A NEW PROOF OF SOME RESULTS OF RÉNYI AND THE ASYMPTOTIC DISTRIBUTION OF THE RANGE OF HIS KOLMOGOROV-SMIRNOV TYPE RANDOM VARIABLES

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1. Introduction and summary. Let $X_1, X_2, \ldots, X_n$ be mutually independent random variables with a common continuous distribution function $F(t)$. Let $F_n(t)$ be the corresponding empirical distribution function, that is $F_n(t) =$ (number of $X_i \leqslant t$, $1 \leqslant i \leqslant n)/n$. In his paper **(12)** A. Rényi proves, among many others, the following two theorems:

THEOREM 1.

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}} \sup_{a\leqslant F(t)} [F_n(t) - F(t)]/F(t) < y\}$$
$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{y(a/[1-a])^{\frac{1}{2}}} \exp(-\tfrac{1}{2}t^2)dt, \quad \textit{if } y > 0,\ 0 < a < 1 \textit{ and zero otherwise.}$$
$$= \Phi(y(a/[1-a])^{\frac{1}{2}}).$$

THEOREM 2.

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}} \sup_{a\leqslant F(t)} |F_n(t) - F(t)|/F(t) < y\}$$
$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \{(-1)^k/(2k+1)\} \exp\{-(2k+1)^2 \pi^2(1-a)/8ay^2\},$$
$$\textit{if } y > 0,\ 0 < a < 1 \textit{ and zero otherwise.}$$
$$= L(y(a/[1-a])^{\frac{1}{2}}).$$

Obviously (see also **(2)**), the following two corollaries are also true:

COROLLARY 1.

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}} \sup_{F(t)\leqslant b} \{F_n(t) - F(t)\}/[1 - F(t)] < y\} = \Phi(y([1-b]/b)^{\frac{1}{2}}),$$

where $\Phi(.)$ *is as defined in Theorem* 1, *and* $0 < b < 1$.

COROLLARY 2.

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}} \sup_{F(t)\leqslant b} |F_n(t) - F(t)|/\{1 - F(t)\} < y\} = L(y([1-b]/b)^{\frac{1}{2}}),$$

where $L(.)$ *is as defined in Theorem* 2, *and* $0 < b < 1$.

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If in these two corollaries we put $b = 1 - a$, $0 < a < 1$, and replace $1 - F(t)$ by $F(t)$ and $1 - F_n(t)$ by $F_n(t)$, then they coincide with Theorems 1 and 2 respectively.

One of the purposes of this paper is to give a simple proof for Corollaries 1 and 2 and thereby for Theorems 1 and 2. In § 3 two more theorems of Rényi are also proved.

We define

$$\begin{aligned} D_n^+(b) &= \sup_{F(t)\leqslant b} \{F_n(t) - F(t)\}/\{1 - F(t)\}, \\ D_n^-(b) &= -\inf_{F(t)\leqslant b} \{F_n(t) - F(t)\}/\{1 - F(t)\}, \end{aligned} \tag{1.1}$$

where $0 < b < 1$ in both cases. Let us put

$$R_n(b) = D_n^+(b) + D_n^-(b); \tag{1.2}$$

$R_n(b)$ is called the range of the two random variables of (1.1).

The distribution of the range of the original Kolmogorov–Smirnov statistic was derived by Kuiper in **(10)**. He also points out that in the case when the probability is distributed on the circumference of a circle and there is therefore no natural starting point for the distribution function, and different starting points will give the Kolmogorov–Smirnov test statistics different values, the range

$$\sup_{0\leqslant t\leqslant 1} \{F_n(t) - F(t)\} - \inf_{0\leqslant t\leqslant 1} \{F_n(t) - F(t)\}$$

is independent of the starting point. This property is not shared by the weighted range in (1.2). In this paper we are going to prove the following two theorems regarding the random variables of (1.1) and (1.2).

THEOREM 3.

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}}D_n^-(b) < x,\ n^{\frac{1}{2}}D_n^+(b) < y\} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left[-\frac{(2k+1)^2\pi^2 b}{2(x+y)^2(1-b)}\right]\sin\frac{(2k+1)\pi y}{x+y},$$

where $x > 0$, $y > 0$.

THEOREM 4.

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}}R_n(b) < r\} = \int_0^r \delta(b/(1-b); u)\, du, \qquad r > 0,$$

where

$$\delta(b/(1-b); r) = 8\sum_{k=1}^{\infty} (-1)^{k-1} k^2 \phi(kr[b/(1-b)]^{-\frac{1}{2}})$$

is the asymptotic density function of the range $R_n(b)$, *and where* $\phi(.)$ *stands for the normal density function with zero mean and unit variance.*

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2. Proof of Corollaries 1 and 2 of § 1. Without loss of generality we may assume that $F(t) = t$ with t uniformly distributed on [0, 1], and that $F_n(t)$ is the empirical distribution function constructed by selecting a random sample of size n from this uniform distribution. Let

$$\xi_n(t) = n^{\frac{1}{2}}(F_n(t) - t), \qquad 0 \leqslant t \leqslant 1.$$

It is known that $\xi_n(t)$ is a Markov process. According to a theorem of Donsker and Doob **(6, 7)** we have

$$\lim_{n\to\infty} P\{\sup_{0\leqslant t\leqslant 1} \xi_n(t) < y\} = P\{\sup_{0\leqslant t\leqslant 1} \xi(t) < y\},$$

where $\xi(t)$ is a Gaussian process with parameter t, $0 \leqslant t \leqslant 1$, and covariance function $r(s, t) = s(1 - t)$, $0 \leqslant s < t \leqslant 1$. In our case we get

$$\lim_{n\to\infty} P\{\sup_{0\leqslant t\leqslant b} \xi_n(t)/(1 - t) < y\} = P\{\sup_{0\leqslant t\leqslant b} \xi(t)/(1 - t) < y\}, \tag{2.1}$$

and the covariance function of the process $\xi(t)/(1 - t)$ is given as

$$r(s, t) = s/(1 - s), \qquad 0 \leqslant s < t \leqslant b.$$

It is convenient here to make a transformation due to Doob **(7)** as follows:

$$X(t) = (t + 1)\xi(t/(t + 1)), \qquad 0 \leqslant t < \infty.$$

The resulting process $X(t)$ is called the Wiener–Einstein process, or the Brownian movement process, with $E(X(t)) = 0$, $r(s, t) = \min(s, t)$, $0 \leqslant s$, $t < \infty$. In our case this transformation and (2.1) give

$$\begin{aligned} P\{\sup\nolimits_{0\leqslant t\leqslant b} \xi(t)/(1 - t) < y\} &= P\{\sup\nolimits_{0\leqslant t\leqslant b/(1-b)} X(t) < y\} \\ &= \delta_1(y, b/(1 - b)), \end{aligned} \tag{2.2}$$

and when $X(t)$ is the above Brownian process, $\delta_1(y; b/(1 - b))$ is known to be equal to $\Phi(y([1 - b]/b)^{\frac{1}{2}})$ of Corollary 1; see, e.g., **(1**, (4.7)**)**.

An immediate verification of (2.2) also follows from the following property of the Brownian process due to Bachelier; see also **(7**, (4.1)**)**. For fixed s

$$P\{\max\nolimits_{0\leqslant t\leqslant T} [X(s + t) - X(s)] \geqslant y\} = 2P\{X(s + T) - X(s) \geqslant y\}. \tag{2.3}$$

Letting $s = 0$, and using the notation of (2.2), we get (on recalling that the sample functions of $X(t)$ are continuous with probability 1):

$$\delta_1(y, T) = P\{\sup\nolimits_{0\leqslant t\leqslant T} X(t) < y\} = 1 - 2P\{X(T) \geqslant y\} = \Phi(yT^{-\frac{1}{2}}),$$

where $\Phi(.)$ is as defined in Theorem 1. When $T = b/(1 - b)$, we get Corollary 1.

In the case of Corollary 2 we are led to consider

$$\delta_2(y, b/(1 - b)) = P\{\sup\nolimits_{0\leqslant t\leqslant b/(1-b)} |X(t)| < y\}, \tag{2.4}$$

and this is the absorption probability problem for the Brownian process $X(t)$ with barriers $\pm y$. In particular it is known that

$$(2.5)\qquad P\{-x < X(t) < y; 0 \leqslant t \leqslant T\} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left[-\frac{(2k+1)^2\pi^2 T}{2(x+y)^2}\right]\sin\frac{(2k+1)\pi y}{x+y}, \quad x > 0,\ y > 0.$$

When $x = y$, (2.5) reduces to $\delta_2(y, T)$ of (2.4) and if $T = b/(1-b)$, we get Corollary 2.

3. Some consequences of (2.3) and (2.5). In his paper **(12)**, Rényi also derives

THEOREM 5.

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}} \sup_{a\leqslant F(t)\leqslant b} [F_n(t) - F(t)]/[1 - F(t)] < y\} = N(y; a, b),$$

where $-\infty < y < +\infty$ *and* $0 < a < b < 1$.

THEOREM 6.

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}} \sup_{a\leqslant F(t)\leqslant b} |F_n(t) - F(t)|/\{1 - F(t)\} < y\} = R(y; a, b),$$

if $y > 0$, $0 < a < b < 1$ *and zero otherwise.*

For the form of $N(.)$ and $R(.)$ we refer the reader to **(12**, (3.6) and (3.7)**)** where our $N(.)$ and $R(.)$ are gained by putting $b = 1 - a$ and $a = 1 - b$, and by replacing $F(t)$ by $1 - F(t)$ and $F_n(t)$ by $1 - F_n(t)$. That is, Theorems 5 and 6 are Corollary 1 and 2 type versions of **(12**, (3.6) and (3.7)**)**. Using (2.3) and (2.5) of § 2 we give a short proof of them here and note also that there is a misprint in the definition of ρ_k of **(12**, (3.7)**)**. Instead of having $[a/(1-a)]^{\frac{1}{2}}$ in the denominator of this ρ_k, we should have $[(1-b)/b]^{\frac{1}{2}}$ in its numerator.

Using the argument and notation of § 2, in case of Theorem 5 we are led to consider

$$(3.1)\qquad P\{\sup_{a/(1-a)\leqslant t\leqslant b/(1-b)} X(t) < y\},$$

and would like to show here that this probability statement is equal to $N(y; a, b)$ of Theorem 5. Considering the Brownian process $X(t)$, we have

$$(3.2)\qquad P\{\sup_{T_1\leqslant t\leqslant T_2} X(t) < y\} = P\{\sup_{0\leqslant t\leqslant T_2-T_1} X(t+T_1) < y\}$$
$$= P\{\sup_{0\leqslant t\leqslant T_2-T_1} [X(t+T_1) - X(T_1)] + X(T_1) < y\}$$
$$= \int_{-\infty}^{+\infty} P\{\sup_{0\leqslant t\leqslant T_2-T_1} [X(t+T_1) - X(T_1)] < y - x\} dP\{X(T_1) < x\},$$

the convolution of the distribution functions of two independent random variables. It follows from (2.3) that

$$P\{\sup_{0\leqslant t\leqslant T_2-T_1}[X(t+T_1)-X(T_1)]<y-x\} = \Phi([y-x][T_2-T_1]^{-\frac{1}{2}}),$$

where $\Phi(.)$ is as defined in Theorem 1 and $X(T_1)$ is Gaussian with zero mean and variance T_1. Letting $T_2 = b/(1-b)$ and $T_1 = a/(1-a)$ we get, through (3.1), Theorem 5.

In case of Theorem 6 we have to consider

$$P\{\sup_{a/(1-a)\leqslant t\leqslant b/(1-b)}|X(t)|<y\}, \tag{3.3}$$

and would like to show that this probability is equal to $R(y; a, b)$ of Theorem 6. Following the lines of the above argument, we have

$$\begin{aligned} &P\{\sup_{T_1\leqslant t\leqslant T_2}|X(t)|<y\} \\ &= P\{-y<[X(t+T_1)-X(T_1)]+X(T_1)<y;\ 0\leqslant t\leqslant T_2-T_1\} \\ &= \int_{-\infty}^{+\infty} P\{-(y+x)<X(t+T_1)-X(T_1)<y-x; \\ &\qquad\qquad 0\leqslant t\leqslant T_2-T_1\}\,dP\{X(T_1)<x\}. \end{aligned} \tag{3.4}$$

It follows from (2.5) that

$$\begin{aligned} &P\{-(y+x)<X(t+T_1)-X(T_1)<y-x;\ 0\leqslant t\leqslant T_2-T_1\} \\ &\quad = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left(-\frac{2k+1)^2\pi^2(T_2-T_1)}{8y^2}\right)\sin\left((2k+1)\frac{\pi(y-x)}{2y}\right), \end{aligned}$$

if $y>0$ and $|x|\leqslant y$, and it is equal to zero if $y\leqslant 0$ or $y>0$ but $|x|>y$. This and (3.4) imply that

$$\begin{aligned} P\{\sup_{T_1\leqslant t\leqslant T_2}|X(t)|<y\} &= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\exp\{-(2k+1)^2\pi^2(T_2-T_1)/8y^2\}}{2k+1} \\ &\quad\times\int_{-y}^{+y}\frac{\exp(-x^2/2T_1)}{\sqrt{(2\pi T_1)}}\sin\left((2k+1)\pi\frac{y-x}{2y}\right)dx. \end{aligned}$$

It can be shown by simple calculations that this last expression is equal to

$$L(yT_2^{-\frac{1}{2}})E_k, \tag{3.5}$$

where

$$\begin{aligned} E_k = 1-\frac{2}{\sqrt{(2\pi)}}\int_{y/T_1^{\frac{1}{2}}}^{\infty}\exp(-\tfrac{1}{2}x^2)dx + \frac{2T_1^{\frac{1}{2}}\exp(-y^2/2T_1)}{\sqrt{(2\pi)}y} \\ \times\int_0^{(2k+1)\pi}\exp(T_1x^2/2y^2)\sin x\,dx, \end{aligned}$$

and where $L(.)$ is the distribution function of Theorem 2. Putting

$$T_2 = b/(1-b) \quad\text{and}\quad T_1 = a/(1-a)$$

in (3.5) we get $R(y; a, b)$ of Theorem 6.

4. Proof of Theorems 3 and 4 of § 1. Using the ideas of Doob's paper **(7)** (cf. Donsker **(6)**), it is easily seen that the statement of Theorem 3 is equivalent to

$$P\{-\inf_{0\leqslant t\leqslant b/(1-b)} X(t) < x, \sup_{0\leqslant t\leqslant b/(1-b)} X(t) < y\} = P\{-x < X(t) < y, 0 \leqslant t \leqslant b/(1-b)\}$$

and this probability statement can be evaluated from (2.5) of § 2 on putting $T = b/(1-b)$. This completes the proof of Theorem 3.

Let us now denote the probability in (2.5) by $V(T; x, y)$. Then the probability density corresponding to (2.5) is given by the mixed derivative

$$v(T; x, y) = V_{xy}(T; x, y), \tag{4.1}$$

and introducing

$$R(T) = \sup_{0\leqslant t\leqslant T} X(t) - \inf_{0\leqslant t\leqslant T} X(t), \tag{4.2}$$

one gets

$$\delta(T; r) = \int_0^r v(T; x, r - x)\, dx, \tag{4.3}$$

the density function of the range $R(T)$ which, with $T = b/(1-b)$, is the asymptotic density function of the range $R_n(b)$ of (1.2). Thus Theorem 4 is proved by performing the calculations indicated in (4.1) and (4.3).

Feller **(8)**, in connection with the asymptotic distribution of the range of sums of independent random variables, has studied the distribution of the range $R(T)$, using an equivalent form of the distribution function $V(T; x, y)$ of (2.5) (we are referring here to **(8**, (3.4) and (3.5)**)**. He has shown in particular that the density function of the range $R(T)$ of (4.2) is **(8**, (3.6)**)**.

$$\delta(T; r) = 8\sum_{k=1}^{\infty} (-1)^{k-1} k^2 \phi(krT^{-\frac{1}{2}}). \tag{4.4}$$

We can thus skip here the calculations indicated in (4.1) and (4.3), and (4.4), with $T = b/(1-b)$, terminates the proof of Theorem 4.

Remark. In **(8)**, Feller also shows that $\delta(T; r)$ of (4.4) is positive and that it is indeed a density function, by pointing out in (3.8) that it is related to the distribution function $L(z)$ which occurs in the original Kolmogorov–Smirnov theorem on empirical distribution functions; see, e.g., **(8**, (2.7)**)**. It is only appropriate, therefore, that in $R_n(b)$ of (1.2) we have found a Kolmogorov–Smirnov type range statistic, whose limit distribution actually produces the density function $\delta(T; r)$ of (4.4).

5. On another range statistic. In relation (1.1) $D_n^+(.)$ and $D_n^-(.)$ could be defined for all t such that $a \leqslant F(t) \leqslant b$, where $0 < a < b < 1$ in both cases, and we could also consider their appropriate range statistics in the sense of (1.2). Then, in terms of our Brownian process $X(t)$ of § 2, we would have the problem of finding

$$(5.1)\quad P\{-\inf_{a/(1-a)\leqslant t\leqslant b/(1-b)} X(t) < -x, \sup_{a/(1-a)\leqslant t\leqslant b/(1-b)} X(t) < y\} \\ = P\{x < X(t) < y; a/(1-a) \leqslant t \leqslant b/(1-b)\},$$

where $y > 0$, $x < 0$. We have, in general, for $y > 0$ and $y - x > 0$,

$$(5.2)\quad P\{x < X(t) < y; T \leqslant t \leqslant T_2\} \\ = P\{-\tfrac{1}{2}(y-x) < X(t) - \tfrac{1}{2}(y+x) < \tfrac{1}{2}(y-x); T_1 \leqslant t \leqslant T_2\} \\ = P\{-\tfrac{1}{2}(y-x) < [X(t+T_1) - X(T_1)] + [X(T_1) - \tfrac{1}{2}(y+x)] \\ < \tfrac{1}{2}(y-x); 0 \leqslant t \leqslant T_2 - T_1\} \\ = \int_{-\infty}^{+\infty} P\{-[\tfrac{1}{2}(y-x)+u] < X(t+T_1) - X(T_1) < \tfrac{1}{2}(y-x) - u; \\ 0 \leqslant t \leqslant T_2 - T_1\}\, dP\{X(T_1) - \tfrac{1}{2}(y+x) < u\},$$

where, for $y - x > 0$ and $|u| \leqslant \frac{1}{2}(y-x)$, the first probability statement under the integral sign can be easily evaluated using (2.5) of § 2, and $X(T_1)$ is Gaussian with zero mean and variance T_1. If in (5.2) $x < 0$ and $T_1 = a/(1-a)$, $T_2 = b/(1-b)$, we get the joint distribution function of (5.1). Performing the calculations indicated in (4.1) and (4.3) in this context now, the density function of the range

$$\sup_{a/(1-a)\leqslant t\leqslant b/(1-b)} X(t) - \inf_{a/(1-a)\leqslant t\leqslant b/(1-b)} X(t)$$

can be computed. However, because of the convolution integral of (5.2), this density function seems to have a complicated form, which I could not reduce to a presentable form.

6. Some further remarks.

Remark 1. Two sample versions of Rényi's Kolmogorov–Smirnov type statistics of **(12)** have been studied in **(3)**. In particular the asymptotic distribution function of the random variable

$$(6.1)\qquad N^{\frac{1}{2}} \sup_{a\leqslant F(t)} \{F_{1n}(t) - F_{2m}(t)\}/F(t), \qquad 0 < a < 1,$$

has been derived, where $F_{1n}(t)$ and $F_{2m}(t)$ are two empirical distribution functions based on two mutually independent random samples of size n and m respectively of mutually independent random variables, $N = nm/(n+m)$, and $n \to \infty$, $m \to \infty$ so that $m/n \to \rho$, a constant. Using the same method of proof as in **(3**, § 4**)**, one can also derive the limit distribution of

$$(6.2)\qquad N^{\frac{1}{2}} \sup_{F(t)\leqslant b} \{F_{1n}(t) - F_{2m}(t)\}/\{1 - F(t)\}, \qquad 0 < b < 1.$$

In fact all the theorems of **(3)** remain true with $a = 1 - b$ in their forms. Combining the method of **(3**, § 4**)** and that of § 4 of this paper and intro-

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ducing definitions in connection with (6.2) on the line of (1.1) and (1.2), it can be easily seen that Theorems 3 and 4 remain true in this 2-sample situation also.

Remark 2. k-sample ($k \geqslant 1$) versions of Rényi's Kolmogorov–Smirnov type statistics have been studied in **(4, 5)**. In particular the asymptotic distribution function of the random variable

$$(6.3) \qquad N_k^{\frac{1}{2}} \sup_{a \leqslant F(t)} \left\{ \prod_{j=1}^{k} F_{n_j}(t) - F^k(t) \right\} \bigg/ F^k(t), \qquad 0 < a < 1,$$

has been derived, where $F_{n_j}(t)$, $j = 1, \ldots, k$, are empirical distribution functions, based on random samples of size n_j, $j = 1, \ldots, k$, of mutually independent random variables,

$$N_k = n_1 \bigg/ \left(\sum_{j=1}^{k} n_1/n_j \right),$$

and $n_j \to \infty$, $k = 1, \ldots, k$, so that $n_1/n_j \to \rho_j$, $j = 2, \ldots, k$, where the ρ_j are constant for each j. Using the same method of proof as in **(5,** § 2**)**, one can also derive the limit distribution of

$$(6.4) \qquad N_k^{\frac{1}{2}} \sup_{F(t) \leqslant b} \left\{ \prod_{j=1}^{k} (1 - F_{n_j}(t)) - (1 - F(t))^k \right\} \bigg/ (1 - F(t))^k,$$

where $0 < b < 1$. In fact, **(5**, Theorems 1, 2, 3, and 4**)**, remain true with $a = 1 - b$ in their forms. Combining the method of **(5,** § 2**)** and that of § 4 of this paper and introducing definitions in connection with (6.4) on the lines of (1.1) and (1.2), it can be easily seen that Theorems 3 and 4 of this paper hold in this k-sample situation also.

Remark 3. If in (1.1) we put $b = 1 - a$, and replace $1 - F(t)$ by $F(t)$ and $1 - F_n(t)$ by $F_n(t)$, then we get

$$(6.5) \qquad \begin{aligned} D_n^+(a) &= \sup_{a \leqslant F(t)} \{F(t) - F_n(t)\}/F(t), \\ D_n^-(a) &= -\inf_{a \leqslant F(t)} \{F(t) - F_n(t)\}/F(t), \qquad 0 < a < 1, \end{aligned}$$

and $R_n(a) = D_n^+(a) + D_n^-(a)$. In this context obvious corollaries to Theorems 3 and 4 then hold, with the same distribution functions, with b replacing $1 - a$ in their form. The content of Remarks 1 and 2 also remains true, *mutatis mutandis*, in the light of this remark when one defines $R_n(a)$ in terms of (6.1) and (6.3) on the lines of (6.5).

Remark 4. Using **(2**, Theorem 5**)**, it can be seen that $F(t)$ in the denominator of the random variables of (1.1), (6.1), (6.2), (6.3), (6.4), and (6.5) can be replaced by appropriate sample distribution functions, and Theorems 3 and 4 and their versions in the sense of Remarks 1, 2, and 3 remain true.

Remark 5. The referee of this paper has pointed out that the probability distribution function in (2.5) is symmetric in the variables x and y, which fact cannot be seen from the formula given in this paper, and has given the following symmetric form of the function:

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left[-\frac{(2k+1)^2\pi^2T}{2(x+y)^2}\right]\cos\left[\frac{(2k+1)\pi}{2}\frac{x-y}{x+y}\right].$$

This form of the function in (2.5) with $T = b/(1-b)$ gives us a symmetric expression in the variables x and y for the limit distribution function of Theorem 3.

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