees endowed with weighted path metrics. Section 3 focuses on trees that resemble jagged combs. Section 4 deals with spherically symmetric trees that have systematically weighted edges. We also make a distinction between *convergent* and *divergent* spherically symmetric trees. In both cases we show that the generalised roundness of such trees can easily be one. In particular, we identify a large class of metric trees of generalised roundness one that have finite diameter.

In Section 5 we examine isometric embedding properties of metric trees of generalised roundness one. We prove that all metric trees of generalised roundness one possess the stronger property of strict 1-negative type. Due to the relationship between generalised roundness and negative type, it also follows that no metric tree of generalised roundness one has p -negative type for any $p > 1$. Taken together, these facts imply the following embedding phase transition: if (T, d) is a metric tree of

generalised roundness one, then (1) the metric transform (T, \sqrt{d}) is isometric to an affinely independent subset of ℓ_2 and (2) the metric transform $(T, \sqrt{d^p})$ does not embed isometrically into ℓ_2 for any $p, 1 < p \leq 2$. Moreover, the only metric trees that satisfy condition (2) are those of generalised roundness one.

2. Comparing the generalised roundness of metric and Banach spaces

In this section we develop a technique for comparing the generalised roundness of metric spaces. In order to do this we introduce a metric space version of the Banach space notion of finite representability (introduced by James [16, 17]). Throughout this section, all Banach spaces are assumed to be real and infinite dimensional unless noted otherwise. The first and second duals of a Banach space X are denoted by X^* and X^{**} , respectively. All L_p -spaces are assumed to be commutative unless noted otherwise.

DEFINITION 2.1. Let X and X' be Banach spaces.

- (1) X is *crudely represented* in X' if there exists an $\varepsilon_0 > 0$ such that for each finite-dimensional subspace $E \subset X$ there exist a finite-dimensional subspace $F \subset X'$ (with $\dim E = \dim F$) and a one-to-one linear mapping $T : E \rightarrow F$ that satisfies $\|T\| \|T^{-1}\| \leq 1 + \varepsilon_0$.
- (2) X is *finitely represented* in X' if for each $\varepsilon > 0$ and each finite-dimensional subspace $E \subset X$ there exist a finite-dimensional subspace $F \subset X'$ (with $\dim E = \dim F$) and a one-to-one linear mapping $T : E \rightarrow F$ that satisfies

$$(1 - \varepsilon)\|x\| \leq \|Tx\| \leq (1 + \varepsilon)\|x\| \quad \text{for all } x \in E. \tag{2.1}$$

It is easy to see that an equivalent reformulation of the condition given in Definition 2.1 is the following: for each $\varepsilon > 0$ and each finite-dimensional subspace $E \subset X$ there exist a finite-dimensional subspace $F \subset X'$ (with $\dim E = \dim F$) and a one-to-one linear mapping $T : E \rightarrow F$ with $\|T\| \|T^{-1}\| \leq 1 + \varepsilon$. While this reformulation makes the relationship between crude and finite representability plain, the metric nature of (2.1) suits our purposes and motivates Definitions 2.2 and 2.3 below.

The notion of crude representability is particularly important in the uniform theory of Banach spaces. Recall that two Banach spaces X and X' are said to be *uniformly homeomorphic* if there exists a bijection $f : X \rightarrow X'$ such that f and f^{-1} are both uniformly continuous. A famous result of Ribe [31] asserts that if a Banach space X is uniformly homeomorphic to a Banach space X' , then X is crudely represented in X' and X' is crudely represented in X (sometimes known as Ribe’s rigidity theorem).

A one-to-one linear mapping $T : E \rightarrow F$ that satisfies (2.1) is said to be a $(1 + \varepsilon)$ -*isomorphism*. A similar notion for metric spaces may be formulated as follows.

DEFINITION 2.2. Let (X, d) and (X', ρ) be metric spaces and suppose that $\varepsilon > 0$. A one-to-one mapping $\phi : X \rightarrow X' : x \mapsto x'$ is called a $(1 + \varepsilon)$ -*scale isomorphism* if there exists a constant $n = n(\varepsilon) > 0$ such that

$$(1 - \varepsilon)nd(a, b) \leq \rho(a', b') \leq (1 + \varepsilon)nd(a, b) \quad \text{for all } a, b \in X.$$

It is worth noting that we will use the notation x' to denote $\phi(x)$ throughout this section.

DEFINITION 2.3. A metric space (X, d) is said to be *locally represented* in a metric space (X', ρ) if for each $\varepsilon > 0$ and each nonempty finite set $X^\# \subseteq X$ there exists a $(1 + \varepsilon)$ -scale isomorphism $\phi : (X^\#, d) \rightarrow (X', \rho)$.

The following lemma notes that for Banach spaces, finite representation implies local representation.

LEMMA 2.4. *Let X and X' be given Banach spaces. If X is finitely represented in X' , then X is locally represented in X' .*

PROOF. Let $X^\#$ be a given nonempty finite subset of X and suppose that $\varepsilon > 0$. Let E denote the linear span of $X^\#$ in X . Then E is a finite-dimensional subspace of X . As X is finitely represented in X' , there exists a $(1 + \varepsilon)$ -isomorphism $T : E \rightarrow X'$. Setting ϕ to be the restriction of T to $X^\#$, we obtain a $(1 + \varepsilon)$ -scale isomorphism $X^\# \rightarrow X'$ (with constant $n = 1$). Hence, X is locally represented in X' . \square

We turn now to the main technical result of this section. It provides a new technique for comparing the generalised roundness of metric spaces.

THEOREM 2.5. *If a metric space (X, d) is locally represented in a metric space (X', ρ) , then every generalised roundness exponent of (X', ρ) is a generalised roundness exponent of (X, d) . Hence, $\wp_{X'} \leq \wp_X$.*

PROOF. It suffices to prove that if p is not a generalised roundness exponent of (X, d) , then p is not a generalised roundness exponent of (X', ρ) .

Suppose that $p \geq 0$ is not a generalised roundness exponent of (X, d) . We immediately have that $p > 0$ because 0 is a generalised roundness exponent of all metric spaces. From our definition, there must be a simplex $[a_i; b_j] \subseteq X$ such that

$$\sum_{i < j} (d(a_i, a_j)^p + d(b_i, b_j)^p) > \sum_{i, j} d(a_i, b_j)^p$$

and so we may choose an $\varepsilon > 0$ so that

$$(1 - \varepsilon)^p \cdot \sum_{i < j} (d(a_i, a_j)^p + d(b_i, b_j)^p) > (1 + \varepsilon)^p \cdot \sum_{i, j} d(a_i, b_j)^p. \tag{2.2}$$

We now let $X^\#$ denote the finite subset of X that consists of the simplex points a_i, b_j . As (X, d) is locally represented in (X', ρ) and $\varepsilon > 0$, there must exist an injection $\phi : X^\# \rightarrow X' : x \mapsto x'$ and a constant $n = n(\varepsilon) > 0$ such that

$$(1 - \varepsilon)nd(a, b) \leq \rho(a', b') \leq (1 + \varepsilon)nd(a, b)$$

for all $a, b \in X^\#$. If we scale the metric on X' by defining $\omega = \rho/n$, we obtain

$$(1 - \varepsilon)d(a, b) \leq \omega(a', b') \leq (1 + \varepsilon)d(a, b) \tag{2.3}$$

for all $a, b \in X^\sharp$. It now follows from (2.2) and (2.3) that p is not a generalised roundness exponent for the scaled metric space (X', ω) . Indeed,

$$\begin{aligned} \sum_{i < j} (\omega(a'_i, a'_j)^p + \omega(b'_i, b'_j)^p) &\geq (1 - \varepsilon)^p \cdot \sum_{i < j} (d(a_i, a_j)^p + d(b_i, b_j)^p) \\ &> (1 + \varepsilon)^p \cdot \sum_{i, j} d(a_i, b_j)^p = \sum_{i, j} ((1 + \varepsilon)d(a_i, b_j))^p \\ &\geq \sum_{i, j} \omega(a'_i, b'_j)^p. \end{aligned}$$

This completes the proof because generalised roundness is preserved under any scaling of the metric ρ . □

For the remainder of this section we will consider the application of Theorem 2.5 to Banach spaces. It is germane to recall a few facts about the generalised roundness of L_p -spaces. If X is an L_p -space, then $\wp_X = p$ if $1 \leq p \leq 2$ and $\wp_X = 0$ if $p > 2$. These results are due to Enflo [8] in the case $1 \leq p \leq 2$ and Lennard *et al.* [25] in the case $p > 2$. With the exception of the Schatten p -classes C_p , the generalised roundness of noncommutative L_p -spaces has not been widely studied. In [25], the authors noted that $\wp_{C_p} = 0$ if $p > 2$. It is only relatively recently that Dahma and Lennard [4] have shown that $\wp_{C_p} = 0$ if $0 < p < 2$.

COROLLARY 2.6. *If a Banach space X is finitely represented in a Banach space X' , then $\wp_{X'} \leq \wp_X$. In particular, if $\wp_X = 0$, then $\wp_{X'} = 0$.*

PROOF. Immediate from Lemma 2.4 and Theorem 2.5. □

Examples of Banach spaces that have generalised roundness zero include $C[0, 1]$, ℓ_∞ , c_0 , ℓ_p if $p > 2$, and the Schatten p -class C_p if $p \neq 2$. For each Banach space X and each integer $n \geq 2$, Dineen [6] has shown that ℓ_∞ is finitely represented in the space $\mathcal{P}^n(X)$ of bounded n -homogeneous polynomials on X . Hence, $\mathcal{P}^n(X)$ has generalised roundness zero by Corollary 2.6. For each $p \in (1, \infty)$, c_0 is finitely represented in the quasi-reflexive James space J_p . (This result is due to Giesy and James [11] in the case $p = 2$ and, for $p \neq 2$, it is due to Bird *et al.* [1].) Hence, for each $p \in (1, \infty)$, J_p has generalised roundness zero by Corollary 2.6.

On the basis of existing theory and Corollary 2.6, we are able to isolate some situations where generalised roundness functions as an invariant in the uniform theory of Banach spaces. For instance, as the next corollary shows, no Banach space of positive generalised roundness is uniformly homeomorphic to ℓ_p for any $p > 2$.

COROLLARY 2.7. *If a Banach space X is uniformly homeomorphic to ℓ_p ($1 \leq p < \infty$), then $\wp_X \leq \wp_{\ell_p}$. In particular, if $p > 2$, then $\wp_X = 0$.*

PROOF. Ribe’s rigidity theorem [31] implies that ℓ_p is crudely represented in X . However, if ℓ_p is crudely represented in X , then ℓ_p is finitely represented in X . This follows from Krivine’s theorem [22] (as noted by Rosenthal [32] and Lemberg [24]) if $1 < p < \infty$, and it is due to James [15] in the case $p = 1$. Thus, $\wp_X \leq \wp_{\ell_p}$ by Corollary 2.6. Moreover, if $p > 2$, then $\wp_{\ell_p} = 0$. So, for $p > 2$, we deduce that $\wp_X = 0$. □

The uniform structure of ℓ_p , $1 < p < \infty$, is particularly well understood. For instance, if a Banach space X is uniformly homeomorphic to ℓ_p , $1 < p < \infty$, then it is linearly isomorphic to ℓ_p . This deep theorem is due to Enflo [9, Theorem 6.3.1] in the case $p = 2$ and Johnson *et al.* [18, Theorem 2.1] when $p \neq 2$. So, if $1 < p < \infty$, one may replace the phrase ‘uniformly homeomorphic’ in the statement of Corollary 2.7 with the phrase ‘linearly isomorphic’ without losing any generality. The situation for c_0 is somewhat similar.

COROLLARY 2.8. *If a Banach space X is uniformly homeomorphic to c_0 , then $\wp_X = 0$. In particular, no Banach space of positive generalised roundness is uniformly homeomorphic to c_0 .*

PROOF. Ribe’s rigidity theorem [31] implies that c_0 is crudely represented in X . However, James [15] has shown that if c_0 is crudely represented in X , then c_0 is finitely represented in X . Thus, $\wp_X \leq \wp_{c_0}$ by Corollary 2.6 and $\wp_{c_0} = 0$ implies that $\wp_X = 0$. \square

The uniform structure of c_0 is more beguiling and less well understood than that of ℓ_p , $1 < p < \infty$. Johnson *et al.* [18, Corollary 3.2] proved that if a complemented subspace of a $C(K)$ space is uniformly homeomorphic to c_0 , then it is linearly isomorphic to c_0 . Godefroy *et al.* [12, Theorem 5.6] have shown that a Banach space which is uniformly homeomorphic to c_0 is an isomorphic predual of ℓ_1 with summable Szlenk index. But it is not known whether a predual of ℓ_1 with summable Szlenk index is linearly isomorphic to c_0 . Thus, unlike ℓ_p ($1 < p < \infty$), it remains unclear whether c_0 is determined by its uniform structure.

COROLLARY 2.9. *If X is a Banach space, then $\wp_X = \wp_{X^{**}}$ and $\wp_{X^*} = \wp_{X^{***}}$.*

PROOF. As X embeds isometrically into X^{**} , we see that $\wp_{X^{**}} \leq \wp_X$. In addition, the principle of local reflexivity (originally due to Lindenstrauss and Rosenthal [28]) implies that X^{**} is finitely represented in X . Hence, $\wp_X \leq \wp_{X^{**}}$ by Corollary 2.6. By combining these two inequalities, we obtain $\wp_X = \wp_{X^{**}}$. By replacing X with X^* , we also see that $\wp_{X^*} = \wp_{X^{***}}$. \square

Examples show that for a Banach space X we may have $\wp_X \neq \wp_{X^*}$. Indeed, if $p \in [1, 2)$ and $X = \ell_p$, then $\wp_X = p$ and $\wp_{X^*} = 0$. However, if $p \neq 2$ and $X = C_p$, then $\wp_X = \wp_{X^*} = 0$. Thus, given a Banach space X , the entries of the sequence $(\wp_X, \wp_{X^*}, \wp_{X^{**}}, \dots)$ take on at most two values (in the interval $[0, 2]$) by Corollary 2.9.

Intimately related to the concept of finite representability is the notion of an ultrapower of a Banach space. Given an ultrafilter \mathcal{U} on a set I and a Banach space X , there is a canonical procedure to construct a large Banach space $(X)_{\mathcal{U}}$ called the *ultrapower* of X . Importantly, $(X)_{\mathcal{U}}$ contains a natural isometric copy of X and it is finitely represented in X . For a detailed construction of $(X)_{\mathcal{U}}$, and a discussion of the interplay between finite representability and ultrapowers, see Hájek and Johanis [13].

COROLLARY 2.10. *Let \mathcal{U} be a given ultrafilter on a set I and let X be a Banach space. Then $\wp_X = \wp_{(X)_{\mathcal{U}}}$.*

PROOF. As X embeds isometrically into $(X)_{\mathcal{U}}$, we see that $\wp_{(X)_{\mathcal{U}}} \leq \wp_X$. In addition, $(X)_{\mathcal{U}}$ is finitely represented in X by Stern [35, Theorem 6.6]. Hence, $\wp_X \leq \wp_{(X)_{\mathcal{U}}}$ by Corollary 2.6. By combining these two inequalities, we obtain $\wp_X = \wp_{(X)_{\mathcal{U}}}$. \square

Lennard *et al.* [26, Theorem 2.3] noticed that if the infimal cotype of a Banach space X is greater than two, then X must have generalised roundness zero. By utilising deep theory and Corollary 2.6, we are able to exhibit a more precise relationship between the supremal type and the infimal cotype of a Banach space and its generalised roundness. The notions of type and cotype have been paramount in the local theory of Banach spaces for quite some time and are defined in the following manner.

DEFINITION 2.11. A Banach space X is said to have type p if there exists a constant $\mathcal{A} \in (0, \infty)$ such that for all integers $n > 0$ and for all finite sequences $(x_j)_{j=1}^n$ in X ,

$$\sum_{\epsilon \in \{-1, +1\}^n} \left\| \sum_{j=1}^n \frac{\epsilon_j x_j}{2^n} \right\|_X \leq \mathcal{A} \left(\sum_{j=1}^n \|x_j\|_X^p \right)^{1/p}. \tag{2.4}$$

Cotype p is defined similarly but with the inequality (2.4) reversed.

It is well known that no Banach space can have type $p > 2$ or cotype $q < 2$. We let $p(X)$ denote the supremum of all p such that X has type p and $q(X)$ denote the infimum of all q such that X has cotype q . For an overview of theory of type and cotype, we refer the reader to Diestel *et al.* [5].

A famous theorem of Maurey and Pisier [29] states that $\ell_{p(X)}$ and $\ell_{q(X)}$ are finitely represented in X . This provides an immediate link to generalised roundness.

COROLLARY 2.12. *If X is a Banach space, then $\wp_X \leq \min\{\wp_{\ell_{p(X)}}, \wp_{\ell_{q(X)}}\}$. In particular, if $q(X) > 2$, then $\wp_X = 0$.*

PROOF. By the Maurey–Pisier theorem, $\ell_{p(X)}$ and $\ell_{q(X)}$ are finitely represented in X . Hence, $\wp_X \leq \wp_{\ell_{p(X)}}$ and $\wp_X \leq \wp_{\ell_{q(X)}}$ by Corollary 2.6. In particular, if $q(X) > 2$, then $\wp_{\ell_{q(X)}} = 0$ and so $\wp_X = 0$. \square

There are some classical Banach spaces for which the inequality in Corollary 2.12 is an equality. For example, if X is an L_p -space, $1 \leq p < \infty$, then $\wp_X = \min\{\wp_{\ell_{p(X)}}, \wp_{\ell_{q(X)}}\}$. In this case, $\wp_X = p$ if $1 \leq p \leq 2$ and $\wp_X = 0$ if $p > 2$. Moreover, it is well known that $p(X) = \min\{p, 2\}$ and $q(X) = \max\{p, 2\}$. So, for example, if $p > 2$, then $\wp_X = 0$ and $q(X) = p$. Thus, $\wp_{\ell_{q(X)}} = 0$. On the other hand, if $X = C_p$, $1 \leq p < \infty$, then $p(X)$ and $q(X)$ have the same values as any L_p -space but, by inspection, $\wp_X = \min\{\wp_{\ell_{p(X)}}, \wp_{\ell_{q(X)}}\}$ if and only if $p \geq 2$.

3. Comb-like graphs of generalised roundness one

In this section we apply Theorems 2.5 and 3.3 to analyse the generalised roundness of countable metric trees that resemble combs. We first give sufficient conditions for the existence of a $(1 + \varepsilon)$ -scale isomorphism $\phi : (T, d) \rightarrow (T, \rho)$ under the assumption that d and ρ are path-weighted metrics on a given finite tree T . In what follows, we let

\mathbb{N} denote the set of all nonnegative integers. Moreover, given a positive integer m , we let $[m]$ denote the segment $\{0, 1, 2, \dots, m\}$.

LEMMA 3.1. *Let d and ρ be two path-weighted metrics on a given finite tree T . Let*

$$m = \max \left\{ \frac{\rho(\mathbf{a}, \mathbf{b})}{d(\mathbf{a}, \mathbf{b})} : \mathbf{a}, \mathbf{b} \in T \text{ and } \mathbf{a} \neq \mathbf{b} \right\}.$$

Then there must be an edge $\{\mathbf{x}, \mathbf{y}\}$ in T such that $m = \rho(\mathbf{x}, \mathbf{y})/d(\mathbf{x}, \mathbf{y})$.

PROOF. Suppose that $\mathbf{a}, \mathbf{c} \in T$ are nonadjacent vertices such that $m = \rho(\mathbf{a}, \mathbf{c})/d(\mathbf{a}, \mathbf{c})$. Then we may choose a strictly intermediate vertex $\mathbf{b} \in T$ on the geodesic from \mathbf{a} to \mathbf{c} . Now let $q = \rho(\mathbf{a}, \mathbf{b})/d(\mathbf{a}, \mathbf{b})$ and $r = \rho(\mathbf{b}, \mathbf{c})/d(\mathbf{b}, \mathbf{c})$. Without loss of generality, we may assume that $q \geq r$. Furthermore, as ρ and d are path metrics on T , we have $\rho(\mathbf{a}, \mathbf{c}) = \rho(\mathbf{a}, \mathbf{b}) + \rho(\mathbf{b}, \mathbf{c})$ and $d(\mathbf{a}, \mathbf{c}) = d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c})$. In particular, it follows that

$$m = \frac{\rho(\mathbf{a}, \mathbf{c})}{d(\mathbf{a}, \mathbf{c})} = \frac{\rho(\mathbf{a}, \mathbf{b}) + \rho(\mathbf{b}, \mathbf{c})}{d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c})} = \frac{qd(\mathbf{a}, \mathbf{b}) + rd(\mathbf{b}, \mathbf{c})}{d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c})} \leq \frac{qd(\mathbf{a}, \mathbf{b}) + qd(\mathbf{b}, \mathbf{c})}{d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c})} = q.$$

Therefore, by definition of m , it must be the case that $m = q$. This shows that we can always pass to a pair of vertices connected by a geodesic with fewer edges and preserve the ratio m . Applying this logic finitely many times gives the lemma. \square

The following analogous lemma for minima may be proved in the same way.

LEMMA 3.2. *Let d and ρ be two path-weighted metrics on a given finite tree T . Let*

$$m^* = \min \left\{ \frac{\rho(\mathbf{a}, \mathbf{b})}{d(\mathbf{a}, \mathbf{b})} : \mathbf{a}, \mathbf{b} \in T \text{ and } \mathbf{a} \neq \mathbf{b} \right\}.$$

Then there must be an edge $\{\mathbf{x}, \mathbf{y}\}$ in T such that $m^ = \rho(\mathbf{x}, \mathbf{y})/d(\mathbf{x}, \mathbf{y})$.*

THEOREM 3.3. *Let $\varepsilon > 0$ be given. Let d and ρ be two path-weighted metrics on a given finite tree T . If there exists a constant $n = n(\varepsilon) > 0$ such that*

$$(1 - \varepsilon)n \leq \frac{\rho(\mathbf{x}, \mathbf{y})}{d(\mathbf{x}, \mathbf{y})} \leq n(1 + \varepsilon) \tag{3.1}$$

for each edge $\{\mathbf{x}, \mathbf{y}\}$ in T , then the identity map $\phi : (T, d) \rightarrow (T, \rho) : x \mapsto x$ is a $(1 + \varepsilon)$ -scale isomorphism.

PROOF. Using the notation of Lemmas 3.1 and 3.2, it follows from (3.1) that

$$(1 - \varepsilon)n \leq m^* \leq m \leq n(1 + \varepsilon).$$

Thus, given any two distinct vertices $\mathbf{a}, \mathbf{b} \in T$, the definitions of m and m^* give $(1 - \varepsilon)n \leq \rho(\mathbf{a}, \mathbf{b})/d(\mathbf{a}, \mathbf{b}) \leq n(1 + \varepsilon)$. Hence, the identity map $\phi : (T, d) \rightarrow (T, \rho) : x \mapsto x$ is a $(1 + \varepsilon)$ -scale isomorphism. \square

We now apply Theorems 2.5 and 3.3 to analyse the generalised roundness of certain countable metric trees that resemble combs.

DEFINITION 3.4. The vertex set V of the *infinite comb* C consists of the points $\mathbf{x}_0, \mathbf{x}_k$ and \mathbf{y}_k , where k is any positive integer. The edge set E of C consists of the unordered pairs $\{\mathbf{x}_{k-1}, \mathbf{x}_k\}$ and $\{\mathbf{x}_k, \mathbf{y}_k\}$, where k is any positive integer.

For each positive integer m , the *m-comb*, denoted by C_m , is the finite subtree of C with vertex set $\{\mathbf{x}_{1+k}, \mathbf{y}_{1+k} \mid k \in [m]\}$.

We are interested in placing various path metrics on the infinite comb C and the m -comb C_m . One way to do this is to adopt the following canonical procedure.

DEFINITION 3.5. Let $f : \mathbb{N} \rightarrow (0, \infty)$ be a function. We define a path metric ρ_f on the infinite comb C in the following manner:

- (1) $\rho_f(\mathbf{x}_{k-1}, \mathbf{x}_k) = f(k - 1)$; and
- (2) $\rho_f(\mathbf{x}_k, \mathbf{y}_k) = f(k)$ for each positive integer k .

All other distances in C are then determined geodesically. The resulting metric tree will be denoted by $C(f)$. If, moreover, we restrict ρ_f to the m -comb, the resulting metric tree will be denoted by $C_m(f)$.

There are some special cases of Definition 3.5 worth highlighting. If $f(k) = 1$ for all $k \geq 0$, the resulting metric trees $C(f)$ and $C_m(f)$ will be denoted by $C(1)$ and $C_m(1)$, respectively. In other words, $C(1)$ and $C_m(1)$ are the combs C and C_m endowed with the usual combinatorial path metric δ . Caffarelli *et al.* [2] have shown that $\varphi_{C_m(1)} \rightarrow 1$ as $m \rightarrow \infty$. Hence, $\varphi_{C(1)} = 1$. We will see presently that by placing mild assumptions on the function f it follows that $\varphi_{C(f)} = 1$.

LEMMA 3.6. *If the m -comb $C_m(1)$ is locally represented in a metric tree (T, d) for all integers $m > 0$, then $\varphi_{(T,d)} = 1$.*

PROOF. All metric trees have generalised roundness at least one. By Theorem 2.5, $\varphi_{(T,d)} \leq \varphi_{C_m(1)}$ for all $m > 0$. Since $\varphi_{C_m(1)} \rightarrow 1$ as $m \rightarrow \infty$ by [2], $\varphi_{(T,d)} = 1$. □

DEFINITION 3.7. A function $f : \mathbb{N} \rightarrow (0, \infty)$ is said to be *additively subexponential* if $\lim_{n \rightarrow \infty} f(n + m)/f(n) = 1$ for each integer $m > 0$.

The class of additively subexponential functions $f : \mathbb{N} \rightarrow (0, \infty)$ is very large. For instance, f could be any rational function that takes positive values on \mathbb{N} . Other interesting possibilities for f include inverse tangent, logarithmic functions (translated suitably) and classically subexponential functions such as $e^{\sqrt{n}}$. Furthermore, note that if $f : \mathbb{N} \rightarrow (0, \infty)$ is an additively subexponential function, then so is $1/f$.

THEOREM 3.8. *Suppose $f : \mathbb{N} \rightarrow (0, \infty)$ is an additively subexponential function. Then $\varphi_{C(f)} = 1$.*

PROOF. By Lemma 3.6, it suffices to prove that $C_m(1)$ is locally represented in $C(f)$ for all $m > 0$. Let $m > 0$ be a given integer. Let $\varepsilon > 0$ be given. As f is additively

subexponential, we may choose an integer $n_0 > 0$ so that $1 - \varepsilon \leq f(n + k)/f(n) \leq 1 + \varepsilon$ for each $k \in [m]$ and all $n \geq n_0$. In particular,

$$1 - \varepsilon \leq \frac{f(n_0 + k)}{f(n_0)} \leq 1 + \varepsilon$$

for each $k \in [m]$. Consider the subtree Y' of $C(f)$ that has vertices \mathbf{x}_{n_0+k} and \mathbf{y}_{n_0+k} for all $k \in [m]$. As simple (unweighted) graphs, Y' and $C_m(1)$ are one and the same graph, namely, C_m . Let $\phi : C_m(1) \rightarrow Y'$ denote this natural identification. Let ρ denote the path metric that Y' inherits from $C(f)$. We may regard the metrics on Y' and $C_m(1)$ as path metrics on C_m . For each edge $\{\mathbf{s}, \mathbf{t}\}$ in C_m , we have, by choice of n_0 ,

$$f(n_0)(1 - \varepsilon) \leq \min_{k \in [m]} f(n_0 + k) \leq \frac{\rho(\phi(\mathbf{s}), \phi(\mathbf{t}))}{\delta(\mathbf{s}, \mathbf{t})} \leq \max_{k \in [m]} f(n_0 + k) < f(n_0)(1 + \varepsilon).$$

It follows from Theorem 3.3 that $\phi : C_m(1) \rightarrow Y'$ is a $(1 + \varepsilon)$ -scale isomorphism. As $C_m(1)$ is a finite metric space and as $\varepsilon > 0$ was arbitrary, we conclude that $C_m(1)$ is locally represented in $C(f)$. □

It also follows from Theorem 3.8 that if f is an additively subexponential function, then $\wp_{C(f)} = \wp_{C(1/f)}$.

4. Convergent and divergent spherically symmetric trees of generalised roundness one

Caffarelli *et al.* [2] considered the generalised roundness of spherically symmetric trees endowed with the usual combinatorial path metric (wherein all edges in the tree are assumed to have unit length). In this section we consider a broader class of spherically symmetric trees by relaxing the requirement that all edges have unit length. This allows us to make a distinction between *convergent* and *divergent* spherically symmetric trees.

Given a vertex \mathbf{v}_0 in a tree T , we let $r(T, \mathbf{v}_0) = \sup\{\delta(\mathbf{v}_0, \mathbf{v}) : \mathbf{v} \in T\}$, where δ denotes the usual combinatorial path metric on T . We call $r(T, \mathbf{v}_0)$ the \mathbf{v}_0 -depth of T . Naturally included here is the possibility that the \mathbf{v}_0 -depth of T may be infinite. A vertex \mathbf{v} of T is a *level- k* vertex of T if $\delta(\mathbf{v}_0, \mathbf{v}) = k$. The *children* of a level- k vertex $\mathbf{v} \in T$ consist of all level- $(k + 1)$ vertices $\mathbf{w} \in T$ such that $\delta(\mathbf{v}, \mathbf{w}) = 1$. We let $d_k(\mathbf{v})$ denote the number of children of \mathbf{v} .

DEFINITION 4.1. A tree T is said to be *spherically symmetric* if we can choose a vertex $\mathbf{v}_0 \in T$ so that for any k , all level- k vertices of T have the same number of children. Such a pair (T, \mathbf{v}_0) will be called a *spherically symmetric tree* (SST).

Notice that if \mathbf{v} is a level- k vertex in a given SST (T, \mathbf{v}_0) , then $d_k = d_k(\mathbf{v})$ depends only upon k . Thus, d_k is the number of children of any level- k vertex in T . We call the (possibly finite) sequence $(d_k)_{0 \leq k < r(T, \mathbf{v}_0)}$ the *downward degree sequence* of (T, \mathbf{v}_0) . We will say that (d_k) is *nontrivial* provided $d_k > 1$ for at least one k such that $0 \leq k < r(T, \mathbf{v}_0)$.

Now suppose that (T, \mathbf{v}_0) is a given SST with downward degree sequence (d_k) . If, for each k such that $0 \leq k < r(T, \mathbf{v}_0)$, l_k is a positive real number, we will call $\ell = (l_0, l_1, l_2, \dots)$ a *downward length sequence* for (T, \mathbf{v}_0) . Given such a sequence ℓ , we may define a path metric ρ_ℓ on T in the following manner. For any k such that $0 \leq k < r(T, \mathbf{v}_0)$, if \mathbf{w} is a child of a level- k vertex $\mathbf{v} \in T$, we define $\rho_\ell(\mathbf{v}, \mathbf{w}) = l_k$. All other ρ_ℓ -distances in T are then determined geodesically. The resulting metric tree (T, ρ_ℓ) is said to be *convergent* if $\sum l_k < \infty$ and *divergent* if $\sum l_k = \infty$. One significance of convergent SSTs is that they have finite diameter.

We proceed to show that large classes of divergent and convergent SSTs have generalised roundness one. The following lemma is a variation of [2, Theorem 2.1]. As the statement of the lemma is complicated, we will comment on the intuition behind this result. Among all n -point metric trees endowed with the usual combinatorial path metric, the complete bipartite graph $K_{1,n-1}$ has the smallest generalised roundness. Moreover, as $n \rightarrow \infty$, the generalised roundness of $K_{1,n-1}$ tends to one. The conditions placed on the SST in the statement of the following lemma ensure that it contains a star-like structure that resembles $K_{1,q}$, $q = d_0 d_1 \cdots d_k$, modulo scaling. Such an SST must have generalised roundness relatively close to one.

LEMMA 4.2. *Let (T, \mathbf{v}_0) be a finite SST with a nontrivial downward degree sequence $(d_0, d_1, \dots, d_{n-1})$. Suppose that $\ell = (l_0, l_1, \dots, l_{n-1})$ is a downward length sequence for (T, \mathbf{v}_0) that satisfies $2l_0 < l_0 + l_1 + \dots + l_{n-1}$. For each k , $1 \leq k \leq n$, set $M_k = \sum_{i=0}^{k-1} l_i$ and let m be the largest integer k such that $M_k < \frac{1}{2}M_n$. Then, for each nonnegative integer $k \leq m$ such that $d_0 d_1 \cdots d_k > 1$,*

$$\wp_{(T, \rho_\ell)} \leq \frac{\ln\left(2 + \frac{2}{(d_0 d_1 \cdots d_k)^{-1}}\right)}{\ln\left(2 - \frac{2M_k}{M_n}\right)}.$$

If $d_0 d_1 \cdots d_m = 1$, then we have the trivial bound $\wp_{(T, \rho_\ell)} \leq 2$.

PROOF. Let (T, \mathbf{v}_0) be a finite SST that satisfies the hypotheses of the lemma. Because at least one $d_j > 1$, there must exist at least one vertex $\mathbf{z} \in T$ with at least two children $\mathbf{x}, \mathbf{y} \in T$. As children in T are ρ_ℓ -equidistant from their parents, we see that

$$\rho_\ell(\mathbf{x}, \mathbf{z}) = \frac{\rho_\ell(\mathbf{x}, \mathbf{y})}{2} = \rho_\ell(\mathbf{z}, \mathbf{y}).$$

The existence of such a metric mid point in (T, ρ_ℓ) ensures that $\wp_{(T, \rho_\ell)} \leq 2$.

Now assume that $d_0 d_1 \cdots d_m > 1$ and consider any nonnegative integer $k \leq m$ such that $d_0 d_1 \cdots d_k > 1$. Then there are $d_0 d_1 \cdots d_{k-1}$ vertices at distance M_k from \mathbf{v}_0 . For each of the d_k children of such a vertex, choose a leaf which is a descendant of that child (or the child itself if it is a leaf). This results in a total of $q = d_0 d_1 \cdots d_k > 1$ distinct leaves, which we label $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_q$. Set $\mathbf{b}_j = \mathbf{v}_0$ for all j such that $1 \leq j \leq q$. For all i and j , we have $\rho_\ell(\mathbf{a}_i, \mathbf{b}_j) = M_n$. Moreover, for all $i \neq j$, we have $\rho_\ell(\mathbf{a}_i, \mathbf{a}_j) \geq 2(M_n - M_k)$ and $\rho_\ell(\mathbf{b}_i, \mathbf{b}_j) = 0$. It follows that any generalised roundness exponent p

of (T, ρ_ℓ) must satisfy

$$\frac{1}{2}q(q-1)(2(M_n - M_k))^p \leq \sum_{i < j} \{\rho_\ell(\mathbf{a}_i, \mathbf{a}_j)^p + \rho_\ell(\mathbf{b}_i, \mathbf{b}_j)^p\} \leq \sum_{i,j} \rho_\ell(\mathbf{a}_i, \mathbf{b}_j)^p = q^2 M_n^p. \tag{4.1}$$

By comparing the left- and right-hand sides of (4.1), it follows that p must satisfy

$$p \leq \frac{\ln(2 + \frac{2}{(d_0 d_1 \dots d_k) - 1})}{\ln(2 - \frac{2M_k}{M_n})}.$$

As p was an arbitrary generalised roundness exponent of (T, ρ_ℓ) , we conclude that the lemma holds. □

THEOREM 4.3. *Let (T, \mathbf{v}_0) be a countable SST with downward degree sequence (d_0, d_1, d_2, \dots) and downward length sequence $\ell = (l_0, l_1, l_2, \dots)$. If $d_i > 1$ for infinitely many i and $\sum l_i = \infty$, then $\wp_{(T, \rho_\ell)} = 1$.*

PROOF. For each positive integer n , let (T_n, \mathbf{v}_0) denote the finite SST with downward degree sequence $(d_0, d_1, \dots, d_{n-1})$ and downward length sequence $\ell = (l_0, l_1, \dots, l_{n-1})$. For each k , $1 \leq k \leq n$, set $M_k = \sum_{i=0}^{k-1} l_i$. As $\sum l_i = \infty$, we have $2l_0 < l_0 + l_1 + \dots + l_{n-1}$ provided n is sufficiently large. Moreover, for each such integer n , we may choose the largest integer $k = k(n)$ such that $M_k \leq \ln M_n$. As $n \rightarrow \infty$, the quantities k, M_k, M_n and $d_0 d_1 \dots d_k$ all tend to ∞ . However, by construction, $(2M_k)/M_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, by Lemma 4.2, $\wp_{(T_n, \rho_\ell)} \rightarrow 1$ as $n \rightarrow \infty$ and so we conclude that $\wp_{(T, \rho_\ell)} = 1$. □

THEOREM 4.4. *Let $f : \mathbb{N} \rightarrow (0, \infty)$ be an additively subexponential function. Let (T, \mathbf{v}_0) be a countable SST with downward degree sequence (d_k) . Let ℓ denote the downward length sequence $(f(k))$. If $d_k > 1$ for each integer $k \geq 0$, then $\wp_{(T, \rho_\ell)} = 1$.*

PROOF. The condition $d_k > 1$ for each integer $k \geq 0$ ensures that the infinite comb $C(f)$ is isometric to a metric subspace of (T, ρ_ℓ) . Thus, $\wp_{(T, \rho_\ell)} \leq \wp_{C(f)}$. Moreover, $\wp_{C(f)} = 1$ by Theorem 3.8 and $\wp_{(T, \rho_\ell)} \geq 1$, thereby forcing $\wp_{(T, \rho_\ell)} = 1$. □

Theorem 4.4 provides examples of convergent SSTs with generalised roundness one. For instance, we may simply set $d_k = 2$ and $f(k) = (k + 1)^{-2}$ for all $k \geq 0$ to obtain a countable SST that is convergent and has generalised roundness one. In particular, such SSTs have finite diameter.

5. Embedding properties of metric trees of generalised roundness one

We conclude this paper with some comments on the special Euclidean embedding properties of metric trees of generalised roundness one that set them apart from all other metric trees. As noted already, the notions of generalised roundness and negative type are equivalent. To make this statement more precise, we recall the following definition, the roots of which can be traced back to an 1841 paper of Cayley [3].

DEFINITION 5.1. Let $p \geq 0$ and let (X, d) be a metric space. Then:

- (1) (X, d) has p -negative type if and only if, for all integers $k \geq 2$, all finite subsets $\{x_1, \dots, x_k\} \subseteq X$ and all choices of real numbers η_1, \dots, η_k with $\eta_1 + \dots + \eta_k = 0$,

$$\sum_{1 \leq i, j \leq k} d(x_i, x_j)^p \eta_i \eta_j \leq 0; \tag{5.1}$$

- (2) (X, d) has strict p -negative type if and only if it has p -negative type and the inequalities (5.1) are all strict except in the trivial case $(\eta_1, \dots, \eta_k) = (0, \dots, 0)$.

Lennard *et al.* [25] proved that for all $p \geq 0$, a metric space (X, d) has p -negative type if and only if p is a generalised roundness exponent of (X, d) . The significance of this result is that it builds a bridge between Enflo’s [8] notion of generalised roundness and classical isometric embedding theory. These connections, in conjunction with contemporary results on strict negative type, yield the following theorem.

THEOREM 5.2. *If (T, d) is a metric tree of generalised roundness one, then:*

- (1) *the metric transform (T, \sqrt{d}) is isometric to an affinely independent subset of ℓ_2 ;*
- (2) *the metric transform $(T, \sqrt{d^p})$ does not embed isometrically into ℓ_2 for any p , $1 < p \leq 2$.*

The only metric trees that satisfy condition (2) are those of generalised roundness one.

PROOF. The vertex set of (T, d) is countable because it is a metric tree of generalised roundness one. Our definitions imply that each finite subset of T is contained in a finite subtree of T . Hjorth *et al.* [14] have shown that all finite metric trees have strict 1-negative type. Hence, each finite metric subspace of (T, d) has strict 1-negative type. This ensures that (T, d) has strict 1-negative type. Equivalently, the metric transform (T, \sqrt{d}) has strict 2-negative type. Therefore, (T, \sqrt{d}) is isometric to an affinely independent subset of ℓ_2 by Kelleher *et al.* [20, Theorem 5.6]. This gives condition (1).

If the metric transform $(T, \sqrt{d^p})$ were to embed isometrically into ℓ_2 for some $p \in (1, 2]$, this would imply that (T, d) has p -negative type. But, by Lennard *et al.* [25], this would mean that p is a generalised roundness exponent of (T, d) , thereby contradicting our assumption that $\varphi_{(T,d)} = 1$. This establishes condition (2).

On the other hand, if a metric tree (Z, d) is not of generalised roundness one, then it must be the case that $\varphi_{(Z,d)} > 1$ (because 1 is a generalised roundness exponent of all metric trees). By Lennard *et al.* [25], this implies that (Z, d) has p -negative type for some $p \in (1, 2]$. Consequently, the metric transform $(T, \sqrt{d^p})$ embeds isometrically into ℓ_2 by Kelleher *et al.* [20, Theorem 5.6]. We conclude that the only metric trees that satisfy condition (2) are those of generalised roundness one. □

The proof of Theorem 5.2 shows that all metric trees have strict 1-negative type. Therefore, every metric tree satisfies Theorem 5.2(1) by the result of Kelleher *et al.* [20].

Acknowledgements

We are indebted to the *Visiting Researcher Support Programme* at Unisa, the US National Science Foundation and the Department of Mathematics and the Center for Applied Mathematics at Cornell University for their support.

References

- [1] A. Bird, G. Jameson and N. J. Laustsen, 'The Giesy–James theorem for general index p , with an application to operator ideals on the p th James space', *J. Operator Theory* **70** (2013), 291–307.
- [2] E. Caffarelli, I. Doust and A. Weston, 'Metric trees of generalized roundness one', *Aequationes Math.* **83** (2012), 239–256.
- [3] A. Cayley, 'On a theorem in the geometry of position', *Cambridge Math. J.* **II** (1841), 267–271. Also in *The Collected Mathematical Papers of Arthur Cayley*, Vol. I (Cambridge University Press, Cambridge, 1889), 1–4.
- [4] A. M. Dahma and C. J. Lennard, 'Generalized roundness of the Schatten class, C_p ', *J. Math. Anal. Appl.* **412** (2014), 676–684.
- [5] J. Diestel, H. Jarchow and A. M. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, 43 (Cambridge University Press, Cambridge, 1995).
- [6] S. Dineen, 'A Dvoretzky theorem for polynomials', *Proc. Amer. Math. Soc.* **123** (1995), 2817–2821.
- [7] A. N. Dranishnikov, G. Gong, V. Lafforgue and G. Yu, 'Uniform embeddings into Hilbert space and a question of Gromov', *Canad. Math. Bull.* **45** (2002), 60–70.
- [8] P. Enflo, 'On a problem of Smirnov', *Ark. Mat.* **8** (1969), 107–109.
- [9] P. Enflo, 'Uniform structures and square roots in topological spaces II', *Israel J. Math.* **8** (1970), 253–272.
- [10] T. Faver, K. Kochalski, M. Murugan, H. Verheggen, E. Wesson and A. Weston, 'Roundness properties of ultrametric spaces', *Glasg. Math. J.* **56** (2014), 519–535.
- [11] D. P. Giesy and R. C. James, 'Uniformly non- ℓ_1 and B -convex Banach spaces', *Studia Math.* **48** (1973), 61–69.
- [12] G. Godefroy, N. J. Kalton and G. Lancien, 'Szlenk indices and uniform homeomorphisms', *Trans. Amer. Math. Soc.* **353** (2001), 3895–3918.
- [13] P. Hájek and M. Johanis, *Smooth Analysis in Banach Spaces*, De Gruyter Series in Nonlinear Analysis and Applications, 19 (De Gruyter, Berlin, 2014), 1–497.
- [14] P. Hjorth, P. Lisoněk, S. Markvorsen and C. Thomassen, 'Finite metric spaces of strictly negative type', *Linear Algebra Appl.* **270** (1998), 255–273.
- [15] R. C. James, 'Uniformly nonsquare Banach spaces', *Ann. of Math. (2)* **80** (1964), 542–550.
- [16] R. C. James, 'Some self-dual properties of normed linear spaces', in: *Sympos. Infinite Dimensional Topology*, Annals of Mathematics Studies, 69 (Princeton University Press, Princeton, NJ, 1972).
- [17] R. C. James, 'Super-reflexive Banach spaces', *Canad. J. Math.* **24** (1972), 896–904.
- [18] W. B. Johnson, J. Lindenstrauss and G. Schechtman, 'Banach spaces determined by their linear structure', *Geom. Funct. Anal.* **6** (1996), 430–470.
- [19] C. Kelleher, D. Miller, T. Osborn and A. Weston, 'Strongly non-embeddable metric spaces', *Topology Appl.* **159** (2011), 749–755.
- [20] C. Kelleher, D. Miller, T. Osborn and A. Weston, 'Polygonal equalities and virtual degeneracy in L_p -spaces', *J. Math. Anal. Appl.* **415** (2014), 247–268.
- [21] J. B. Kelly, 'Hypermetric spaces', in: *Proc. Conf. Geometry of Metric and Linear Spaces*, Lecture Notes in Mathematics, 490 (Springer, Berlin, 1975), 17–31.
- [22] J. L. Krivine, 'Sous-espaces de dimension finie des espaces de Banach réticulé', *Ann. of Math. (2)* **104** (1976), 1–29.
- [23] J.-F. Lafont and S. Prassidis, 'Roundness properties of groups', *Geom. Dedicata* **117** (2006), 137–160.

- [24] H. Lemberg, 'Nouvelle démonstration d'un théorème de J. L. Krivine sur la finie représentation de ℓ_p dans un espace de Banach', *Israel J. Math.* **39** (1981), 341–348.
- [25] C. J. Lennard, A. M. Tonge and A. Weston, 'Generalized roundness and negative type', *Michigan Math. J.* **44** (1997), 37–45.
- [26] C. J. Lennard, A. M. Tonge and A. Weston, 'Roundness and metric type', *J. Math. Anal. Appl.* **252** (2000), 980–988.
- [27] H. Li and A. Weston, 'Strict p -negative type of a metric space', *Positivity* **14** (2010), 529–545.
- [28] J. Lindenstrauss and H. Rosenthal, 'The \mathcal{L}_p spaces', *Israel J. Math.* **7** (1969), 325–349.
- [29] B. Maurey and G. Pisier, 'Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach', *Studia Math.* **58** (1976), 45–90.
- [30] E. Prassidis and A. Weston, 'Manifestations of non linear roundness in analysis, discrete geometry and topology', in: *Limits of Graphs in Group Theory and Computer Science*, Research Proceedings of the École Polytechnique Fédérale de Lausanne (eds. G. Arzhantseva and A. Valette) (CRC Press, Boca Raton, FL, 2009), 141–170.
- [31] M. Ribe, 'On uniformly homeomorphic normed spaces', *Ark. Mat.* **14** (1976), 237–244.
- [32] H. P. Rosenthal, 'On a theorem of J. L. Krivine concerning block finite representability of ℓ_p in general Banach spaces', *J. Funct. Anal.* **28** (1978), 197–225.
- [33] S. Sánchez, 'On the supremal p -negative type of a finite metric space', *J. Math. Anal. Appl.* **389** (2012), 98–107.
- [34] I. J. Schoenberg, 'On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space', *Ann. of Math. (2)* **38** (1937), 787–793.
- [35] J. Stern, 'Some applications of model theory in Banach space theory', *Ann. Math. Logic* **9** (1976), 49–122.
- [36] A. Weston, 'On the generalized roundness of finite metric spaces', *J. Math. Anal. Appl.* **192** (1995), 323–334.

LUKIEL LEVY-MOORE, Department of Economics,
New York University, New York, NY 10012, USA
e-mail: LukiellLevyMoore@nyu.edu

MARGARET NICHOLS, Department of Mathematics,
University of Chicago, Chicago, IL 60637, USA
e-mail: mnichols@math.uchicago.edu

ANTHONY WESTON, Department of Mathematics and Statistics,
Canisius College, Buffalo, NY 14208, USA
and
Department of Decision Sciences, University of South Africa,
Unisa 0003, South Africa
e-mail: westona@canisius.edu, westoar@unisa.ac.za