ON THE GROWTH OF BLASCHKE PRODUCTS

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It is well known that the distribution of the zeros of an analytic function affects its rate of growth. The literature is too extensive to indicate here. We only point out (**1**, p. 27; **2**; **3**; **5**), where the angular distribution of the zeros plays a role, as it will in this paper. In private communication, A. Zygmund has raised the following related question, which is the subject of our investigation here.

Let $\{z_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of non-zero complex numbers of modulus less than 1, such that $\sum(1 - |z_n|) < \infty$, and consider the Blaschke product

$$(1) \qquad B(z) = \prod \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z}.$$

Let

$$(2) \qquad I(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log|B(re^{i\theta})|)^2 \, d\theta.$$

What are the sequences $\{z_n\}$ for which $I(r)$ is a bounded function of r? In case $I(r)$ is bounded, we say that the Blaschke product has a uniformly square-integrable log modulus. Clearly, this property is a monotone property of the sequence $\{z_n\}$, in the sense that if $\{z_n'\}$ is a subsequence of a sequence $\{z_n\}$ with this property, then $\{z_n'\}$ also has this property.

We first give a necessary and sufficient condition on $\{z_n\}$ that $I(r)$ be bounded, but this condition is so complicated that it is difficult to see when it is fulfilled. However, if we specialize to the case where the $\{z_n\}$ have only finitely many arguments among them, we find a simple necessary and sufficient condition. Further, we find a simple sufficient condition for the general case that is, in one sense, best possible. Finally, we show by an example that the uniform square integrability does indeed depend on the angular distribution of the $\{z_n\}$.

Theorem 1. *A necessary and sufficient condition that $I(r)$ be bounded is that $J(r)$ be bounded, where*

$$(3) \qquad J(r) = \sum_{k=1}^{\infty} k^{-2} \left| (r^k - r^{-k}) \sum_{|z_n| \leqq r} \bar{z}_n^{\,k} + r^k \sum_{|z_n| > r} (\bar{z}_n^{\,k} - z_n^{-k}) \right|^2.$$

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THEOREM 2. *Let $n(r)$ be the number of z_n satisfying $|z_n| \leqq r$. If*

$$n(r) = O\left(\frac{1}{\sqrt{(1-r)}}\right), \tag{4}$$

then $I(r)$ is bounded.

THEOREM 3. *Suppose that the $\{z_n\}$ lie on finitely many rays through the origin. Then $I(r)$ is bounded if and only if condition* (4) *holds.*

COROLLARY. *There exists a convergent Blaschke product with real positive zeros for which $I(r)$ is unbounded.*

Remark. Condition (4) is easily seen to be equivalent to

$$\sum_{|z_n|>r} (1 - |z_n|) = O(\sqrt{(1-r)}). \tag{5}$$

These conditions imply that

$$\sum_{|z_n|\leqq r} (1 - |z_n|)^{1/2} = O\left(\log \frac{1}{1-r}\right) \tag{6}$$

and

$$\sum (1 - |z_n|)^{\alpha} < \infty \quad \text{if } \alpha > \tfrac{1}{2}. \tag{7}$$

THEOREM 4. *There exists a convergent Blaschke product B for which $I(r) \to 0$ as $r \to 1$ but for which*

$$\sum (1 - |z_n|)^{1-\epsilon} = \infty \tag{8}$$

for each $\epsilon > 0$.

COROLLARY. *If B' is the Blaschke product with zeros $\{|z_n|\}$, where $\{z_n\}$ are the zeros of the Blaschke product B of Theorem* 4, *then $I'(r)$ is not bounded.*

Proof of Theorem 1. As in (**4**), we write

$$\log |B(re^{i\theta})| = \sum c_k(r) e^{ik\theta}, \tag{9}$$

where the Fourier coefficients $c_k(r)$ are given by

$$c_0(r) = \sum_{|z_n|\leqq r} \log \frac{r}{|z_n|} + \log |B(0)|, \tag{10}$$

$$c_k(r) = \frac{1}{2}\alpha_k r^k + \frac{1}{2k} \sum_{|z_n|\leqq r} \left\{\left(\frac{r}{z_n}\right)^k - \left(\frac{\bar{z}_n}{r}\right)^k\right\}, \qquad k = 1, 2, 3, \ldots, \tag{11}$$

$$c_{-k}(r) = \overline{c_k(r)}, \tag{12}$$

and the numbers α_k are given by

$$\log B(z) = \text{const} + \sum_{k=1}^{\infty} \alpha_k z^k, \qquad |z| < |z_0|. \tag{13}$$

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A direct calculation then yields

$$c_0(r) = \sum_{|z_n|\leqq r} \log \frac{r}{|z_n|} + \sum_{n=1}^{\infty} \log |z_n| = -\int_r^1 \frac{n(t)}{t}\, dt, \tag{14}$$

$$c_k(r) = \frac{1}{2k} \sum_{n=1}^{\infty} (\bar{z}_n^{\,k} - z_n^{-k}) r^k + \frac{1}{2k} \sum_{|z_n|\geqq r} \left\{ \left(\frac{r}{z_n}\right)^k - \left(\frac{\bar{z}_n}{r}\right)^k \right\}. \tag{15}$$

It is easy to verify that all the series are absolutely convergent as a result of the condition $\sum (1 - |z_n|) < \infty$. Furthermore, $c_0(r)$ is bounded, and approaches 0 as $r \to 1-$ so that Parseval's Theorem then yields the result, after a slight rearrangement of the sums in (15).

Before proving the later results, we need a lemma.

LEMMA 1. *Let*

$$u(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \qquad |x| \leqq 1. \tag{16}$$

Then

$$[u(1) - 2u(r) + u(r^2)] = O(1 - r) \quad \textit{as } r \to 1-. \tag{17}$$

Proof. We have:

$$u(x) = \int_0^x \frac{1}{t} \log \frac{1}{1-t}\, dt.$$

On integrating by parts, we obtain:

$$u(1) - u(x) = u(1 - x) + \log(1 - x) \log x.$$

Taking expansions of $u(1 - x)$ and $\log x$ in powers of $(1 - x)$, we obtain:

$$u(1) - u(x) = -(1 - x)\log(1 - x) + (1 - x) + O((1 - x)^2 \log(1 - x)),$$

from which (17) easily follows.

Proof of Theorem 2. We suppose that $n(r) = O((1 - r)^{-1/2})$. Now,

$$I(r) \leqq c_0(r)^2 + \sum_{k=1}^{\infty} \frac{1}{4k^2} (|U_k|^2 + |V_k|^2),$$

where

$$U_k = \frac{1 - r^{2k}}{r^k} \sum_{|z_n|\leqq r} \bar{z}_n^{\,k}, \qquad V_k = r^k \sum_{|z_n|>r} \frac{1 - |z_n|^{2k}}{\bar{z}_n^{\,k}}.$$

As remarked earlier, $c_0(r)$ is bounded, and we can forget about it. Now, writing $O(1)$ for a generic constant independent of k, we have:

$$|U_k| \leqq \frac{1 - r^{2k}}{r^k} \int_0^r t^k\, dn(t) \leqq \frac{1 - r^{2k}}{r^k} r^k n(r) \leqq O(1) \frac{1 - r^{2k}}{\sqrt{(1 - r)}}.$$

Therefore,

$$E(r) = \sum_{k=1}^{\infty} \frac{|U_k|^2}{k^2} \leqq O(1) \frac{1}{1 - r} \sum_{k=1}^{\infty} \frac{(1 - r^{2k})^2}{k^2}$$

$$= O(1) \frac{1}{1 - r} [u(1) - 2u(r^2) + u(r^4)],$$

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and by Lemma 1, $E(r)$ is bounded.

Now

$$|V_k| \leqq r^k \int_r^1 (t^{-k} - t^k)\,dn(t) = \int_r^1 \frac{t^{-k} - t^k}{t^{-1} - t}(t^{-1} - t)\,dn(t).$$

Writing $t^{-1} = e^x$ so that

$$\frac{t^{-k} - t^k}{t^{-1} - t} = \frac{\sinh kx}{\sinh x}$$

and using the fact that $\sinh kx/\sinh x$ is an increasing function of x, we see that

$$\begin{aligned}\int_r^1 (t^{-k} - t^k)\,dn(t) &\leqq \frac{r^{-k} - r^k}{r^{-1} - r}\int_r^1 (t^{-1} - t)\,dn(t)\\ &= \frac{r^{-k}(1 - r^{2k})}{r^{-1}(1 - r^2)}\int_r^1 \frac{1 - t^2}{t}\,dn(t)\\ &\leqq O(1)\frac{1 - r^k}{r^k(1 - r)}\int_r^1 (1 - t)\,dn(t)\\ &= O(1)\frac{1 - r^k}{r^k(1 - r)}\left[-(1 - r)n(r) + \int_r^1 n(t)\,dt\right],\end{aligned}$$

from which we see that

$$|V_k| \leqq O(1)\frac{1 - r^k}{\sqrt{(1 - r)}},$$

and proceeding as above, we see that $\sum k^{-2}|V_k|^2 = O(1)$, and the theorem is proved.

Proof of Theorem 3. The "if" part follows from Theorem 2. For the converse, we suppose, for simplicity, that each $z_n = x_n$ is real and positive, the general case following by a simple argument. We write

$$J(r) = \sum_{k=1}^{\infty} k^{-2}(U_k + V_k)^2,$$

where

$$U_k = (r^{-k} - r^k)\sum_{x_n \leqq r} x_n^{\,k}, \qquad V_k = r^k \sum_{x_n \geqq r}(x_n^{\,-k} - x_n^{\,k}).$$

Since U_k and V_k are both non-negative, we see that $J(r)$ is bounded if and only if $\sum k^{-2}U_k^2$ and $\sum k^{-2}V_k^2$ are both bounded. Now

$$k^{-1}r^k \sum_{x_n > r}(x_n^{\,-k} - x_n^{\,k}) = k^{-1}r^k \sum_{x_n > r} x_n^{\,-k}(1 - x_n^{\,2k}) \geqq k^{-1}r^k \sum_{x_n > r} x_n^{\,-k}(1 - x_n^{\,k}).$$

However, $1 - x^k \geqq (1 - x)kx^{k-1}$ for $0 < x < 1$ so that

$$k^{-1}r^k \sum_{x_n > r}(x_n^{\,-k} - x_n^{\,k}) \geqq r^k \sum_{x_n > r} x_n^{\,-k}x_n^{\,k-1}(1 - x_n) \geqq r^k \sum_{x_n > r}(1 - x_n).$$

Hence, we see that

$$\left(\sum r^{2k}\right)\left(\sum_{x_n>r}(1-x_n)\right)^2$$

is bounded, from which (5) follows. As we have remarked, (5) is equivalent to (4), and the result is proved. Notice that if $s \leqq r$, we have:

$$k^{-1}\sum_{x_n\leqq r} x_n^{\ k}(r^{-k}-r^k) \geqq k^{-1}\sum_{s\leqq x_n\leqq r} x_n^{\ k}(r^{-k}-r^k)$$
$$\geqq k^{-1}(n(r)-n(s))s^k(r^{-k}-r^k),$$

so that we have:

(18) $$\sup J(r) \geqq (n(r)-n(s))^2[u((s/r)^2)-2u(s^2)+u((rs)^2)]$$

which gives us an estimate on $n(r) - n(s)$.

Proof of Theorem 4. We let

(19) $$\begin{cases} z_{n,\nu} = \rho_n\omega_n^{\ \nu}, \qquad \omega_n = e^{2\pi i/\lambda_n}, \\ \beta_n(z) = \prod_{\nu=0}^{\lambda_n-1} \frac{\bar{z}_{n\nu}}{|z_{n\nu}|}\frac{z_{n\nu}-z}{1-\bar{z}_{n\nu}z} = \frac{\rho_n^{\ \lambda_n}-z^{\lambda_n}}{1-\rho^{\lambda_n}z^{\lambda_n}}, \\ B(z) = \prod_{n=1}^{\infty}\beta_n(z), \\ \lambda_n = (n!)^3, \quad 1-\rho_n = \frac{1}{cn^2(n!)^3}, \qquad n = 1, 2, 3, \ldots, \end{cases}$$

where c is a positive constant to be chosen soon. Let $\{z_n\}$ be the $z_{n\nu}$ chosen in some order. It is clear that $\sum(1-|z_n|) < \infty$ but that $\sum(1-|z_n|)^{1-\epsilon} = \infty$ for each $\epsilon > 0$. We choose the constant c so that

$$\rho_n^{\ \lambda_n} = \left(1-\frac{1}{cn^2(n!)^3}\right)^{(n!)^3} = 1-\frac{1}{cn^2}+O\left(\frac{1}{n^4}\right)$$

satisfies

(20) $$1-\frac{1}{2n^2} \leqq \rho_n^{\ \lambda_n} \leqq 1-\frac{1}{2cn^2}, \qquad n = 1, 2, 3, \ldots.$$

Since, writing $z = re^{i\theta}$,

$$|\beta_n(z)|^{-2} = 1+\frac{(1-\rho_n^{\ 2\lambda_n})(1-r^{2\lambda_n})}{(\rho_n^{\ \lambda_n}-r^{\lambda_n})^2+4\rho_n^{\ \lambda_n}r^{\lambda_n}\sin^2(\frac{1}{2}\lambda_n\theta)},$$

we have:

(21) $$-\log|\beta_n(z)| = \frac{1}{2}\log\left[1+\frac{(1-\rho_n^{\ 2\lambda_n})(1-r^{2\lambda_n})}{(\rho_n^{\ \lambda_n}-r^{\lambda_n})^2+4\rho_n^{\ \lambda_n}r^{\lambda_n}\sin^2(\frac{1}{2}\lambda_n\theta)}\right].$$

Let $N = N(z) = N(r)$ be determined by

$$\rho_N \leqq r < \rho_{N+1}.$$

For $n > N + 1$,

$$(22) \quad -\log|\beta_n(z)| \leqq \frac{1}{2}\log\left[1 + \frac{(1-\rho_n^{2\lambda_n})(1-r^{2\lambda_n})}{(\rho_n^{\lambda_n} - \rho_{N+1}^{\lambda_n})^2}\right] \leqq \frac{(1-\rho_n^{2\lambda_n})(1-r^{2\lambda_n})}{2(\rho_n^{\lambda_n} - \rho_{N+1}^{\lambda_n})^2}.$$

Hence,

$$(23) \quad \sum_{n>N+1} -\log|\beta_n(z)| \leqq k \sum_{n>N+1} (1-\rho_n^{2\lambda_n}) \leqq \sum_{n>N+1} \frac{1}{2n^2} = O\left(\frac{1}{N}\right).$$

For $n < N$, we have, similarly to (22),

$$-\log|\beta_n(z)| \leqq \frac{(1-\rho_n^{2\lambda_n})(1-r^{2\lambda_n})}{2(\rho_N^{\lambda_n} - \rho_n^{\lambda_n})^2}$$

so that

$$(24) \quad \sum_{n<N} -\log|\beta_n(z)| \leqq k \sum_{n<N} n^4(1-\rho_n^{2\lambda_n})(1-r^{2\lambda_n})$$

$$\leqq k \sum_{n<N} n^4(1-\rho_n^{2\lambda_n})(1-\rho_N^{2\lambda_n}) \leqq \frac{k}{N^2}\sum_{n<N}\frac{1}{n} = O\left(\frac{1}{N}\right).$$

Combining (23) and (24) we see that

$$-\log|B(z)| = -\log|\beta_N(z)| - \log|\beta_{N+1}(z)| + O\left(\frac{1}{N}\right)$$

so that in order to prove that $I(r) \to 0$ it is enough to prove that

$$\int_0^{2\pi} \log^2|\beta_N(z)|\, d\theta \quad \text{and} \quad \int_0^{2\pi} \log^2|\beta_{N+1}(z)|\, d\theta$$

both tend to zero. For the first one, we have:

$$\int_0^{2\pi} \log^2|\beta_N(z)|\, d\theta \leqq \int_0^{2\pi} \log^2\left[1 + \frac{(1-\rho_N^{2\lambda_N})(1-r^{2\lambda_N})}{4\rho_N^{\lambda_N} r^{\lambda_N}\sin^2(\frac{1}{2}\lambda_N\theta)}\right] d\theta$$

$$\leqq 2\lambda_n \int_0^{\pi/\lambda_N} \log^2\left[1 + \frac{\pi^2(1-\rho_N^{2\lambda_N})(1-r^{2\lambda_N})}{\rho_N^{\lambda_N} r^{\lambda_N}\lambda_N^2\theta^2}\right] d\theta$$

$$= 2\int_0^{\pi} \log^2\left[1 + \frac{\pi^2(1-\rho_N^{2\lambda_N})(1-r^{2\lambda_N})}{\rho_N^{\lambda_N} r^{\lambda_N} t^2}\right] dt.$$

Now, using the easily derived estimate

$$(25) \quad \int_0^{\pi} \log^2\left(1 + \frac{\beta^2}{\theta^2}\right) d\theta \leqq k\beta,$$

we have:

$$\int_0^{2\pi} \log^2|\beta_N(z)|\, d\theta \leqq k\left[\frac{(1-\rho_N^{2\lambda_N})(1-r^{2\lambda_N})}{\rho_N^{\lambda_N} r^{\lambda_N}}\right]^{1/2} = O\left(\frac{1}{N^2}\right).$$

Finally, for $\int_0^{2\pi} \log^2|\beta_{N+1}(z)|\, d\theta$, we distinguish two cases, namely

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$$r^{\lambda_{N+1}} \leqq \tfrac{1}{2}\rho_{N+1}{}^{\lambda_{N+1}} \quad \text{and} \quad r^{\lambda_{N+1}} > \tfrac{1}{2}\rho_{N+1}{}^{\lambda_{N+1}}.$$

In the first case, starting with (21) we have:

$$\int_0^{2\pi} \log^2 |\beta_{N+1}(z)|\, d\theta \leqq \frac{1}{4}\int_0^{2\pi} \frac{(1-\rho_{N+1}{}^{2\lambda_{N+1}})^2(1-\rho_N{}^{2\lambda_{N+1}})^2}{(\frac{1}{2}\rho_{N+1}{}^{\lambda_{N+1}})^4}\, d\theta$$

$$\leqq k\,\frac{(1-\rho_{N+1}{}^{2\lambda_{N+1}})^2}{\rho_{N+1}{}^{4\lambda_{N+1}}} = O\left(\frac{1}{N}\right).$$

In the second case,

$$\int_0^{2\pi} \log^2 |\beta_{N+1}(z)|\, d\theta \leqq \frac{1}{4}\int_0^{2\pi} \log^2\left[1 + \frac{1-\rho_{N+1}{}^{2\lambda_{N+1}}}{2\rho_{N+1}{}^{2\lambda_{N+1}}\sin^2(\frac{1}{2}\lambda_{N+1}\theta)}\right] d\theta$$

$$\leqq \frac{1}{4}\, 2\lambda_{N+1}\int_0^{\pi/\lambda_{N+1}} \log^2\left[1 + \frac{\pi^2(1-\rho_{N+1}{}^{2\lambda_{N+1}})}{2\rho_{N+1}{}^{2\lambda_{N+1}}\lambda_{N+1}{}^2\theta^2}\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \log^2\left[1 + \frac{\pi^2(1-\rho_{N+1}{}^{2\lambda_{N+1}})}{2\rho_{N+1}{}^{2\lambda_{N+1}}t^2}\right] dt = O\left(\frac{1}{N}\right)$$

by (25), and the proof is complete.

In closing, we note that the methods used here can be used to obtain related results. For example, if $n(r) \leqq \nu(r)(1-r)^{-\alpha}$, where $0 < \alpha < 1$ and either $\nu(r) \downarrow 0$ or $\nu(r) = \text{const}$, then

$$I(r) = O(\nu^2(r)(1-r)^{1-2\alpha}).$$

In particular, if $n(r) = o((1-r)^{-1/2})$, then $I(r) = o(1)$.

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