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AXIOMS FOR CONVEXITY

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The basic elementary results about convex sets are derived successively from various properties of segments. The complete set of properties is shown to form a natural set of axioms characterising the convex sets in a real vector space.

1. INTRODUCTION

The problem of axiomatising the concept of convexity may be approached from quite different viewpoints. A *convexity space* is frequently defined to be a family $\mathbf{F}$ of subsets of a set X which contains both X itself and the empty set $\emptyset$ and which is closed under arbitrary intersections. In a sense there is no loss of generality in requiring also that any collection of subsets in $\mathbf{F}$ which is totally ordered by inclusion has its union again in $\mathbf{F}$; see Sierksma [**16**]. Such a notion of convexity is clearly so broad that few general statements can be made. However, the value of this approach is that it leads us to ask questions in new situations for which the answers are already known in the standard case of convex sets in real n-dimensional space. The answers in the new situations may be quite different. An attractive introduction to this approach is contained in Jamison-Waldner [**10**].

Alternatively, one may consider the problem of formulating a set of axioms for 'convex' sets which will completely characterise them as the convex sets in a vector space over the real numbers. For this approach see Hammer [**8**] and Whitfield and Yong [**20**].

There is also an approach intermediate between these two. In the standard case a set is convex if and only if it contains the whole segment joining each two of its points. However, the notion of segment arises quite naturally in other situations. For example,

(i) X is a Banach space and, if $x, y \in X$, the segment $[x, y]$ is the set of all $z \in X$ such that

$$|x - z| + |z - y| = |x - y|;$$

(ii) X is a finite connected graph and, if x, y are vertices of X, the segment $[x, y]$ is the set of all vertices of X which lie on shortest paths from x to y;

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(iii) X is a vector space over the field of p-adic numbers and, if $x, y \in X$, the segment $[x, y]$ is the set of all $z \in X$ such that $z = \lambda x + \mu y$, where λ and μ are p-adic integers.

These examples are discussed in Boltyanskii and Soltan [**2**], Soltan [**17**] and Monna [**14**] respectively.

By axiomatising the properties of segments we can obtain a concept of convexity which is independent of linearity and topology, but which resembles the standard case more closely than an arbitrary convexity space. This last approach is treated at length in Prenowitz and Jantosciak [**15**]. However, the primary interest of these authors was in Euclidean geometry, and their axioms are not altogether appropriate for other situations. (They also choose to exclude from a segment its endpoints.)

In the present work the basic elementary results about convex sets in a real vector space are derived successively from certain properties of segments. If these properties are regarded as postulates, then the validity of a result depends only on those postulates which precede its statement. The complete set of postulates is shown to characterise convex sets in a real vector space by deducing all the axioms of Whitfield and Yong [**20**].

The merit of this approach, we believe, is that it exhibits the logical structure of the subject and forces us to formulate definitions and construct proofs in the 'right' way. It may also have some shock-therapeutic value to meet the hyperplane separation theorem for convex sets at the end of the development, instead of at the beginning.

2. Postulates and propositions

Let X be a real vector space and let $[x, y]$ denote the closed straight line segment with endpoints $x, y \in X$. Then, obviously,

P0 *$[x, y]$ is a nonempty subset of X for all $x, y \in X$.*

We can define convexity without any further requirements. A set $C \subseteq X$ is said to be *convex* if $[x, y] \subseteq C$ for all $x, y \in C$. From this definition and P0 we obtain immediately

Proposition 1. *Convex sets have the following properties:*

(i) *the whole space X and the empty set $\emptyset$ are convex sets,*
(ii) *the intersection of any collection of convex sets is again a convex set,*
(iii) *the union of any collection of convex sets which is totally ordered by inclusion is again a convex set.*

For any set $S \subseteq X$, we define the *convex hull* $[S]$ of S to be the intersection of all convex sets which contain S. From this definition and Proposition 1 we can deduce

Proposition 2. *Convex hulls have the following properties:*

(o) $[\emptyset] = \emptyset$,
(i) $S \subseteq [S]$,
(ii) $S \subseteq T$ *implies* $[S] \subseteq [T]$,
(iii) $[[S]] = [S]$,
(iv) *the convex hull of any set is the union of the convex hulls of all its finite subsets.*

Proof: The only property whose derivation is not immediate is (iv). For its proof we refer to Cohn [**3**, Chapter II.1], since the purely set-theoretic ideas are not our primary concern. □

[In the same reference it is shown that, conversely, given any map $\mathbf{C}: S \to [S]$ of the subsets of a set X with the properties in Proposition 2, if we define a set S to be 'convex' when $[S] = S$ then the properties in Proposition 1 hold. Moreover, the correspondence between maps $\mathbf{C}$ with the properties in Proposition 2 and systems of 'convex' sets with the properties in Proposition 1 is bijective.]

Let A be a subset of X and E a subset of A. Then E is said to be an *extreme subset* of A if, for every $B \subseteq A$,

$$[B] \cap E \subseteq [B \cap E]. \tag{1}$$

It is worth noting that (1) holds for every $B \subseteq A$ if (and only if) it holds for every finite $B \subseteq A$. For, if $B \subseteq A$ and $x \in [B] \cap E$, then $x \in [F]$ for some finite set $F \subseteq B$. Hence, by (1) with B replaced by F,

$$x \in [F] \cap E \subseteq [F \cap E] \subseteq [B \cap E].$$

Extreme subsets have the following basic properties:

Proposition 3. *For any set* $A \subseteq X$,

(i) A *and* $\emptyset$ *are extreme subsets of* A,
(ii) *the intersection of any collection of extreme subsets of* A *is again an extreme subset of* A,
(iii) *the union of any collection of extreme subsets of* A *is again an extrene subset of* A,
(iv) *if* E *is an extreme subset of* A *and* E^* *is an extreme subset of* E, *then* E^* *is an extreme subset of* A,
(v) *if* $E \subseteq \tilde{A} \subseteq A$ *and* E *is an extreme subset of* A, *then* E *is an extreme subset of* $\tilde{A}$.

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PROOF: We give the proof of (ii) only, since (i) and (v) follow immediately from the definition, and (iii) and (iv) are easy exercises in the algebra of sets.

Let $\{E_i : i \in I\}$ be any collection of extreme subsets of A and put $E = \bigcap_{i \in I} E_i$. Suppose $B \subseteq A$ and $x \in [B] \cap E$. Then $x \in [F]$ for some finite $F \subseteq B$. Moreover, we may assume that $x \notin [F^*]$ for every proper subset F^* of F. Then, since $x \in [F] \cap E_i \subseteq [F \cap E_i]$, we have $F \cap E_i = F$ for all $i \in I$. Hence $F \subseteq E_i$ for all $i \in I$, and so $F \subseteq E$. Consequently $x \in [F] \subseteq [B \cap E]$. ∎

A point $e \in A$ is said to be an *extreme point* of A if $E = \{e\}$ is an extreme subset of A. The next result shows that testing for extreme points is much simpler than testing for arbitrary extreme subsets.

PROPOSITION 4. *Let $A \subseteq X$ and $e \in A$. Then e is an extreme point of A if and only if $e \notin [A \setminus e]$.*

PROOF: Suppose first that e is an extreme point of A and let $B = A \setminus e$. Then $B \cap e = \emptyset$ and hence, by (1), $e \notin [B]$.

Conversely, suppose that $e \notin [A \setminus e]$ and $B \subseteq A$. If $e \notin B$, then $B \subseteq A \setminus e$ and $[B] \subseteq [A \setminus e]$, hence $e \notin [B]$. Therefore $[B] \cap e \subseteq [B \cap e]$ for $e \notin B$, as well as for $e \in B$. ∎

From Proposition 4 we immediately obtain

PROPOSITION 5. *Let C be a convex set and $e \in C$. Then e is an extreme point of C if and only if $C \setminus e$ is convex.*

We will denote by $\mathrm{E}(A)$ the set of all extreme points of the set A. It can be characterised in the following way:

PROPOSITION 6. *If $A \subseteq X$, then $\mathrm{E}(A)$ is the intersection of all subsets of A which have the same convex hull as A.*

PROOF: It follows at once from (1) that if e is an extreme point of A, and if $B \subseteq A$ with $[B] = [A]$, then $e \in B$.

On the other hand, if $e \in A$ is not an extreme point of A then $e \in [A \setminus e]$, by Proposition 4. Hence $A \subseteq [A \setminus e]$ and $[A] = [A \setminus e]$. But $e \notin A \setminus e$. ∎

The reason for giving P0 its zero rating is that it will now be superseded by the stronger, but equally obvious, property

P1 $\quad x, y \in [x, y] \subseteq X$ *for all* $x, y \in X$.

If $x, y \in X$, the segment $[x, y]$ is characterised analytically as the set of all $z \in X$ which can be represented in the form $z = \lambda x + (1 - \lambda)y$ with $0 \leqslant \lambda \leqslant 1$. We use this characterisation to derive the next property of segments, whose geometrical significance is illustrated in Figure 1.

P2 *If* $z_1 \in [x, y_1]$, $z_2 \in [x, y_2]$ *and* $z \in [z_1, z_2]$, *then* $z \in [x, y]$ *for some* $y \in [y_1, y_2]$.

PROOF: We have $z = \theta z_1 + (1-\theta)z_2$, where

$$z_1 = \lambda x + (1-\lambda)y_1, \quad z_2 = \mu x + (1-\mu)y_2,$$

and $\theta, \lambda, \mu \in [0, 1]$. Evidently we may assume that $\theta \in (0, 1)$ and that at least one of $\lambda, \mu \neq 1$. Then

$$\nu := 1 - \theta\lambda - (1-\theta)\mu \in (0, 1].$$

Moreover $z = (1-\nu)x + \nu y$, where

$$y = \theta(1-\lambda)\nu^{-1}y_1 + (1-\theta)(1-\mu)\nu^{-1}y_2 \in [y_1, y_2].$$

∎

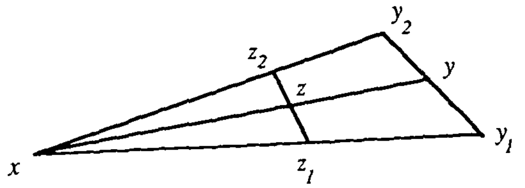

Figure 1

The property of *join-hull commutativity* can now be deduced.

PROPOSITION 7. *For any point* $x \in X$ *and any nonempty set* $S \subseteq X$,

$$[x \cup S] = \bigcup_{y \in [S]} [x, y].$$

PROOF: The right side is certainly contained in the left, since if $y \in [S]$ then $y \in [x \cup S]$, $x \in [x \cup S]$ and hence also $[x, y] \subseteq [x \cup S]$. On the other hand, the right side contains $x \cup S$, since it contains x and $[S]$. Consequently, to show that the right side contains the left we need only show that it is convex. But this follows immediately from P2. ∎

The geometrical significance of the next property of segments is illustrated in Figure 2.

P3 *If* $z_1 \in [x, y_1]$ *and* $z_2 \in [x, y_2]$, *then*

$$[y_1, z_2] \cap [y_2, z_1] \neq \emptyset.$$

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PROOF: For some $\lambda, \mu \in [0, 1]$ we have

$$z_1 = \lambda y_1 + (1-\lambda)x, \quad z_2 = \mu y_2 + (1-\mu)x.$$

If $\lambda\mu < 1$ then, putting $\rho = (1-\lambda\mu)^{-1}$, we have

$$(1-\mu)\lambda\rho y_1 + (1-\lambda)\rho z_2 = (1-\mu)\rho z_1 + (1-\lambda)\mu\rho y_2.$$

If $\lambda\mu = 1$ the result is trivial, since then $\lambda = \mu = 1$ and $z_1 = y_1$, $z_2 = y_2$. ∎

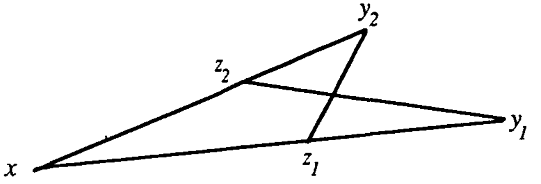

Figure 2

A basic separation property of convex sets can now be deduced.

PROPOSITION 8. *If C and D are disjoint convex subsets of X, then there exist disjoint convex sets C' and D' with $C' \cup D' = X$ such that $C \subseteq C'$, $D \subseteq D'$.*

PROOF: Let $\mathbf{F}$ be the family of all convex sets C'' which contain C but are disjoint from D. Then $\mathbf{F}$ is nonempty, since it contains C. If we partially order $\mathbf{F}$ by inclusion then, by Hausdorff's maximality theorem, $\mathbf{F}$ contains a maximal totally ordered subfamily $\mathbf{F}_0$. The union C' of all the sets in $\mathbf{F}_0$ is again a convex set containing C but disjoint from D.

Since C' is maximal, for every $x \notin C'$ we have

$$[x \cup C'] \cap D \neq \emptyset.$$

We will show that, for every $x \notin C'$,

$$C' \cap [x \cup D] = \emptyset.$$

Assume on the contrary that, for some $x \notin C'$, there exists a point $c' \in C' \cap [x \cup D]$ and let $d'' \in [x \cup C'] \cap D$. By Proposition 7 we have $c' \in [x, d']$ for some $d' \in D$ and $d'' \in [x, c'']$ for some $c'' \in C'$. Hence, by P3,

$$[c'', c'] \cap [d', d''] \neq \emptyset.$$

Since $[c'', c'] \subseteq C'$ and $[d', d''] \subseteq D$, this is a contradiction.

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Consider now the family $\mathbf{G}$ of all convex sets D'' which contain D but are disjoint from C'. Then $\mathbf{G}$ is nonempty and contains a maximal totally ordered subfamily $\mathbf{G}_0$. The union D' of all the sets in $\mathbf{G}_0$ is again a convex set containing D but disjoint from C'. In the same way, for every $y \notin D'$ we have

$$[y \cup D'] \cap C' \neq \emptyset$$

and

$$D' \cap [y \cup C'] = \emptyset.$$

Since C' is maximal, it follows that $y \in C'$. That is, C' is the complement of D'. □

We shall say that a set $H \subseteq X$ is a *hemispace* if both H and $X \setminus H$ are convex. Clearly X itself is a hemispace, and the complement of a hemispace is again a hemispace.

From Proposition 8 we can obtain by induction a separation theorem for any finite number of sets:

PROPOSITION 9. *For any sets $S_1, \ldots, S_m \subseteq X$,*

$$\bigcap_{i=1}^{m} [S_i] = \emptyset \tag{2}$$

if and only if there exist hemispaces $H_1, \ldots, H_m$ such that $S_i \subseteq H_i$ $(i = 1, \ldots, m)$ and $\bigcap_{i=1}^{m} H_i = \emptyset$.

PROOF: The condition is obviously sufficient, since $S_i \subseteq H_i$ implies $[S_i] \subseteq H_i$. Suppose, on the other hand, that (2) holds. We need only consider $m \geqslant 2$, since if $m = 1$ we can take $H_1 = \emptyset = S_1$. Assume that for some $k \in \{1, \ldots, m-1\}$ there exist k hemispaces $H_1, \ldots, H_k$ such that $S_i \subseteq H_i$ $(i = 1, \ldots, k)$ and $\bigcap_{i=1}^{k} H_i$ is disjoint from $\bigcap_{i=k+1}^{m} [S_i]$. For $k = 1$ this is guaranteed by Proposition 8.

The assumption implies that the convex sets $[S_{k+1}]$ and $\left(\bigcap_{i=1}^{k} H_i\right) \cap \left(\bigcap_{i=k+2}^{m} [S_i]\right)$ are disjoint. Hence there exists a hemispace H_{k+1} such that $S_{k+1} \subseteq [S_{k+1}] \subseteq H_{k+1}$ and $\bigcap_{i=1}^{k+1} H_i$ is disjoint from $\bigcap_{i=k+2}^{m} [S_i]$. Thus the assumption continues to hold when k is replaced by $k+1$, and so it holds for $k = m-1$. Now, by Proposition 8 again, there exists a hemispace H_m such that $S_m \subseteq H_m$ and $\bigcap_{i=1}^{m} H_i = \emptyset$. □

To make further progress we introduce another property of segments:

P4 *If $u \in [\nu, x]$ and $\nu \in [u, y]$, where $u \neq \nu$, then $u \in [x, y]$.*

PROOF: We have

$$u = \lambda \nu + (1 - \lambda)x, \qquad \nu = \mu u + (1 - \mu)y,$$

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where $\lambda, \mu \in [0, 1)$. Substituting the second equation in the first we obtain $u = \theta x + (1-\theta)y$, where $\theta = (1-\lambda)(1-\lambda\mu)^{-1} \in [0, 1]$. ∎

Under the same hypotheses P4 itself implies that $\nu \in [y, x]$, as may be seen by interchanging u with ν and x with y.

With the help of this property we can now obtain

PROPOSITION 10. *For any set $S \subseteq X$ and any point $a \notin [S]$, the set $[a \cup S] \setminus a$ is convex.*

PROOF: Since $[a \cup S]$ is convex, we need only show that if $x, y \in [a \cup S]$ and $x, y \neq a$, then $a \notin [x, y]$.

Assume on the contrary that $a \in [x, y]$. By Proposition 7 we have $x \in [a, u]$ and $y \in [a, \nu]$ for some $u, \nu \in [S]$. Hence, by P4, $a \in [y, u]$ and so, by P4 again, $a \in [u, \nu]$. Since $a \notin [S]$, this is a contradiction. ∎

Proposition 10 implies the *anti-exchange* property of convex sets:

PROPOSITION 11. *For any convex set C and any $x, y \notin C$ with $x \neq y$, if $y \in [x \cup C]$ then $x \notin [y \cup C]$.*

PROOF: The set $D := [y \cup C] \setminus y$ is convex, by Proposition 10. Moreover $C \subseteq D$, since $C \subseteq [y \cup C]$ and $y \notin C$. Assume $x \in [y \cup C]$. Then $x \in D$ and hence $[x \cup C] \subseteq D$. Since $y \in [x \cup C]$, but $y \notin D$, this is a contradiction. ∎

We can now also prove

PROPOSITION 12. *A set and its convex hull have the same extreme points.*

PROOF: Let $A \subseteq X$. If e is an extreme point of $[A]$ then $e \in A$, by (1) with $B = A$ and $E = \{e\}$. Hence e is an extreme point of A, by Proposition 3(v).

On the other hand, let e be an extreme point of A. If we put $S = A \setminus e$ then, by Proposition 4, $e \notin [S]$. Therefore, by Proposition 10, $[A] \setminus e$ is convex. Consequently, by Proposition 5, e is an extreme point of $[A]$. ∎

The next property of segments has so far been conspicuous by its absence:

P5 $[x, x] = \{x\}$ *for every* $x \in X$.

It tells us that singletons are convex sets. Moreover it now follows from Proposition 7 that the set $S = \{x, y\}$ has convex hull $[S] = [x, y]$. Thus our notation is consistent and we also obtain

PROPOSITION 13. *$[x, y]$ is a convex set and $[x, y] = [y, x]$ for all $x, y \in X$.*

The definition of a convex set can now be reformulated in the following way: *a set is convex if and only if it contains the convex hull of every pair of its points.* We can also give Proposition 7 a more general, and more symmetrical, form:

PROPOSITION 14. *For any nonempty sets $S, T \subseteq X$,*

$$[S \cup T] = \bigcup_{x \in [S],\, y \in [T]} [x, y].$$

PROOF: By Proposition 2(iv) it is sufficient to prove the result for finite sets S, T. If T is a singleton the result follows from P5 and Proposition 7. We use induction on the cardinality of T and assume that the result holds as written. If $z \notin T$ then, by the induction hypothesis,

$$\begin{aligned}
[S \cup z \cup T] &= \bigcup_{w \in [S \cup z],\, y \in [T]} [w, y] \\
&= \bigcup_{x \in [S],\, y \in [T]} \bigcup_{w \in [x, z]} [w, y] \\
&= \bigcup_{x \in [S],\, y \in [T]} [\{y, x, z\}] \\
&= \bigcup_{x \in [S],\, y \in [T]} \bigcup_{\nu \in [y, z]} [x, \nu] \\
&= \bigcup_{x \in [S],\, \nu \in [z \cup T]} [x, \nu].
\end{aligned}$$

Thus the result holds also when the cardinality of T is increased by 1. ∎

From P4 and P5 we immediately obain

PROPOSITION 15. *If $z \in [y, x]$ and $y \in [z, x]$, then $y = z$.*

In agreement with the usual notation for intervals in $\mathbb{R}$ we set, for any $x, y \in X$,

$$[x, y) = [x, y] \setminus y, \qquad (x, y] = [x, y] \setminus x, \qquad (x, y) = [x, y] \setminus \{x, y\}.$$

PROPOSITION 16. *$(a, b]$ and (a, b) are convex sets, for all $a, b \in X$.*

PROOF: We may assume $a \neq b$, by P5. Then the convexity of $(a, b]$ follows from Proposition 10 with $S = \{b\}$.

Suppose $x, y \in (a, b)$. Then $a \notin [y, b]$, by Proposition 15. Since $a \in [x, y]$ would imply $a \in [y, b]$, by P4, it follows that $a \notin [x, y]$. Similarly $b \notin [x, y]$. Hence $[x, y] \subseteq (a, b)$. This establishes the convexity of (a, b). ∎

The defining property of extreme subsets can now be given a simpler, and more familiar, form:

PROPOSITION 17. *A subset E of a set $A \subseteq X$ is extreme if and only if $x, y \in A$ and $(x, y) \cap E \neq \emptyset$ together imply $x, y \in E$.*

PROOF: On account of P5, the relation (1) always holds if B is a singleton. If $B = \{x, y\}$, where $x \neq y$, then (1) is equivalent to "$(x, y) \cap E \neq \emptyset$ implies $x, y \in E$".

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Hence, to complete the proof we assume this property and show that if (1) holds for a finite set B and if $x \in A \setminus B$, then

$$[x \cup B] \cap E \subseteq [(x \cup B) \cap E].$$

By Proposition 7,

$$[x \cup B] \cap E = \bigcup_{y \in [B]} [x, y] \cap E.$$

If $x \notin E$, then $[x, y] \cap E = \{y\} \cap E$ and hence

$$[x \cup B] \cap E = [B] \cap E \subseteq [B \cap E] = [(x \cup B) \cap E].$$

On the other hand, if $x \in E$ then

$$\begin{aligned} [x \cup B] \cap E &= \bigcup_{y \in [B] \cap E} [x, y] \\ &\subseteq \bigcup_{y \in [B \cap E]} [x, y] \\ &= [x \cup (B \cap E)] \\ &= [(x \cup B) \cap E]. \end{aligned}$$

$\square$

The linear nature of segments is brought out in the next property:

P6 *If $z \in [x, y]$, then $[x, y] = [x, z] \cup [y, z]$.*

As a consequence of this property we have the following counterpart:

Proposition 18. *If $z \in [x, y]$, then $[x, z] \cap [y, z] = \{z\}$.*

Proof: Suppose $w \in [x, z] \cap [y, z]$. Then $w \in [x, y]$, by Proposition 13, and hence, by P6, either $z \in [x, w]$ or $z \in [y, w]$. In both cases Proposition 15 implies that $w = z$. $\square$

It will now be shown that the points of any segment can be totally ordered. If $c \in [a, d]$ we write $c \leqslant_a d$, or simply $c \leqslant d$ if there is no danger of confusion. Then $c \leqslant c$, by P1. If $c \leqslant d$ and $d \leqslant c$ then $c = d$, by Proposition 15. If $c \leqslant d$ and $d \leqslant e$, then $[a, d] \subseteq [a, e]$ by Proposition 13, and hence $c \leqslant e$. Finally, if $c, d \in [a, b]$ then either $c \leqslant d$ or $d \leqslant c$. For if $c \notin [a, d]$ then $c \in [b, d]$, by P6, therefore $d \notin [b, c]$ by Proposition 15, and hence $d \in [a, c]$ by P6.

Proposition 19. *If $c \in [a, b]$ and $d \in [a, c]$, then $c \in [b, d]$.*

Proof: by P1 we may assume $c \neq d$. Then $c \notin [a, d]$, by Proposition 15. Since $d \in [a, b]$, by Proposition 13, it now follows from P6 that $c \in [b, d]$. $\square$

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Another simple property of segments which we require is

P7 *If* $[x, y] \cap [x, z] \neq \{x\}$, *then* $y, z \in [x, w]$ *for some* w.

It follows that we can choose $w \in \{y, z\}$, as we now show.

PROPOSITION 20. *If* $[x, y] \cap [x, z] \neq \{x\}$, *then either* $z \in [x, y]$ *or* $y \in [x, z]$.

PROOF: We may assume $y \neq z$. Choose w as in P7. If $z \notin [x, y]$ then, by P6, $z \in [w, y]$. Similarly, if $y \notin [x, z]$ then $y \in [w, z]$. The result now follows from Proposition 15. $\square$

The next result is one of the axioms of Whitfield and Yong [20].

PROPOSITION 21. *The union of two segments with more than one common point is again a segment.*

PROOF: Let $[a, b]$ and $[c, d]$ be two segments whose intersection contains the distinct points x, y. To prove that $[a, b] \cup [c, d]$ is a segment we may obviously assume that neither segment is contained in the other.

By P6 we may suppose the notation chosen so that $y \in [x, d]$ and $x \in [y, a]$. Then, by P4 and Proposition 13, $x, y \in [a, d]$. Since $y \notin [x, a]$, by Proposition 15, it follows from P6 that $y \in [x, b]$. Hence, by Proposition 20, either $b \in [x, d]$ or $d \in [x, b]$. Similarly, since $x \notin [y, d]$, we have $x \in [y, c]$ and hence either $a \in [y, c]$ or $c \in [y, a]$.

If $c \in [y, a]$ then $b \in [x, d]$, since $d \in [x, b]$ would imply $[c, d] \subseteq [a, b]$. Hence $b, c \in [a, d]$ and $b \in [c, d]$. It follows that $[a, b] \subseteq [a, d]$, $[c, d] \subseteq [a, d]$, and

$$\begin{aligned} [a, d] &= [a, b] \cup [b, d] \\ &\subseteq [a, b] \cup [c, d]. \end{aligned}$$

Consequently $[a, b] \cup [c, d] = [a, d]$.

If $a \in [y, c]$ then $d \in [x, b]$ and a similar argument applies. $\square$

If a and b are distinct points, we define the *line* $\langle a, b\rangle$ to be the set of all points c such that either $c \in [a, b]$ or $a \in [b, c]$ or $b \in [c, a]$. Hence $\langle a, b\rangle = \langle b, a\rangle$ and $[a, b] \subseteq \langle a, b\rangle$. In particular, $a, b \in \langle a, b\rangle$.

It follows at once from the definition of a line that if a, b, c are distinct points such that $c \in \langle a, b\rangle$, then also $a \in \langle b, c\rangle$ and $b \in \langle c, a\rangle$. Thus the property depends only on the triple $\{a, b, c\}$ and we say that the three points are *collinear*. From P4, P6 and Propositions 19, 20 we obtain, simply by enumeration of cases,

PROPOSITION 22. *If* $\{a, b, c\}$ *and* $\{a, b, d\}$ *are collinear triples, and if* $c \neq d$, *then* $\{b, c, d\}$ *is also a collinear triple.*

PROPOSITION 23. *Suppose there exist three points* a, b, c *which are not collinear. Then for any distinct points* x, y *there exists a point* $z \neq x, y$ *such that* x, y, z *are not collinear.*

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PROOF: If $x \in \langle a, b\rangle$, $y \notin \langle a, b\rangle$ we can take $z = b$ if $x = a$ and $z = a$ if $x \neq a$, since then $\langle x, z\rangle = \langle a, b\rangle$. Also, it follows from Proposition 22 that if $x \in \langle a, b\rangle$, $y \in \langle a, b\rangle$ we can take $z = c$. Thus we may now assume that $x, y \notin \langle a, b\rangle$. If $a \notin \langle x, y\rangle$ we can take $z = a$. If a, x, y are collinear, then b, x, y are not collinear and we can take $z = b$. ∎

A set $L \subseteq X$ is said to be *linear* if $\langle x, y\rangle \subseteq L$ for all distinct points $x, y \in L$. It follows at once from this definition that singletons are linear sets and that Proposition 1 continues to hold if throughout its statement 'convex' is replaced by 'linear'.

Similarly, we define the *linear hull* $\langle S\rangle$ of a set $S \subseteq X$ to be the intersection of all linear sets which contain S. Then Proposition 2 continues to hold if throughout its statement 'convex' is replaced by 'linear' and '[ ]' by '$\langle\ \rangle$'. Again our notation is consistent, since the line $\langle x, y\rangle$ is the linear hull of the set $\{x, y\}$. [Note also that if $x, y \in X$, the line $\langle x, y\rangle$ is the set of all $z \in X$ which can be represented in the form $z = \lambda x + (1 - \lambda)y$ with $\lambda \in \mathbb{R}$.]

A linear set L is said to be a *hyperplane* if $L \neq X$ and if the only linear sets which contain L are L itself and X. Equivalently, a linear set $L \neq X$ is a hyperplane if $\langle x \cup L\rangle = X$ for every $x \in X \setminus L$.

Some notions of a topological nature will now be introduced. We define the *intrinsic interior* C^i of a convex set C to be the set of all $x \in C$ such that, for every $y \in C \setminus x$, there exists some $z \in C$ for which $x \in (y, z)$.

Thus $C^i \subseteq C$, and $C^i = C$ if C contains at most one point.

PROPOSITION 24. *Let C be a convex set and C^i its intrinsic interior. If $x \in C^i$ and $y \in C$, then $[x, y) \subseteq C^i$. In particular, C^i is also a convex set.*

PROOF: Suppose $z \in (x, y)$. We wish to show that, for any $u \in C \setminus z$, there exists some $w \in C \setminus z$ such that $z \in [u, w]$. Hence we may suppose that $u \neq x, y$ and that $z \notin [u, y]$. For some $\nu \in C \setminus x$ we have $x \in [u, \nu]$. Then, by P2, $z \in [u, w]$ for some $w \in [\nu, y]$. If $w \neq z$ there is nothing more to do. Hence we may suppose $z \in [\nu, y]$. Since $z \notin [u, y]$, it follows from P4 that $z \neq \nu$.

If $\nu \in [x, y]$ then $x \in [u, y]$, by P4, and hence $z \in [x, y] \subseteq [u, y]$, contrary to hypothesis. Therefore $\nu \notin [x, y]$. We may suppose also $y \notin [u, \nu]$, since $y \in [u, \nu]$ implies $z \in [x, y] \subseteq [u, \nu]$.

Since $z \in [x, y] \cap [\nu, y]$ and $\nu \notin [x, y]$, it follows from Proposition 20 that $x \in [\nu, y]$. Thus $x \in [\nu, y] \cap [u, \nu]$. Since $y \notin [u, \nu]$, it follows from Proposition 20 again that $u \in [\nu, y]$. Hence, by P6, $[\nu, y] = [\nu, u] \cup [u, y]$. Since $z \notin [u, y]$, it follows that $z \in [u, \nu]$. ∎

We further define the *convex closure* $\overline{C}$ of a convex set C to be C itself if $C^i = \emptyset$ and otherwise to be the set of all $y \in X$ such that $[x, y) \subseteq C^i$ for every $x \in C^i$. In

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either event $C \subseteq \overline{C}$, by Proposition 24.

PROPOSITION 25. *If C is a convex set, then $\overline{C}$ is also a convex set.*

PROOF: Obviously we may assume that $C^i \neq \emptyset$. Suppose $x, y \in \overline{C}$, where $x \neq y$. We wish to show that if $z \in (x,\, y)$, then also $z \in \overline{C}$. Thus we may assume that $z \notin C$.

Let $c \in C^i$ and $w \in (c,\, z)$. Suppose $w \in [x,\, y]$. Then either $z \in [x,\, w]$ or $z \in [y,\, w]$, by P6. If $z \in [x,\, w]$, then $z \in [c,\, x]$ by P4. Since $z \neq x$, it follows that $z \in C^i$, which is contrary to assumption. If $z \in [y,\, w]$ we obtain a contradiction similarly.

Therefore $w \notin [x,\, y]$. On the other hand, by P2, $w \in [d,\, y]$ for some $d \in [c,\, x]$. Since necessarily $d \neq x$, it follows that $d \in C^i$ and hence $w \in C^i$. This proves that $z \in \overline{C}$, and thus that $\overline{C}$ is convex. ∎

To obtain further results of this nature an additional property of segments must be invoked:

P8 *If $x \neq y$, then $(x,\, y) \neq \emptyset$.*

PROOF: Put $z = (1/2)x + (1/2)y$. Then $z \in [x,\, y]$ and $z \neq x, y$. ∎

This property is used in the proof of the next two results.

PROPOSITION 26. *If C is a convex set, then $\overline{C}^i = C^i$ and $\overline{C} = \overline{\overline{C}}$.*

PROOF: Obviously we may assume that $\overline{C} \neq C$, and then $C^i \neq \emptyset$. Suppose $x \in \overline{C}^i$. Then for every $y \in \overline{C} \setminus x$ there exists $z \in \overline{C}$ such that $x \in (y,\, z)$. In particular this holds for every $y \in C^i \setminus x$ and then $(y,\, z) \subseteq C^i$, by the definition of $\overline{C}$. Thus $x \in C^i$ and $\overline{C}^i \subseteq C^i$.

Suppose, on the other hand, that $x \in C^i$ and $y \in \overline{C} \setminus x$. Then $(x,\, y) \subseteq C^i$ and, by P8, there exists some $w \in (x,\, y)$. By the definition of C^i there exists $z \in C$ such that $x \in (w,\, z)$. Then $x \in (y,\, z)$, by P4. Thus $x \in \overline{C}^i$ and $C^i \subseteq \overline{C}^i$.

This proves the first assertion of the proposition. To prove the second we need only show that $\overline{\overline{C}} \subseteq \overline{C}$, since the reverse inclusion is trivial. Suppose $x \in \overline{\overline{C}}$. Then $(x,\, y] \subseteq \overline{C}^i$ for every $y \in \overline{C}^i$. Since $\overline{C}^i = C^i$, by what we have just proved, this implies $x \in \overline{C}$. ∎

PROPOSITION 27. *If C is a convex set, then $\left(C^i\right)^i = C^i$. Moreover, if $C^i \neq \emptyset$ then $\overline{C^i} = \overline{C}$.*

PROOF: To prove the first assertion we may assume that C^i contains more than one point. Suppose $x, y \in C^i$, where $x \neq y$. Then there exists $z \in C$ such that $x \in (y,\, z)$. Moreover, by P8, there exists some $w \in (x,\, z)$. Then $w \in C^i$, by Proposition 24, and $x \in [y,\, w]$, by Proposition 19. Hence $w \neq y$ and $x \in (y,\, w)$. Thus $x \in C^{ii}$ and $C^{ii} = C^i$.

Since $C^i \neq \emptyset$, $u \in \overline{C}$ if and only if $(u, v] \subseteq C^i$ for every $v \in C^i$. Since $C^{ii} = C^i$, $u \in \overline{C^i}$ if and only if the same condition is satisfied. $\square$

It is of interest that the postulate P3 is actually a consequence of the other postulates which have by now been introduced. In establishing this we are free to use any of the preceding propositions except Propositions 8 and 9, since an examination of our discussion reveals that P3 played no direct or indirect role in the proofs of the other propositions.

Proposition 28. *P1-2 and P4-8 imply P3.*

Proof: It is sufficient to show that $[y_1, z_2] \cap [y_2, z_1] \neq \emptyset$ if $z_1 \in (x, y_1)$, $z_2 \in (x, y_2)$ and $z_1 \neq z_2$. Moreover, we may assume that $z_1 \notin [y_1, z_2]$ and $z_2 \notin [y_2, z_1]$. Then, by Proposition 19, $z_2 \notin [x, z_1]$ and $z_1 \notin [x, z_2]$. Furthermore $x \notin [z_1, z_2]$, by P4. Consequently, by Proposition 20, $[x, z_1] \cap [z_1, z_2] = \{z_1\}$ and $[x, z_2] \cap [z_1, z_2] = \{z_2\}$.

By P8 we can choose $w \in (z_1, z_2)$. (This is the only place where we use P8.) Thus $w \neq x$. By P1 and P2, $w \in [x, u]$ for some $u \in [y_2, z_1]$ and also $w \in [x, v]$ for some $v \in [y_1, z_2]$. Hence, by Proposition 20, either $u \in [x, v]$ or $v \in [x, u]$. Without loss of generality assume $u \in [x, v]$. Then, by Proposition 19, $u \in [v, w]$.

Thus $u \in [\{y_1, z_2, z_1\}]$ and hence $u \in [z_1, t]$ for some $t \in [y_1, z_2]$. If $t \in [y_2, z_1]$ we are finished. We will assume $t \notin [y_2, z_1]$ and derive a contradiction. Since $u = z_1$ would imply $w \in [x, z_1] \cap (z_1, z_2)$, we must have $u \neq z_1$. Since $u \in [z_1, t] \cap [z_1, y_2]$, it follows from Proposition 20 that $y_2 \in [z_1, t]$. But, since $[y_1, z_2] \subseteq [\{y_1, x, y_2\}]$, we have also $t \in [y_2, s]$ for some $s \in [x, y_1]$. Hence, by P4, $y_2 \in [z_1, s]$. Since $[z_1, s] \subseteq [x, y_1]$, by Proposition 13, it follows that $y_2 \in [x, y_1]$ and so $z_2 \in [x, y_1]$. Since $z_1 \notin [y_1, z_2]$, we obtain from P6 that $z_1 \in [x, z_2]$. It now follows from Proposition 19 that $z_2 \in [y_2, z_1]$. This is the desired contradiction. $\square$

Figure 3 may provide assistance in understanding the motivation for this proof.

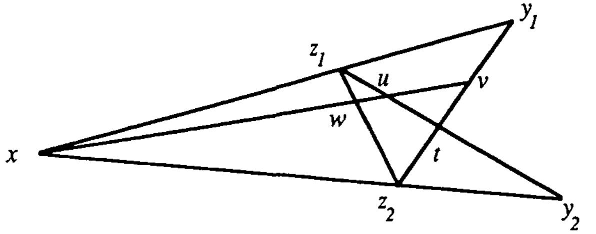

Figure 3

The next property of segments is a counterpart to P8:

P9 *If $x \neq y$, then there exists a point z such that $x \in (y, z)$.*

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Actually, as we now show, it virtually supersedes P8.

Proposition 29. *Suppose there exist three points which are not collinear. Then P1-2, P4-7 and P9 together imply P8 (and hence also P3, by Proposition 28).*

Proof: Let y_1, y_2 by any two distinct points. By Proposition 23 we can choose $z \neq y_1, y_2$ so that y_1, y_2, z are not collinear. By P9 we can now choose z_1 so that $z \in (y_2, z_1)$ and then x so that $z_1 \in (x, y_1)$. It follows from P2, with $z_2 = y_2$, that $z \in [x, y]$ for some $y \in [y_1, y_2]$. Hence we need only show that $y \neq y_1, y_2$.

Since y_2, z, z_1 are collinear but y_1, y_2, z are not, it follows from Proposition 22 that z, z_1, y_1 are not collinear. Hence $z \neq x$. Moreover, since x, z_1, y_1 are collinear, it follows that x, z, y_1 are not collinear. Hence $y \neq y_1$. Similarly it may be shown that $y \neq y_2$. ∎

We use P9 in the proof of the next two results.

Proposition 30. *If C is a hemispace, then C^i is also a hemispace. Moreover,*

$$\overline{X \setminus C^i} = X \setminus C^i.$$

Proof: Obviously we may assume that $C^i \neq \emptyset, X$. Let $D = X \setminus C$ and $E = X \setminus C^i$. Suppose $x, y \in E$ and $z \in (x, y)$. We wish to show that also $z \in E$.

Assume on the contrary that $z \in C^i$. By P9 we can choose u so that $x \in (u, z)$ and ν so that $y \in (z, \nu)$. Then $u \in D$, since $u \in C$ would imply $x \in C^i$, and similarly $\nu \in D$. But, by P4, $x, y \in (u, \nu)$. Since D is convex, it follows that $x, y \in D$ and hence also $z \in D$. But this is a contradiction.

Thus C^i is a hemispace. To prove that also $\overline{E} = E$ we may assume that $E^i \neq \emptyset$ and $C^i \neq \emptyset$. Let $a \in E^i$ and suppose there exists a point $b \in \overline{E} \setminus E$. By P9 there exists a point c such that $b \in (a, c)$ and also a point d such that $a \in (b, d)$. Then $c \in C^i$, since $c \in E$ would imply $b \in E^i$. Similarly $d \in D \subseteq E$, because $b \in C^i$ and so $d \in C$ would imply $a \in C^i$. Since $a \in E^i$, there exists $e \in E$ such that $a \in (d, e)$. Since $b \in [d, e]$ would imply $b \in E$, by the convexity of E, we must have $e \in (b, d)$ and actually $e \in (a, b)$. Then $b \in (e, c)$, by Proposition 19. On the other hand, since $b \in C^i$ there exists $f \in C$ such that $b \in (c, f)$. Moreover $e \notin (c, f)$, since $e \in (c, f)$ would imply $e \in C^i$. Hence, by Proposition 20, $f \in [c, e]$ and actually $f \in (b, e]$. Thus $f \in (a, b)$ and hence $f \in E^i$. Consequently there exists $g \in E$ such that $f \in (a, g)$. Since $b \in (a, g)$ would imply $b \in E^i$, we must have $g \in (a, b)$ and actually $g \in (b, f)$. Since $f \in E^i$ and $b \in \overline{E}$, this implies $g \in E^i$. On the other hand, since $f \in C$ and $b \in C^i$, it implies $g \in C^i$. Thus we have a contradiction. ∎

Proposition 31. *If C is a nonempty hemispace such that $C = C^i$, and if $D = X \setminus C$, then $\overline{D} = D$ and $\overline{C} \setminus C = D \setminus D^i$.*

PROOF: The relation $\overline{D} = D$ follows at once from Proposition 30. To prove the other relation we show first that $\overline{C} \cap D^i = \emptyset$.

Assume, on the contrary, that there exists a point $x \in \overline{C} \cap D^i$ and choose some $y \in C$. By P9 there exists a point z such that $x \in (y, z)$. Moreover $z \in D$, since $z \in C$ would imply $x \in C$. Since $x \in D^i$, there exists a point $w \in D$ such that $x \in (z, w)$. But $y \notin [w, z]$, since $y \in C$. Hence $w \in (y, z)$ and actually $w \in (x, y)$. Since $x \in \overline{C}$ and $C^i = C$, it follows that $w \in C$, which is a contradiction.

Hence $\overline{C} \setminus C \subseteq D \setminus D^i$. To complete the proof we must show that $D \setminus D^i \subseteq \overline{C}$. Choose any $a \in D \setminus D^i$. Then, for some $b \in D$, $a \in (b, c)$ implies $c \in C$. Since $C = C^i$, we need only show that $(a, d) \subseteq C$ for any $d \in C$ not collinear with a, b. Fix some c such that $a \in (b, c)$ and choose $e \in C$ so that $d \in (c, e)$. Then, by P2, for any $f \in (a, d)$ there exists $g \in [a, c]$ such that $f \in [e, g]$. Moreover $g \neq a$, since a, c, d are not collinear. Consequently $g \in C$ and hence $f \in C$. ∎

Let C be a nonempty convex set. A point $c_0 \in C$ will be said to be an *X-interior point* if for any $x \in X \setminus c_0$ there exist $c_1, c_2 \in C$ with $c_0 \in (c_1, x)$ and $c_0 \in (c_1, c_2)$.

PROPOSITION 32. *Let C be a nonempty convex set such that $C = C^i$. If one point of C is an X-interior point, then every point of C is an X-interior point.*

PROOF: By hypothesis there exists a point $c_0 \in C$ such that, for any $x \in X \setminus c_0$ there exist $c_1, c_2 \in C$ with $c_0 \in (c_1, x)$ and $c_0 \in (c_1, c_2)$. We wish to show that any $b_0 \in C \setminus c_0$ has a corresponding property. If b_0 is collinear with c_0, x this follows directly from the fact that $C = C^i$. Thus we may assume that b_0 is not collinear with c_0, x. Again since $C = C^i$, there exists a point $c \in C$ such that $b_0 \in (c_1, c)$. By P3 there exists a point $b_2 \in [c_0, c] \cap [x, b_0]$. In fact $b_2 \in (x, b_0)$, by Proposition 22. Since $b_2 \in C$, there exists a point $b_1 \in C$ such that $b_0 \in (b_1, b_2)$. Since also $b_0 \in (b_1, x)$, by P4, this is what we wanted to show. ∎

We define a nonempty convex set C to be a *convex body* if every point of C is an X-interior point. In order to prove the separation theorem for convex bodies we introduce our final property of segments.

P10 *If C is a convex subset of $[x, y]$ such that $x \in C$, $y \notin C$, then there exists a point $z \in [x, y]$ such that $[x, z) \subseteq C$ and $(z, y] \subseteq X \setminus C$.*

PROOF: This follows at once from the Dedekind cut property for $\mathbb{R}$. ∎

PROPOSITION 33. *If A_1, A_2 are disjoint convex bodies, then X is the union of pairwise disjoint nonempty sets H, C_1, C_2, where H is a hyperplane and C_1, C_2 are hemispaces with $C_1 = C_1^i$, $C_2 = C_2^i$ such that $A_1 \subseteq C_1$, $A_2 \subseteq C_2$.*

PROOF: By Proposition 8 there exists a hemispace C such that $A_1 \subseteq C$, $A_2 \subseteq D = X \setminus C$. In the present case $A_1 \subseteq C^i$, since A_1 is a convex body. Consequently, by Proposition 30, we may assume that $C = C^i$ and $\overline{D} = D$. Since A_2 is a convex body, we now have $A_2 \subseteq D^i$. If we put $H = D \setminus D^i$, then X is the disjoint union of the set H and the hemispaces $C_1 = C$, $C_2 = D^i$. Moreover, $C_1 = C_1^i$ and $C_2 = C_2^i$, by Proposition 27. The set $H = D \cap (X \setminus D^i)$ is convex, since D^i is a hemispace. Hence to prove that H is actually a linear set it is enough to show that $x, y \in H$ and $x \in (y, z)$ together imply $z \in H$. Since $y \in \overline{C}$, by Proposition 31, $z \in C$ would imply $x \in C$. Therefore $z \in D$. Since $z \in D^i$ would imply $x \in D^i$, we actually have $z \in H$.

Suppose $b \in C_1$ and $c \in C_2$. Then by P10, the segment $[b, c]$ contains a point d such that $[b, d) \subseteq C_1$ and $(d, c] \subseteq C_2$. Then $d \notin C_1$, since $C_1 = C_1^i$, and similarly $d \notin C_2$. Hence $d \in H$. This proves, in particular, that $H \neq \emptyset$.

It remains to show that H is a hyperplane. If $b \in X \setminus H$, then $b \in C_1 \cup C_2$. Without loss of generality assume $b \in C_1$. We wish to show that $\langle b \cup H \rangle = X$. Let $a \in H$ and choose c so that $a \in (b, c)$. Then $c \in C_2$, since H is linear and C_1 is convex. For any $c_1 \in C_1$ the segment $[c, c_1]$ contains a point $d \in H$. Consequently $C_1 \subseteq \langle b \cup H \rangle$, and similarly $C_2 \subseteq \langle b \cup H \rangle$. Thus $\langle b \cup H \rangle = X$. $\square$

A well-known example which illustrates that Proposition 33 can fail to hold without P10 is obtained by taking X to be the field $\mathbb{Q}$ of rational numbers, A_1 to be the set of all $x \in \mathbb{Q}$ with $x^2 > 2$ and $x > 0$ and $A_2 = X \setminus A_1$.

All the axioms of Whitfield and Yong [20] are now immediate consequences of our postulates or of propositions which have already been established. Consequently, by their main result, our postulates characterise X as a linearly open convex subset of a real vector space, provided X has dimension greater than 2. (The restriction on the dimension is necessitated by the existence of non-desarguesian planes.) Without giving any general definition of dimension, we can formulate this in the following way:

Proposition 34. *Let X be a set with subsets $[x, y]$, defined for all $x, y \in X$, possessing the properties P1-2, P4-7 and P9-10. Assume, in addition, that there exists a set $S \subseteq X$ containing four points such that some point of the convex hull of S is not contained in the convex hull of any proper subset of S.*

Then X can be identified with a convex subset X^* of a real vector space V, whose intersection with every line in V is an open segment in V, so that the subset $[x, y]$ coincides with the closed segment with endpoints x, y.

Proof: We merely point out that the additional assumption implies that there exist three distinct points which are not collinear since, by Proposition 13, no point of S is contained in a segment whose endpoints are two other points of S. Hence P8 holds, by Proposition 29. $\square$

Whitfield and Yong give examples to show that each of their axioms is independent of the remaining axioms. Since the example which claims to show this for the axiom REG(i), that is, our P8, is two-dimensional, this contradicts Proposition 29. However, it is easily seen that in fact this example also fails to satisfy the axiom JHC (and our P2). Whitfield and Yong also do not mention that the axiom REG(i) is implied by the other hypotheses of their main result.

3. Concluding remarks

Our purpose has been to present the basic properties of convex sets, as expounded in Eggleston [5], Valentine [19] or Lay [12], in a logical order under minimal hypotheses. It would be impossible to do this in a coherent manner without repeating results which are already known, in some cases in the same generality as here. To provide guidance for the uninitiated, and to avoid criticism from the initiated, we now give some additional references.

Our discussion of extreme subsets owes much to Lassak [11]. Proposition 8 was already proved by Ellis [6]. Proposition 9 is due to Martinez-Legaz and Singer [13, p.177]. It is included here, although we make no later use of it, because it may be regarded as the abstract basis for the method of Lagrange multipliers; see Boltyanskii [1]. The anti-exchange property for finite convex geometries, or antimatroids, is emphasised in Edelman and Jamison [4]. What we have called the intrinsic interior of a convex set is also known as its intrinsic core; our non-topological definition of the convex closure of a convex set seems to be new.

The problem of axiomatising geometry has an extensive literature, beginning with Euclid [7]. Besides the classic work of Hilbert [9], we mention the more recent book by Vaisman [18]. Whether one set of axioms is regarded as 'better' than another will depend on one's interests. Part of the motivation for our work has been the feeling that convex geometry is more primitive than affine geometry since segments, unlike lines, are bounded objects. In our view also the postulates P2 and P10 appear more naturally here than their counterparts, Pasch's axiom and the continuity axiom, in the traditional approach.

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