

TORAL AUTOMORPHISMS AND ANTIAUTOMORPHISMS OF ROTATION ALGEBRAS

HU YAOHUA AND P.J. STACEY

If U, V are the generators of a rational or irrational rotation C^* -algebra then an automorphism ϕ of the algebra is determined by $\phi(U) = \lambda U^a V^c$ and $\phi(V) = \mu U^b V^d$ where λ, μ are complex numbers of modulus 1 and a, b, c, d are integers with $ad - bc = 1$. If $ad - bc = -1$, then these formulae determine an antiautomorphism of the algebra. The classification of such automorphisms and antiautomorphisms up to conjugacy by arbitrary automorphisms is studied and an almost complete classification is obtained.

1. INTRODUCTION

Let A_θ be the universal C^* -algebra generated by a pair U, V of unitaries satisfying $VU = \rho UV$ where $\rho = e^{2\pi i\theta}$ and $0 \leq \theta < 1$. Brenken [2] and Watatani [8] introduced an $SL(2, \mathbb{Z})$ action on A_θ , which can be described in the following way. If $A \in SL(2, \mathbb{Z})$ and $\lambda, \mu \in S^1$ then an automorphism $\phi_{A, \lambda, \mu}$ of A_θ is determined, using the universal property, by $\phi_{A, \lambda, \mu}(U) = \lambda U^a V^c$ and $\phi_{A, \lambda, \mu}(V) = \mu U^b V^d$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Following Watatani, such automorphisms of A_θ will be called *toral*. The particular choice $\lambda = e^{\pi iac}$ and $\mu = e^{\pi ibd}$ gives an $SL(2, \mathbb{Z})$ action by toral automorphisms.

A series of papers by Farsi and Watling, including [3] and [4], obtained properties of the fixed point algebras and cross product algebras associated with toral automorphisms. The purpose of the present paper is to include a study of antiautomorphisms and to study the conjugacy problem for both toral automorphisms and antiautomorphisms.

2. CONJUGACY OF TORAL AUTOMORPHISMS

The description of toral automorphisms up to conjugacy was mostly completed in the paper [4]. The following propositions make those results explicit as well as completing the description. Under the isomorphism of $K_1(A_\theta)$ with \mathbb{Z}^2 given by $[U^n V^m] \mapsto (n, m)$ the map $(\phi_{A, \lambda, \mu})_*$ corresponds to multiplication by A and hence a necessary condition for $\phi_{A, \lambda, \mu}$ and $\phi_{A', \lambda', \mu'}$ to be conjugate automorphisms is that A and A' are conjugate in $SL(2, \mathbb{Z})$. The following results specify the extra conditions needed to ensure conjugacy of the automorphisms.

Received 22nd July, 1998

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/99 \$A2.00+0.00.

PROPOSITION 1. (Farsi and Watling). *If $\text{trace}(A) \neq 2$ then $\phi_{A,\lambda,\mu}$ is conjugate to $\phi_{A,1,1}$. Furthermore $\phi_{A,1,1}$ is conjugate to $\phi_{B,1,1}$ if and only if A is conjugate to B in $SL(2, \mathbb{Z})$.*

PROOF: This is established in the course of the proofs of [4, Proposition 3 and Lemma 4]. □

If $\text{trace}(A) = 2$ then it is well-known that A is conjugate in $SL(2, \mathbb{Z})$ to $M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ for some $m \in \mathbb{Z}$ and that m is uniquely determined. The following explicit description of the conjugacy when $m \neq 0$ is needed for Proposition 2. Let $A = \begin{pmatrix} 1-a & b \\ c & 1+a \end{pmatrix}$ with $a^2 = -bc$. If $c = 0$ then A is already of the required form (with $a = 0$), if $b = 0$ then $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1-a & 0 \\ c & 1+a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+a & -c \\ 0 & 1-a \end{pmatrix}$ is of the required form (with $a = 0$) and otherwise each of a, b, c are non-zero. In the latter case let $|m| = (b, c)$ with $bm > 0$. From the condition $a^2 = -bc$ it follows that $b = r^2m$ and $c = -t^2m$ for some $r, t \in \mathbb{Z}$ with $(r, t) = 1$. Furthermore the signs of r, t can be chosen so that $a = mrt$. If u, v are chosen to satisfy $ru - tv = 1$ then $A = \begin{pmatrix} r & v \\ t & u \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & -v \\ -t & r \end{pmatrix}$. This last equation and hence the equations $a = mrt, b = r^2m$ and $c = -t^2m$ hold even when $b = 0$ or $c = 0$.

PROPOSITION 2. (i) (Farsi and Watling). *If $A = \begin{pmatrix} 1-a & b \\ c & 1+a \end{pmatrix}$, $a = mrt, b = r^2m$ and $c = -t^2m$ with $(r, t) = 1$ and $m \neq 0$ then $\phi_{A,\lambda,\mu}$ is conjugate to $\phi_{M,\gamma,1}$ where $M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $\gamma = \lambda^r \mu^t \rho^{[-c-ac-a^2]/2}$.*

(ii) $\phi_{M,\gamma,1}$ is conjugate to $\phi_{M,\gamma',1}$ if and only if $\gamma = \gamma'$ or $\gamma = \gamma'^{-1}$.

PROOF: (i) This is shown in the proof of [4, Lemma 16].

(ii) Suppose that $\phi_{M,\gamma,1}\beta = \beta\phi_{M,\gamma',1}$ for some automorphism β of A_θ . Then $\phi_{M,\gamma,1}\beta(U) = \gamma'\beta(U)$. Following the proof of [3, Theorem 1], let $c_{p,q}$ be the Fourier coefficient of $\beta(U)$ corresponding to U^pV^q , defined to be the usual Fourier coefficient of $E(\beta(U)V^{*q})$ corresponding to U^p , where E is the canonical conditional expectation from A_θ onto $C^*(U)$. Then $\sum \gamma'c_{p,q}U^pV^q = \sum \gamma^pc_{p,q}U^p(U^mV)^q = \sum \gamma^pc_{p,q}\rho^sU^{p+qm}V^q$ for some s , so $|c_{p,q}| = |c_{p+qm,q}|$ for each p, q and thus, since $|c_{p,q}| \rightarrow 0$ as $p \rightarrow \infty$, $c_{p,q} = 0$ unless $q = 0$. It follows that $\beta(U)$ has Fourier series $\sum c_pU^p$ and that $c_p\gamma' = c_p\gamma^p$ for each p . If γ has infinite order then c_p is non-zero for only one value of p and thus $\beta(U) = c_pU^p$. If γ has order k then $\beta(U) = U^p f(U^k)$ for some $f : S^1 \rightarrow S^1$ and some p such that $\gamma' = \gamma^p$. If $\beta_* : K_1(A_\theta) \rightarrow K_1(A_\theta)$ corresponds to β then, when $K_1(A_\theta)$ is identified with \mathbb{Z}^2 in the usual way, it follows that $\beta_*(1, 0) = (p + k\ell, 0)$ for some $\ell \in \mathbb{Z}$ (or $\beta_*(1, 0) = (p, 0)$ when γ has infinite order). On the other hand, from $\phi_{M,\gamma,1}\beta = \beta\phi_{M,\gamma',1}$ it follows that $MB = BM$

where β_* has matrix B . However if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $c = 0$ and $a = d$ and so, from $ad - bc = 1$, $B = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b \in \mathbb{Z}$. Hence $\beta_*(1, 0) = \pm(1, 0)$ and so $(p + k\ell, 0) = \pm(1, 0)$. If $k \neq 1$ (that is, $\gamma \neq 1$) then $\ell = 0$ and $p = \pm 1$ whereas if $k = 1$ then $\gamma' = \gamma = 1$. Thus in all cases $\gamma = \gamma'^{\pm 1}$. Conversely, if $\gamma' = \gamma^{-1}$ and β is defined by $\beta(U) = \rho^m U^*$ and $\beta(V) = V^*$ then $\phi_{M, \gamma, 1} \beta = \beta \phi_{M, \gamma', 1}$, as required. \square

The final case to be considered is when A is conjugate to, and hence equal to, I_2 . The behaviour of $\phi_{A, \lambda, \mu}$ depends on the nature of the subgroup G of S^1 generated by λ, μ . This can either be a cyclic subgroup of order m (where m is the least common multiple of the orders of λ, μ), an infinite cyclic group (when λ or μ has infinite order and $\lambda^k = \mu^\ell$ for some k, ℓ not both zero), a group $\mathbb{Z}_m \times \mathbb{Z}$ where $1 < m$ (when $\lambda^k \mu^{-\ell}$ is of finite order for some k, ℓ , not both zero, but either λ or μ has infinite order) or $\mathbb{Z} \times \mathbb{Z}$ (when $\lambda^k \mu^{-\ell}$ is of infinite order unless $k = \ell = 0$).

PROPOSITION 3. (i) (Farsi and Watling). If λ, μ generate a cyclic subgroup of order m in S^1 then $\phi_{I, \lambda, \mu}$ is conjugate to $\phi_{I, \omega, 1}$ where $\omega = e^{2\pi i/m}$.

(ii) If λ, μ generate an infinite cyclic subgroup in S^1 then $\phi_{I, \lambda, \mu}$ is conjugate to $\phi_{I, 1, \gamma}$ for some γ of infinite order in S^1 . Furthermore $\phi_{I, 1, \gamma}$ is conjugate to $\phi_{I, 1, \gamma'}$ if and only if $\gamma = \gamma'$ or $\gamma = \gamma'^{-1}$.

(iii) If λ, μ generate a subgroup of S^1 isomorphic to $\mathbb{Z}_m \times \mathbb{Z}$ then $\phi_{I, \lambda, \mu}$ is conjugate to $\phi_{I, \omega^n, \gamma}$ for some n with $1 \leq n < m$ and $(n, m) = 1$ and some γ of infinite order, where $\omega = e^{2\pi i/m}$. Furthermore $\phi_{I, \omega^n, \gamma}$ is conjugate to $\phi_{I, \omega^{n'}, \gamma'}$ if and only if either $\omega^n = \omega^{n'}$ and $\gamma = \omega^s \gamma'$ for some $s \in \mathbb{Z}$ or $\omega^n = \omega^{-n'}$ and $\gamma = \omega^s \gamma'^{-1}$ for some $s \in \mathbb{Z}$.

(iv) If λ, μ generate a subgroup of S^1 isomorphic to $\mathbb{Z} \times \mathbb{Z}$ then the toral automorphisms conjugate to $\phi_{I, \lambda, \mu}$ are those of the form $\phi_{I, \lambda^a \mu^b, \lambda^r \mu^s}$ where $sa - rb = 1$.

PROOF: (i) From the proof of [4, Theorem 20] there exists $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ such that $\lambda^a \mu^c = \omega$ and $\lambda^b \mu^d = 1$. Then $\phi_{I, \lambda, \mu} \phi_{A, 1, 1} = \phi_{A, 1, 1} \phi_{I, \omega, 1}$, as required.

(ii), (iii). Let $\lambda^k \mu^{-\ell}$ be a generator of $\{1, \dots, \omega^{m-1}\}$ (where $m = 1$ in case (ii)), let $k' = k/(k, \ell)$ and let $\ell' = \ell/(k, \ell)$. Then in case (ii) $\lambda^{k'} \mu^{-\ell'} = 1$ and in case (iii) $\lambda^{k'} \mu^{-\ell'} = \omega^n$ for some $1 \leq n < m$ with $(n, m) = 1$. There exist $b, d \in \mathbb{Z}$ with $dk' + b\ell' = 1$ and then $A = \begin{pmatrix} k' & b \\ -\ell' & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ and $\phi_{I, \lambda, \mu} \phi_{A, 1, 1} = \phi_{A, 1, 1} \phi_{I, \omega^n, \lambda^b \mu^d}$ (with $n = 0$ in case (ii)).

Suppose that there exists an automorphism β with $\beta \phi_{I, \omega^n, \gamma} = \phi_{I, \omega^n, \gamma} \beta$ (with $n = n' = 0$ in case (ii)). If the Fourier coefficient of $\beta(U)$ corresponding to $U^r V^s$ is $c_{r, s}$ then $\omega^{n'} c_{r, s} = \omega^{nr} \gamma^s c_{r, s}$ and so, since γ is of infinite order, $c_{r, s} = 0$ unless $s = 0$. Thus

$\beta(U) \in C^*(U)$. Similarly, if the Fourier coefficient of $\beta(V)$ corresponding to $U^s V^t$ is $d_{s,t}$ then $\gamma' d_{s,t} = \omega^{sn} \gamma^t d_{s,t}$ and, since γ is of infinite order, there exists a unique t such that $d_{s,t} \neq 0$. Thus $\beta(V) = WV^t$ for some $W \in C^*(U)$. The matrix corresponding to $\beta_* : K_1(A_\theta) \rightarrow K_1(A_\theta)$ is therefore of the form $\begin{pmatrix} * & * \\ 0 & t \end{pmatrix}$ and, since β_* is invertible, it follows that $t = \pm 1$. Hence $\gamma = \gamma'$ or $\gamma = \gamma'^{-1}$ in case (ii) and $\gamma = \omega^s \gamma'$ for some $s \in \mathbb{Z}$ or $\gamma = \omega^s \gamma'^{-1}$ for some $s \in \mathbb{Z}$ in case (iii).

By reversing the roles of n, γ and n', γ' , it follows that not only $\beta(U) \in C^*(U)$ but also $\beta^{-1}(U) \in C^*(U)$. Hence the automorphism β restricts to an automorphism of $C^*(U)$ implementing a conjugacy between the restrictions of $\phi_{I, \omega^{n'}, \gamma'}$ and $\phi_{I, \omega^n, \gamma}$. However unless $\omega^n = \omega^{\pm n'}$, the associated rational rotation algebras are non-isomorphic. The case $\omega^n = \omega^{n'}$ corresponds to $\beta_*[U] = [U]$ and hence to $t = 1$, whereas $\omega^n = \omega^{-n'}$ corresponds to $\beta_*[U] = -[U]$ and hence to $t = -1$. Thus either $\omega^n = \omega^{n'}$ and $\gamma = \omega^s \gamma'$ for some $s \in \mathbb{Z}$ or $\omega^n = \omega^{-n'}$ and $\gamma = \omega^s \gamma'^{-1}$ for some $s \in \mathbb{Z}$.

Conversely, in case (iii) let $\beta(U) = U$ and $\beta(V) = U^s V$ where $ns' \equiv s \pmod{m}$, which exists because $(n, m) = 1$. Then $\phi_{I, \omega^n, \gamma} \beta = \beta \phi_{I, \omega^n, \omega^s \gamma}$. Similarly let $\beta(U) = U^*$ and $\beta(V) = V^*$ in case (ii) to obtain $\phi_{I, 1, \gamma} \beta = \beta \phi_{I, 1, \gamma^{-1}}$ or $\beta(U) = U^*$ and $\beta(V) = U^s V^*$ in case (iii) to obtain $\phi_{I, \omega^n, \gamma} \beta = \beta \phi_{I, \omega^{-n}, \omega^s \gamma^{-1}}$.

(iv) Let $\phi_{I, \lambda, \mu} \beta = \beta \phi_{I, \lambda', \mu'}$. Then $\phi_{I, \lambda, \mu} \beta(U) = \lambda' \beta(U)$ and $\phi_{I, \lambda, \mu} \beta(V) = \mu' \beta(V)$ so if the Fourier coefficient of $\beta(U)$ corresponding to $U^a V^b$ is $c_{a,b}$ and that of $\beta(V)$ corresponding to $U^r V^s$ is $d_{r,s}$, then $\lambda' c_{a,b} = \lambda^a \mu^b c_{a,b}$ and $\mu' d_{r,s} = \lambda^r \mu^s d_{r,s}$. Since no two distinct pairs $\lambda^a \mu^b$ are equal it follows that $\beta(U) = c_{a,b} U^a V^b$ for some a, b and similarly $\beta(V) = d_{r,s} U^r V^s$. Hence $\lambda' = \lambda^a \mu^b$, $\mu' = \lambda^r \mu^s$ and, from the invertibility of β_* on $K_1(A_\theta)$, $sa - rb = 1$. Conversely, if a, b, r, s satisfy the condition $sa - rb = 1$ then $\phi_{I, \lambda, \mu} \beta = \beta \phi_{I, \lambda^a \mu^b, \lambda^r \mu^s}$ when $\beta(U) = U^a V^b$ and $\beta(V) = U^r V^s$. □

3. ANTIAUTOMORPHISMS

In order to establish the existence of an antiautomorphism of A_θ , observe that the opposite algebra A_θ^{op} is generated by the elements U, V but subject to the relation $VU = \bar{\rho}UV$ rather than $VU = \rho UV$. Hence A_θ^{op} is isomorphic to $A_{1-\theta}$ and, by the universal property, there exists an isomorphism α from A_θ onto A_θ^{op} with $\alpha(V) = U$ and $\alpha(U) = V$. Composing this with the identity antiautomorphism from A_θ^{op} to A_θ gives an antiautomorphism of A_θ , denoted by $\phi_{B, 1, 1}$ where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{Z})$. If

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ with $ad - bc = -1$ then the composite $\phi_{AB, \mu, \lambda} \phi_{B, 1, 1}$ is an anti-automorphism $\phi_{A, \lambda, \mu}$ with $\phi_{A, \lambda, \mu}(U) = \lambda U^a V^c$ and $\phi_{A, \lambda, \mu}(V) = \mu U^b V^d$.

PROPOSITION 4. (i) *If $A, A' \in GL(2, \mathbb{Z})$ with $\det(A) = 1$ then*

$$\phi_{A,\lambda,\mu}\phi_{A',\lambda',\mu'} = \phi_{AA',\gamma,\delta}$$

where

$$\gamma = \lambda' \lambda^a \mu^c \rho^{(bdc'(c'-1))/2+(aca'(a'-1))/2+a'bcc'} \quad \text{and} \quad \delta = \mu' \lambda^b \mu^d \rho^{(bdd'(d'-1))/2+(ab'c(b'-1))/2+bb'cd'}$$

(ii) *If $A, A' \in GL(2, \mathbb{Z})$ with $\det(A) = -1$ then*

$$\phi_{A,\lambda,\mu}\phi_{A',\lambda',\mu'} = \phi_{AA',\gamma,\delta}$$

where

$$\gamma = \lambda' \lambda^a \mu^c \rho^{(bdc'(c'-1))/2+(aa'c(a'-1))/2+aa'c'd} \quad \text{and} \quad \delta = \mu' \lambda^b \mu^d \rho^{(bdd'(d'-1))/2+(ab'c(b'-1))/2+ab'dd'}$$

(iii) *The map from $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\psi_A = \phi_{A,\exp(\pi iac),\exp(\pi ibd)}$ is a $GL(2, \mathbb{Z})$ action by toral automorphisms and antiautomorphisms.*

(iv) *When $GL(2, \mathbb{Z})$ acts on T^2 by $(\lambda, \mu)\theta_A = (\lambda^a \mu^c, \lambda^b \mu^d)$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\phi_{I,\lambda,\mu}\psi_A = \psi_A\phi_{I,(\lambda,\mu)\theta_A}$ and hence $(\lambda, \mu, A) \mapsto \psi_A\phi_{I,\lambda,\mu}$ is an action of the semidirect product $T^2 \rtimes GL(2, \mathbb{Z})$.*

PROOF: These are all straightforward calculations. □

A major component of the classification of the toral antiautomorphisms up to conjugacy is the classification up to conjugacy of elements in $GL(2, \mathbb{Z})$ with negative determinant. If two such elements A, B satisfy $A = QBQ^{-1}$ with $Q \in GL(2, \mathbb{Z})$ then, replacing Q by QB if necessary, they are conjugate by some $Q \in SL(2, \mathbb{Z})$.

The following result shows that, as for automorphisms when $\text{trace}(A) \neq 2$, the conjugacy problem for toral antiautomorphisms (via automorphisms) when $\text{trace}(A) \neq 0$ reduces to the conjugacy problem in $GL(2, \mathbb{Z})$.

PROPOSITION 5. *If $\text{trace}(A) \neq 0$ and $\det(A) = -1$ then $\phi_{A,\lambda,\mu}$ is conjugate to $\phi_{A,1,1}$. Furthermore $\phi_{A,1,1}$ is conjugate to $\phi_{B,1,1}$ if and only if A is conjugate to B in $GL(2, \mathbb{Z})$.*

PROOF: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $\gamma = \lambda^{(1-d)/(a+d)} \mu^{c/(a+d)}$ and let $\delta = \lambda^{b/(a+d)} \mu^{(1-a)/(a+d)}$. Then $\phi_{A,\lambda,\mu}\phi_{I,\gamma,\delta} = \phi_{I,\gamma,\delta}\phi_{A,1,1}$. If $A = QBQ^{-1}$ with $Q \in SL(2, \mathbb{Z})$ then $\phi_{Q,1,1}\phi_{B,1,1}\phi_{Q,1,1}^{-1} = \phi_{A,\lambda,\mu}$ for some $\lambda, \mu \in S^1$ and hence, by the first part, is conjugate to $\phi_{A,1,1}$. □

If $A \in GL(2, \mathbb{Z})$ with $\det(A) = -1$ and $\text{trace}(A) = 0$ then $A^2 = I$. Hence, from [5, Lemma 5.5, p.166], A is conjugate to either $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (which are not conjugate to one another).

PROPOSITION 6. *If $\text{trace}(A) = 0$ then either $\phi_{A,\lambda,\mu}$ is conjugate to $\phi_{M,\gamma,1}$ for some $\gamma \in S^1$ or to $\phi_{N,\gamma,1}$ for some $\gamma \in S^1$.*

PROOF: Let $Q \in \text{SL}(2, \mathbb{Z})$ with $Q A Q^{-1} = M$ (or $Q A Q^{-1} = N$). Then $\phi_{Q,1,1} \phi_{A,\lambda,\mu} \phi_{Q,1,1}^{-1} = \phi_{M,\lambda',\mu'}$ for some $\lambda', \mu' \in S^1$ (or $\phi_{N,\lambda',\mu'}$ for some $\lambda', \mu' \in S^1$). However $\phi_{I,\mu,1} \phi_{N,\lambda,\mu,1} = \phi_{N,\lambda,\mu} \phi_{I,\mu,1}$ and, if $\delta^2 = \mu$, then $\phi_{I,1,\delta} \phi_{M,\lambda,\mu} = \phi_{M,\lambda,1} \phi_{I,1,\delta}$. □

PROPOSITION 7. *$\phi_{N,\gamma,1}$ is conjugate to $\phi_{N,\gamma',1}$ if and only if $\gamma = \gamma'$ or $\gamma = \gamma'^{-1}$.*

PROOF: Let $\beta \phi_{N,\gamma,1} = \phi_{N,\gamma',1} \beta$. Then $\beta \phi_{I,\gamma,\gamma} = \beta \phi_{N,\gamma,1}^2 = \phi_{N,\gamma',1}^2 \beta = \phi_{I,\gamma',\gamma'} \beta$. Thus when γ has infinite order then, by Proposition 3(ii), either $\gamma = \gamma'$ or $\gamma = \gamma'^{-1}$. When γ has finite order m then, by Proposition 3(i), $\gamma = \gamma'^n$ for some $1 \leq n < m$ with $(n, m) = 1$. Then $\phi_{I,\gamma',\gamma'}(U^{*n} \beta(U)) = \gamma'^{-n} U^{*n} \phi_{I,\gamma',\gamma'} \beta(U) = \gamma^{-1} U^{*n} \beta \phi_{I,\gamma,\gamma}(U) = U^{*n} \beta(U)$ and similarly $\phi_{I,\gamma',\gamma'}(V^{*n} \beta(V)) = V^{*n} \beta(V)$, from which it follows that $U^{*n} \beta(U) \in \text{Fix}(\phi_{I,\gamma',\gamma'}) = C^*(U^m, UV^*)$ and $V^{*n} \beta(V) \in C^*(U^m, UV^*)$. Hence the induced map β_* on $K_1(A_\theta) \cong \mathbb{Z}^2$ satisfies $\beta_*(1, 0) = (n + \ell m + k, -k)$ for some $\ell, k \in \mathbb{Z}$ and $\beta_*(0, 1) = (k' + \ell' m, n - k')$ for some $\ell', k' \in \mathbb{Z}$ that is,

$$\beta_* = \begin{pmatrix} n + \ell m + k & k' + \ell' m \\ -k & n - k' \end{pmatrix}.$$

However $\beta_* N = N \beta_*$, where $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, from which it follows that

$\beta_* \in \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. If $\beta_* = \pm I$, it follows that $k = 0$ and $n + \ell m = \pm 1$ and hence $n = 1$ or $m - 1$, as required. Similarly, if $\beta_* = \pm N$ then $k' + \ell' m = \pm 1$ and $n = k'$, so $n = 1$ or $m - 1$, as required. Conversely, $\phi_{-I,1,1} \phi_{N,\gamma,1} = \phi_{N,\gamma^{-1},1} \phi_{-I,1,1}$ for each $\gamma \in S^1$. □

PROPOSITION 8. (i) *If $\phi_{M,\gamma,1}$ is conjugate to $\phi_{M,\gamma',1}$ then $\gamma^2 = \gamma'^2$ or $\gamma^2 = \gamma'^{-2}$*
 (ii) *If $\gamma = \gamma'$ or $\gamma = \gamma'^{-1}$ then $\phi_{M,\gamma,1}$ is conjugate to $\phi_{M,\gamma',1}$.*

PROOF: (i) Let $\beta \phi_{M,\gamma,1} = \phi_{M,\gamma',1} \beta$. Then $\beta \phi_{I,\gamma^2,1} = \beta \phi_{M,\gamma,1}^2 = \phi_{M,\gamma',1}^2 \beta = \phi_{I,\gamma'^2,1} \beta$. Thus, when γ has infinite order then, by Proposition 3(ii), either $\gamma^2 = \gamma'^2$ or $\gamma^2 = \gamma'^{-2}$. When γ^2 has finite order $m > 1$ then, by Proposition 3(i), $\gamma^2 = \gamma'^{2n}$ for some $1 \leq n < m$ with $(n, m) = 1$. Then $\phi_{I,\gamma'^2,1}(U^{*n} \beta(U)) = U^{*n} \beta(U)$ and $\phi_{I,\gamma'^2,1} \beta(V) = \beta(V)$ so $U^{*n} \beta(U) \in C^*(U^m, V)$ and $\beta(V) \in C^*(U^m, V)$. Hence $\beta_* : K_1(A_\theta) \rightarrow K_1(A_\theta)$ equals $\begin{pmatrix} n + km & mk' \\ \ell & \ell' \end{pmatrix}$ for some $k, \ell, k', \ell' \in \mathbb{Z}$. However, the condition $\beta_* M = M \beta_*$ implies $\beta_* \in \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ and hence $n = 1$ or $m - 1$. Finally, when $\gamma^2 = 1$ then $\gamma'^2 = 1$, so the condition also holds in this case.

(ii) This is immediate since $\phi_{-I,1,1} \phi_{M,\gamma,1} = \phi_{M,\gamma^{-1},1} \phi_{-I,1,1}$. □

The sole remaining question is therefore whether or not $\phi_{M,\gamma,1}$ is conjugate to $\phi_{M,-\gamma,1}$.

PROPOSITION 9. (i) Let γ be of infinite order in S^1 . If $\phi_{M,\gamma,1}$ is conjugate to $\phi_{M,-\gamma,1}$ then $\phi_{M,\lambda,1}$ is conjugate to $\phi_{M,-\lambda,1}$ for all $\lambda \in S^1$.

(ii) Let γ be either of odd order k or of order $2k$ where k is odd. If $\phi_{M,\gamma,1}$ is conjugate to $\phi_{M,-\gamma,1}$ then $\phi_{M,1,1}$ is conjugate to $\phi_{M,-1,1}$.

PROOF: (i) Suppose that $\phi_{M,\gamma,1}\beta = \beta\phi_{M,-\gamma,1}$, where β is an automorphism. Then $\phi_{M,\gamma,1}\beta(U) = -\gamma\beta(U)$ and $\phi_{I,\gamma^2,1}\beta(U) = \gamma^2\beta(U)$ and hence, if $c_{n,m}$ is the Fourier coefficient of $\beta(U)$ corresponding to U^nV^m then $\gamma^2c_{n,m} = \gamma^{2n}c_{n,m}$, so $c_{n,m} = 0$ unless $n = 1$, and $-\gamma c_{n,m}U^nV^m = \gamma^n\rho^{mn}c_{n,-m}V^mU^n$, so $c_{1,m} = -\rho^m c_{1,-m}$. It follows that $\beta(U) = U \sum c_m V^m$ where $\sum c_m V^m$ is a unitary for which $c_m = -\rho^m c_{-m}$ for each $m \in \mathbb{Z}$. On the other hand, if such a unitary exists then let β be the automorphism of A_θ defined by $\beta(U) = U \sum c_m V^m$ and $\beta(V) = V$. Then, for any $\lambda \in S^1$, $\phi_{M,\lambda,1}\beta = \beta\phi_{M,-\lambda,1}$, as required.

(ii) If k is odd and γ is of order k , then $\phi_{M,\gamma,1}^k = \phi_{M,1,1}$ whereas $\phi_{M,-\gamma,1}^k = \phi_{M,-1,1}$. If γ is of order $2k$ where k is odd then $\gamma^k = -1$ and hence $\phi_{M,\gamma,1}^k = \phi_{M,-1,1}$ and $\phi_{M,-\gamma,1}^k = \phi_{M,1,1}$. □

PROPOSITION 10. (i) The real algebra $\{a \in A_\theta : \phi_{M,1,1}(a) = a^*\}$ is isomorphic to the crossed product $C(S^1, \mathbb{R}) \times_{\alpha_\theta} \mathbb{Z}$ where α_θ is the irrational rotation on $C(S^1, \mathbb{R})$, defined by $(\alpha_\theta f)(t) = f(\rho t)$.

(ii) The real algebra $\{a \in A_\theta : \phi_{M,-1,1}(a) = a^*\}$ is isomorphic to the crossed product $\{f \in C(S^1) : f(t) = \overline{f(-t)} \text{ for all } t \in S^1\} \times_{\alpha_\theta} \mathbb{Z}$.

PROOF: When A_θ is viewed as $C(S^1) \times_{\alpha_\theta} \mathbb{Z}$ via the identification of $C^*(U)$ with $C(S^1)$, then $\phi_{M,1,1}$ restricts to the identity on $C(S^1)$ whereas $\phi_{M,-1,1}$ restricts to the automorphism ϕ given by $(\phi f)(t) = f(-t)$. The results are therefore immediate. □

PROPOSITION 11. Let R_{-1} be the real algebra associated with $\phi_{M,-1,1}$ and let R_1 be the real algebra associated with $\phi_{M,1,1}$. Then $K_0(R_1)$ is isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}_2$ whereas $K_0(R_{-1})$ is isomorphic to \mathbb{Z}^2 .

PROOF: The K -theory of $R_1 = C(S^1, \mathbb{R}) \times_{\alpha} \mathbb{Z}$ is calculated in [7, Corollary 1.5.7]. The K -theory of $A \times_{\alpha} \mathbb{Z}$, where $A = \{f \in C(S^1) : f(t) = \overline{f(-t)} \text{ for all } t \in S^1\}$, can be calculated using the Pimsner-Voiculescu sequence for real C^* -algebras, described in [7, 1.4.2]. Note firstly that the involution $t \mapsto -t$ on S^1 gives rise to the real K -theory $KR(S^{2,0})$ of Atiyah [1] and that, by [1, (3.6)], there is an exact sequence

$$\dots \rightarrow KR^{2-q}(pt) \rightarrow KR^{-q}(pt) \rightarrow KR^{-q}(S^{2,0}) \rightarrow KR^{3-q}(pt) \rightarrow \dots$$

Using the values for $KR^{-n}(pt)$ from [6, Chapter III, Theorem 5.19], yields the values $K_4(A) = K_0(A) = \mathbb{Z}$, $K_5(A) = K_1(A) = \mathbb{Z}_2$, $K_6(A) = K_2(A) = 0$ and $K_7(A) = K_3(A) = \mathbb{Z}$. From the real Pimsner-Voiculescu sequence,

$$\dots \rightarrow K_0(A) \xrightarrow{\theta} K_0(A) \rightarrow K_0(A \times_{\alpha} \mathbb{Z}) \rightarrow K_7(A) \xrightarrow{\theta} K_7(A) \rightarrow \dots$$

where $\theta = \alpha_* - id$. Since $\alpha_*[1] = [1]$ it is immediate that, on $K_0(A)$, $\alpha_* = id$. From the identification of K_1 classes with winding numbers, it follows that, on $K_7(C(S^1)) = K_1(C(S^1))$, $\alpha_* = id$. However the action of α_* on $K_7(A)$ commutes with the complexification map $c : K_7(A) \rightarrow K_7(C(S^1))$, from which it follows, using $K_7(A) = \mathbb{Z}$ and $K_7(C(S^1)) = \mathbb{Z}$, that $\alpha_* = id$ on $K_7(A)$. Hence,

$$0 \rightarrow K_0(A) \rightarrow K_0(A \times_{\alpha} \mathbb{Z}) \rightarrow K_7(A) \rightarrow 0$$

from which it follows that $K_0(R_{-1}) = \mathbb{Z}^2$, as claimed. □

PROPOSITION 12. *$\phi_{M,\gamma,1}$ is not conjugate to $\phi_{M,-\gamma,1}$ when γ is either of infinite order, odd order or of order $2k$ where k is odd.*

PROOF: It follows from Proposition 9 that if $\phi_{M,\gamma,1}$ is conjugate to $\phi_{M,-\gamma,1}$ then $\phi_{M,1,1}$ is conjugate to $\phi_{M,-1,1}$. Hence, in the notation of Proposition 11, R_1 is isomorphic to R_{-1} , which contradicts that Proposition. □

The final remaining problem is to determine whether or not $\phi_{M,\gamma,1}$ is conjugate to $\phi_{M,-\gamma,1}$ when the order of γ is a multiple of 4. The following proposition describes how the problem can be resolved in certain cases when θ is rational. Recall that, when $\theta = p/q$ with $(p, q) = 1$ and $0 \leq p < q$, then the algebra A_{θ} can be identified with the C^* -subalgebra of $C_b(\mathbb{R}^2, M_q)$ generated by U, V where $U(\lambda, \mu) = e^{2\pi i \lambda/q} U_0$ and $V(\lambda, \mu) = e^{2\pi i \mu/q} V_0$ with

$$U_0 = \begin{pmatrix} 1 & & & \\ & \rho & & \\ & & \ddots & \\ & & & \rho^{q-1} \end{pmatrix} \quad \text{and} \quad V_0 = \begin{pmatrix} O & I_{q-1} \\ 1 & 0 \end{pmatrix}.$$

If $\gamma = e^{2\pi i m/n}$ with $(m, n) = 1$, then the antiautomorphism $\phi_{M,\gamma,1}$ is given, in this description, by $(\phi_{M,\gamma,1} f)(\lambda, \mu) = f(\lambda + (mq/n), -\mu)^{tr}$, where tr denotes the transpose. To see this, note that this formula defines an antiautomorphism of $C_b(\mathbb{R}^2, M_q)$ mapping U to γU and V to V^* .

The algebra A_{θ} can be explicitly described as $\{f \in C(\mathbb{R}^2, M_q) : f(\lambda + m, \mu + n) = W_1^n W_2^m f(\lambda, \mu) W_2^{*m} W_1^{*n}\}$, where $W_1 = U_0^p$ and $W_2 = V_0^{p''}$ with $pp' \equiv -1 \pmod{q}$ and $pp'' \equiv 1 \pmod{q}$, from which it follows that, for each $f \in A_{\theta}$, $(\lambda, \mu) \mapsto \det f(\lambda, \mu)$ gives a well-defined function on T^2 . When restricted to unitary elements it gives a mapping from T^2 to S^1 . The winding numbers $w_1(f), w_2(f)$ of the associated maps $\lambda \mapsto \det f(\lambda, 0)$ and $\mu \mapsto \det f(0, \mu)$ are homotopy invariants and hence (recalling that $K_1(A_{\theta})$ is isomorphic to \mathbb{Z}^2 with generators $[U]$ and $[V]$) the K_1 element $[f]$ corresponding to the unitary element f of A_{θ} is $(w_1(f), w_2(f))$.

PROPOSITION 13. *Let $\theta = p/q$ where $(p, q) = 1$ and let γ have order n where $(q, n) = 1$. Then, for each unitary f in A_{θ} with $\phi_{M,\gamma,1}(f) = \gamma^r f$, the K_1 class $[f]$ belongs to $(r + n\mathbb{Z}) \times \mathbb{Z}$.*

PROOF: Let $\gamma = e^{2\pi im/n}$ where $(m, n) = 1$. As described above, $f(\lambda + (mq/n), -\mu)^{tr} = \gamma^r f(\lambda, \mu)$ for each $\lambda, \mu \in \mathbb{R}$ and hence $\det f(\lambda + (mq/n), 0) = \gamma^{r^q} \det f(\lambda, 0)$. Let $m'm \equiv 1 \pmod{n}$, so that $\det f(\lambda + (q/n), 0) = \gamma^{r q m'} \det f(\lambda, 0) = e^{2\pi i r q/n} \det f(\lambda, 0)$ and then $\det f(\lambda + (1/n), 0) = e^{2\pi i r/n} \det f(\lambda, 0)$ using $(q, n) = 1$. The winding number of $\{\det f(\lambda, 0) : \lambda_0 \leq \lambda \leq \lambda_0 + 1/n\}$ followed by $\{e^{-2\pi i r t/n} : 0 \leq t \leq 1\}$ is a continuous function of λ_0 and hence constant, say N . Then the winding number of $\lambda \mapsto \det f(\lambda, 0)$ is $r + nN$, so $[f] \in (r + n\mathbb{Z}) \times \mathbb{Z}$. \square

COROLLARY. $\phi_{M, \gamma, 1}$ is not conjugate to $\phi_{M, -\gamma, 1}$ when γ is of order $4k$ and $\theta = p/q$ with $(p, q) = 1$ and $(q, 4k) = 1$, unless $k = 1$.

PROOF: If $\phi_{M, \gamma, 1} \beta = \beta \phi_{M, -\gamma, 1}$ then $\phi_{M, \gamma, 1} \beta(U) = -\gamma \beta(U) = \gamma^{2k+1} \beta(U)$ and hence, by the proposition, $[\beta(U)] \in ((2k+1) + 4k\mathbb{Z}) \times \mathbb{Z}$. However, by the proof of Proposition 8(i), $[\beta(U)] = \pm(1, 0)$. Hence $k = 1$ (and $\beta(U) = (-1, 0)$). \square

The situation $k = 1$ occurring in the last corollary always gives $\phi_{M, \gamma, 1}$ conjugate to $\phi_{M, -\gamma, 1}$, for then $\gamma = \pm i$ and so $-\gamma = \gamma^{-1}$. We conjecture that this is the only situation in which $\phi_{M, \gamma, 1}$ is conjugate to $\phi_{M, -\gamma, 1}$.

REFERENCES

- [1] M.F. Atiyah, 'K-theory and reality', *Quart. J. Math. Oxford Ser. 2* **17** (1966), 367–86.
- [2] B. Brenken, 'Representations and automorphisms of the irrational rotation algebra', *Pacific J. Math.* **111** (1984), 257–282.
- [3] C. Farsi and N. Watling, 'Trivial fixed point subalgebras of the rotation algebra', *Math. Scand.* **72** (1993), 298–302.
- [4] C. Farsi and N. Watling, 'C*-algebras of dynamical systems on the non-commutative torus', *Math. Scand.* **75** (1994), 101–110.
- [5] D.E. Flath, *Introduction to number theory* (J. Wiley & Sons, New York, 1989).
- [6] M. Karoubi, *K-theory: an introduction* (Springer-Verlag, Berlin, Heidelberg, New York, 1978).
- [7] H. Schröder, *K-theory for real C*-algebras and applications*, Pitman Research Notes on Mathematics **290** (Pitman, New York, 1993).
- [8] Y. Watatani, 'Toral automorphisms on the irrational rotation algebra', *Math. Japon.* **26** (1981), 479–484.

School of Mathematics
 La Trobe University
 Bundoora Vic 3083
 Australia
 e-mail: matyh@lure.latrobe.edu.au
 P.Stacey@latrobe.edu.au