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A Note on Noncommutative Interpolation

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Abstract. In this paper we formulate and solve Nevanlinna-Pick and Carathéodory type problems for tensor algebras with data given on the N-dimensional operator unit ball of a Hilbert space. We develop an approach based on the displacement structure theory.

1 Introduction

Interpolation problems for bounded analytic functions were studied quite intensively due to their many applications, for instance to the wave propagation in layered media, circuit synthesis, and robust control. On the mathematical side, they are strongly related to dilation theory and selfadjoint extensions of symmetric operators. Accordingly, there was some interest in generalizing the framework in which similar problems could be formulated. In this paper we deal with interpolation problems for tensor algebras. Problems of this type were already considered in the literature [2], [9], [13]. Several methods were developed in these and other papers, each of them being of a specific interest. Here we consider similar formulations for data on the N-dimensional unit ball

$$\mathcal{B}_N(\mathcal{E}) = \left\{ Z = \begin{bmatrix} Z_1 & \cdots & Z_N \end{bmatrix} \in \mathcal{L}(\mathcal{E})^N \;\middle|\; \sum_{k=1}^{N} Z_k^* Z_k < I_{\mathcal{E}} \right\},$$

where $\mathcal{E}$ is a Hilbert space. When considering this framework, we need to introduce an evaluation of an element of the tensor algebra at a point of $\mathcal{B}_N(\mathcal{E})$, as well as appropriate derivations on the algebra. The goal of this paper is to define all these elements and formulate and solve Nevanlinna-Pick and Carathéodory type problems on $\mathcal{B}_N(\mathcal{E})$. All of these are done by using the approach based on the displacement structure theory, as suggested in [7]. This approach turns out to be quite elementary and has other benefits, some of which are presented in our companion paper [6].

The paper is organized as follows. In Section 2 we introduce a Szegö type kernel for $\mathcal{B}_N(\mathcal{E})$ and some additional notation. Section 3 contains the description of our approach to interpolation on $\mathcal{B}_N(\mathcal{E})$ which is then illustrated by an application to a Nevanlinna-Pick type problem. The last section introduces some natural derivations on the tensor algebra and the formulation and solution of a Carathéodory type problem. Another consequence of this paper is that noncommutative interpolation problems can be dealt with as special interpolation problems in the upper triangular algebra.

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2 Szegö Kernels

Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{L}(\mathcal{E})$ denote the set of all bounded linear operators on $\mathcal{E}$. If N is a positive integer and $Z = \begin{bmatrix} Z_1 & \cdots & Z_N \end{bmatrix}$, $W = \begin{bmatrix} W_1 & \cdots & W_N \end{bmatrix}$ are two elements in $\mathcal{L}(\mathcal{E})^N$, then we define

$$(Z|W) = \sum_{k=1}^{N} Z_k^* W_k,$$

and

$$\mathcal{B}_N(\mathcal{E}) = \left\{ Z = \begin{bmatrix} Z_1 & \cdots & Z_N \end{bmatrix} \in \mathcal{L}(\mathcal{E})^N \;\middle|\; (Z|Z) < I_{\mathcal{E}} \right\},$$

where $I_{\mathcal{E}}$ denotes the identity operator on $\mathcal{E}$.

We introduce a Szegö type kernel on $\mathcal{B}_N(\mathcal{E})$ by using some simple ideas from displacement structure theory (and which, in the case $N = 1$ and $\mathcal{E} = \mathbb{C}$, would give the classical Szegö kernel $K(z, w) = \frac{1}{1-\bar{z}w}$). Let $Z_1, \ldots, Z_n$ be elements in $\mathcal{B}_N(\mathcal{E})$ and consider

$$F_k = \bigoplus_{l=1}^{n} Z_{l,k}^*, \quad k = 1, \ldots N, \tag{2.1}$$

the diagonal matrix with the diagonal made of the k-th components of $Z_1^*, \ldots, Z_n^*$. Also, define

$$U = [\underbrace{I_{\mathcal{E}} \quad \cdots \quad I_{\mathcal{E}}}_{n \text{ terms}}]^*. \tag{2.2}$$

It is easily seen that the so-called *displacement equation*

$$A - \sum_{k=1}^{N} F_k A F_k^* = UU^*, \tag{2.3}$$

admits a unique positive solution A. In fact, it is simple to write the explicit form of the solution. Thus, let $\mathbb{F}_N^+$ be the unital free semigroup with N generators $1, \ldots, N$. The empty word is the identity element of $\mathbb{F}_N^+$. The length of the word σ is denoted by $|\sigma|$ and we consider the lexicographic order on $\mathbb{F}_N^+$. We associate new Hilbert spaces to a Hilbert space $\mathcal{E}$ by the following recursion: $\mathcal{E}_0 = \mathcal{E}$ and for $k \geq 1$,

$$\mathcal{E}_k = \underbrace{\mathcal{E}_{k-1} \oplus \cdots \oplus \mathcal{E}_{k-1}}_{N \text{ terms}} = \mathcal{E}_{k-1}^{\oplus N}. \tag{2.4}$$

Then $U_k = [F_\sigma U]_{|\sigma|=k}$ gives a bounded operator from $\mathcal{E}_k$ into $\mathcal{E}$, where F_σ is a notation for the operator $F_{i_1} \cdots F_{i_k}$ provided that $\sigma = i_1 \cdots i_k$ (we set $F_\varnothing = I_{\mathcal{E}}$). One easily checks that $U_\infty^* = [U_k]_{k=0}^\infty$ is a bounded operator from $\bigoplus_{k=0}^\infty \mathcal{E}_k$ into $\mathcal{E}$ and the solution of (2.3) is given by the formula $A = U_\infty^* U_\infty$. We introduce the notation $Z_\sigma^* = Z_{i_1}^* \cdots Z_{i_k}^*$ for $\sigma = i_1 \cdots i_k$; Z_σ^* should be distinguished from $(Z_\sigma)^*$, the adjoint of Z_σ. Also define $L(Z) = [Z_\sigma^*]_{|\sigma|=0}^\infty$ for Z in $\mathcal{B}_N(\mathcal{E})$. Then the solution of (2.3) can be written in the form

$$A = [L(Z_j)L(Z_k)^*]_{j,k=1}^n.$$

It was suggested by one referee to mention that this formula is just a form of a Neumann series: the map $\Phi(A) = \sum_{k=1}^N F_k A F_k^*$ is a completely positive map with $\Phi(I) < I$, so $\|\Phi\|_{cb} < 1$ (see [12]). Therefore, $(\mathrm{Id} - \Phi)^{-1} = \sum_{k\geq 0} \Phi^k$, which gives the previous formula for A.

This formula for A suggests to introduce the positive definite kernel

$$K(Z,W) = L(Z)L(W)^*, \quad Z, W \in \mathcal{B}_N(\mathcal{E}), \tag{2.5}$$

as another generalization of the classical Szegö kernel. Also, when $\mathcal{E} = \mathbb{C}$ and $N > 1$, we obtain that

$$K(Z,W) = \big(1 - (Z|W)\big)^{-1}, \tag{2.6}$$

which is a positive definite kernel on the unit ball in the complex N-dimensional space $\mathbb{C}^N$ that was studied quite intensively in recent years [13], [3]. Most notably, the kernel (2.6) has a universality property with respect to the Nevanlinna-Pick problem, as explained in [1]. Note that for $\dim \mathcal{E} > 1$, the kernel (2.6) is no longer positive definite.

3 Nevanlinna-Pick Interpolation

In this section we formulate and solve a Nevanlinna-Pick interpolation problem for the noncommutative analytic Toeplitz algebras as discussed in [13]. These algebras represent a multidimensional generalization of the classical Toeplitz algebra associated to the Hardy space H^∞. One reason for their study is that they are Banach algebras containing the tensor algebra. The associative tensor algebra $\mathcal{T}(\mathcal{H})$ generated by the complex vector space $\mathcal{H} = \mathbb{C}^N$ is defined by the algebraic direct sum

$$\mathcal{T}(\mathcal{H}) = \bigoplus_{k\geq 0} \mathcal{H}^{\otimes k},$$

where $\mathcal{H}^{\otimes k} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{k \text{ factors}}$ is the k-fold algebraic tensor product of $\mathcal{H}$ with itself. Also, the Hilbert direct sum

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{k\geq 0} \mathcal{H}^{\otimes k}$$

is the full Fock space associated to $\mathcal{H}$ (on each $\mathcal{H}^{\otimes k}$, $k > 1$, we consider the tensor Hilbert space structure induced by the Euclidean norm on $\mathcal{H} = \mathbb{C}^N$; also $\mathcal{H}^{\otimes 0} = \mathbb{C}$ and $\mathcal{H}^{\otimes 1} = \mathcal{H}$—see [12] for more details on Fock space constructions). The noncommutative Toeplitz algebra, [13], can be identified with the set of those $\phi \in \mathcal{F}(\mathcal{H})$ such that

$$\sup\{\|\phi \otimes p\|_{\mathcal{F}(\mathcal{H})} \mid p \in \mathcal{T}(\mathcal{H}), \|p\|_{\mathcal{F}(\mathcal{H})} \leq 1\} < \infty.$$

We notice that each $\mathcal{H}^{\otimes k}$ can be identified with the Hilbert space $\mathcal{H}_k$ defined by (2.4) with $\mathcal{E} = \mathbb{C}$, and the noncommutative Toeplitz algebra is isometrically isomorphic to the algebra $\mathcal{U}_{\mathcal{T}}(\mathcal{H})$ of upper triangular operators $T = [T_{ij}]_{i,j=0}^{\infty} \in \mathcal{L}(\bigoplus_{k=0}^{\infty} \mathcal{H}_k)$ with the property that for $i \leq j$ and $i, j \geq 1$,

$$T_{ij} = T_{i-1,j-1}^{\oplus N}, \tag{3.1}$$

where for an operator T, $T^{\oplus N} = \underbrace{T \oplus \cdots \oplus T}_{N \text{ terms}}$. This is just a matricial way to say that T commutes with $I_{\mathcal{E}_0} \otimes R$, where R is the right regular representation on the Fock space. It allows us to work within the upper triangular algebra. We also use the notation $\operatorname{diag}[T]$ for the direct sum of a certain number (or ∞) of copies of T. Denote by $\mathcal{S}(\mathcal{H})$ the Schur class of all contractions in $\mathcal{U}_{\mathcal{T}}(\mathcal{H})$. Given a Hilbert space $\mathcal{E}$, we can introduce the algebra $\mathcal{U}_{\mathcal{T}}(\mathcal{H}, \mathcal{E})$ to be the set of all $T = [T_{ij}]_{i,j=0}^{\infty} \in \mathcal{L}(\bigoplus_{k=0}^{\infty} \mathcal{E}_k)$ satisfying (3.1). The corresponding Schur class of all contractions in $\mathcal{U}_{\mathcal{T}}(\mathcal{H}, \mathcal{E})$ is denoted by $\mathcal{S}(\mathcal{H}, \mathcal{E})$. Note that $\mathcal{E}_k$ can be identified with $\mathcal{H}_k \otimes \mathcal{E}$, which justifies our notation.

Since the noncommutative Toeplitz algebras were viewed as generalizations of the classical Toeplitz algebra, it was quite natural to study bounded interpolation problems in this setting. To that end, a ”point evaluation” was introduced and studied in [3], [9], [13] and references therein. At about the same time, bounded interpolation problems were studied for the algebra of upper triangular operators in [4], [10], [14] (see [5] for details and other related references) and a point evaluation was introduced in this setting too. Since $\mathcal{U}_{\mathcal{T}}(\mathcal{H}, \mathcal{E})$ is an algebra of upper triangular operators we can use the later approach as follows: for $T \in \mathcal{U}_{\mathcal{T}}(\mathcal{H}, \mathcal{E})$ and $Z \in \mathcal{B}_N(\mathcal{E})$ define the operator

$$T(Z) = P_{\mathcal{E}} T L(Z)^*, \tag{3.2}$$

where $P_{\mathcal{E}}$ denotes the orthogonal projection of $\bigoplus_{k=0}^{\infty} \mathcal{E}_k$ onto $\mathcal{E}$ $(= \mathcal{E}_0)$. The basic property that qualifies this operator as a point evaluation is given by the following result.

Lemma 3.1 *If $T \in \mathcal{U}_{\mathcal{T}}(\mathcal{H}, \mathcal{E})$, then $TL(Z)^* = \operatorname{diag}[T(Z)]L(Z)^*$.*

Proof We write $L(Z) = \begin{bmatrix} I & \tilde{Z}_1^* & \tilde{Z}_2^* & \cdots \end{bmatrix}$, where $\tilde{Z}_k^* = \begin{bmatrix} Z_\sigma^* \end{bmatrix}_{|\sigma|=k}$. Due to the properties of the lexicographic order we deduce that $\tilde{Z}_k^* = \begin{bmatrix} Z_1^* \tilde{Z}_{k-1}^* & \cdots & Z_N^* \tilde{Z}_{k-1}^* \end{bmatrix}$. Therefore the k-th block entry of $TL(Z)^*$ is

$$\sum_{l=0}^{\infty} T_{k,k+l} \tilde{Z}_{k+l} = \sum_{l=0}^{\infty} T_{k-1,k+l-1}^{\oplus N} \begin{bmatrix} \tilde{Z}_{k-1} Z_1 \\ \vdots \\ \tilde{Z}_{k-1} Z_N \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{l=0}^{\infty} T_{k-1,k+l-1} \tilde{Z}_{k-1} Z_1 \\ \vdots \\ \sum_{l=0}^{\infty} T_{k-1,k+l-1} \tilde{Z}_{k-1} Z_N \end{bmatrix}.$$

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This formula can be used in an inductive argument in order to conclude the proof. ∎

We can now formulate the following Nevanlinna-Pick type problem:

Problem 3.2 Determine for which $Z_1, \dots, Z_n$ in $\mathcal{B}_N(\mathcal{E})$ and $B_1, \dots, B_n$ in $\mathcal{L}(\mathcal{E})$ there is a $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ such that $T(Z_k) = B_k$, $k = 1, \dots, n$.

This problem can be solved by using the methods in [13], but here we indicate an elementary approach based on the displacement structure theory. More precisely, we use a result announced in [7] and proved in detail in [6] that gives the solution of a so-called scattering experiment associated to the data of the Nevanlinna-Pick problem. We will show that this data can be encoded by a displacement equation of the following type:

$$A - \sum_{k=1}^{N} F_k A F_k^* = GJG^*, \tag{3.3}$$

where $F_k \in \mathcal{L}(\mathcal{G})$, $k = 1, \dots, N$, are given contractions on the Hilbert space $\mathcal{G}$. Also $G = \begin{bmatrix} U & V \end{bmatrix} \in \mathcal{L}(\mathcal{E}^2, \mathcal{G})$ and $J = \left[\begin{smallmatrix} I_{\mathcal{E}} & 0 \\ 0 & -I_{\mathcal{E}} \end{smallmatrix}\right]$. The wave operators associated to (3.3) are introduced by the formulae: $U_\infty^* = [U_k]_{k=0}^\infty$, $V_\infty^* = [V_k]_{k=0}^\infty$, where $U_k = [F_\sigma U]_{|\sigma|=k} \colon \mathcal{E}_k \to \mathcal{G}$, $V_k = [F_\sigma V]_{|\sigma|=k} \colon \mathcal{E}_k \to \mathcal{G}$, and $\mathcal{H} = \mathbb{C}^N$. We will assume that both U_∞ and V_∞ are bounded and also that $\lim_{k\to\infty} \sum_{|\sigma|=k} \|F_\sigma^* g\| = 0$ for all $g \in \mathcal{G}$. Under these assumptions we deduce that (3.3) has a unique solution given by

$$A = U_\infty^* U_\infty - V_\infty^* V_\infty. \tag{3.4}$$

Theorem 3.3 *The solution* (3.4) *of the displacement equation* (3.3) *is positive if and only if there exists* $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ *such that* $V_\infty = TU_\infty$.

Proof For the sake of completeness we indicate the main ideas of the proof. This will also show that our approach is quite elementary.

Assume $A = U_\infty^* U_\infty - V_\infty^* V_\infty \geq 0$ and let $A = LL^*$ be a factorization of A with $L \in \mathcal{L}(\mathcal{F}, \mathcal{G})$ for some Hilbert space $\mathcal{F}$. From (3.3) we deduce that

$$LL^* + VV^* = \sum_{k=1}^{N} F_k LL^* F_k^* + UU^*.$$

In matrix form,

$$\begin{bmatrix} L & V \end{bmatrix} \begin{bmatrix} L^* \\ V^* \end{bmatrix} = \begin{bmatrix} F_1 L & \cdots & F_N L & U \end{bmatrix} \begin{bmatrix} L^* F_1^* \\ \vdots \\ L^* F_N^* \\ U^* \end{bmatrix}. \tag{3.5}$$

Defining $A^* = \begin{bmatrix} L & V \end{bmatrix}$ and $B^* = \begin{bmatrix} F_1L & \cdots & F_NL & U \end{bmatrix}$, we deduce from (3.5) that there exists a unitary operator $\theta_0 \in \mathcal{L}\big(\overline{\mathcal{R}(B)}, \overline{\mathcal{R}(A)}\big)$ such that $A = \theta_0 B$. It follows that there exist Hilbert spaces $\mathcal{R}_1$, $\mathcal{R}_2$, and a unitary extension $\theta \in \mathcal{L}(\mathcal{F}^{\oplus N} \oplus \mathcal{E} \oplus \mathcal{R}_1, \mathcal{F} \oplus \mathcal{E} \oplus \mathcal{R}_2)$ of θ_0, hence this extension satisfies the relation

$$\begin{bmatrix} A \\ 0_{\mathcal{R}_2} \end{bmatrix} = \theta \begin{bmatrix} B \\ 0_{\mathcal{R}_1} \end{bmatrix}. \tag{3.6}$$

Let θ_{ij}, $i \in \{1, 2, 3\}$, $j \in \{1, 2, \dots, N + 2\}$, be the matrix coefficients of θ. It is convenient to rename some of these coefficients. Thus, we set

$$X_k = \theta_{1k}, \quad k = 1, \dots, N, \quad Z = \theta_{1,N+1},$$
$$Y_k = \theta_{2k}, \quad k = 1, \dots, N, \quad W = \theta_{2,N+1}.$$

From (3.6) we deduce that

$$L^* = \sum_{k=1}^{N} X_k L^* F_k^* + ZU^*$$

and

$$V^* = \sum_{k=1}^{N} Y_k L^* F_k^* + WV^*.$$

By induction we deduce that

$$V^* = WU^* + \sum_{k=1}^{N} \sum_{|\sigma|=0}^{n} Y_k X_\sigma Z U^* F_{k\sigma}^* + \sum_{|\tau|=n+1} Q_\tau L^* F_\tau^*, \tag{3.7}$$

where Q_τ are monomials of length $|\tau|$ in the variables $X_1, \dots, X_N, Y_1, \dots, Y_N$. Since θ is unitary it follows that all Q_τ are contractions.

We define $T_{00} = W$ and for $j > 0$,

$$T_{0j} = [Y_k X_\sigma Z]_{|\sigma|=j-1; k=1,\dots,N}.$$

Then we define T_{ij}, $i > 0$, $j \geq i$, by the formula (3.1) and $T_{ij} = 0$ for $i > j$. It can be checked that $T = [T_{ij}]_{i,j=0}^{\infty}$ belongs to $\mathcal{S}(\mathcal{H}, \mathcal{E})$. Also, since $\lim_{k\to\infty} \sum_{|\sigma|=k} \|F_\sigma^* g\| = 0$ for all $g \in \mathcal{G}$, we deduce from (3.7) that $V_\infty = TU_\infty$. ∎

We can now give a solution to Problem (3.2).

Theorem 3.4 *Let $Z_1, \dots, Z_n$ be distinct elements in $\mathcal{B}_N(\mathcal{E})$ and $B_1, \dots, B_n$ in $\mathcal{L}(\mathcal{E})$. Then there is a $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ such that $T(Z_k) = B_k$, $k = 1, \dots, n$, if and only if the Pick matrix*

$$P = \big[L(Z_j)\,\mathrm{diag}[I - B_j^* B_k] L(Z_k)^*\big]_{j,k=1}^{n}$$

is positive.

Proof First assume that P is positive. Define the operators F_k, $k = 1, \dots, N$, by the formula (2.1). Let $V = \begin{bmatrix} B_1 & \cdots & B_n \end{bmatrix}^*$ and set $G = \begin{bmatrix} U & V \end{bmatrix}$, where U was defined by (2.2). Then the unique solution of the displacement equation

$$A - \sum_{k=1}^{N} F_k A F_k^* = GJG^*$$

is P, which is positive. By Theorem 3.3, there is $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ such that $V_\infty = TU_\infty$. We can check that

$$U_\infty = \begin{bmatrix} L(Z_1)^* & \cdots & L(Z_n)^* \end{bmatrix},$$

$$V_\infty = \begin{bmatrix} \operatorname{diag}[B_1]L(Z_1)^* & \cdots & \operatorname{diag}[B_n]L(Z_n)^* \end{bmatrix}.$$

From $V_\infty = TU_\infty$ we deduce that $TL(Z_k)^* = \operatorname{diag}[B_k]L(Z_k)^*$, $k = 1, \dots, n$. By Lemma 3.1 we deduce that $\operatorname{diag}[B_k]L(Z_k)^* = \operatorname{diag}[T(Z_k)]L(Z_k)^*$, which implies that $T(Z_k) = B_k$, $k = 1, \dots, n$.

Conversely, assume that there is a $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ such that $T(Z_k) = B_k$, $k = 1, \dots, n$. Then, with the previous notation, we deduce that $V_\infty = TU_\infty$ and so,

$$P = U_\infty^* U_\infty - V_\infty^* V_\infty = U_\infty^*(I - T^*T)U_\infty \geq 0$$

since T is a contraction. ■

If $Z_k = \operatorname{diag}[z_k]$ for some $z_k \in \mathbb{C}$, $k = 1, \dots, n$, then Problem 3.2 reduces to the Nevanlinna-Pick type problem formulated in [2], [9] and [13]. In this case, Theorem 3.4 reduces to the results given in [2], [9] and [13] for the solution of this problem.

4 Differentiation on $\mathcal{U}_{\mathcal{T}}(\mathcal{H}, \mathcal{E})$

We introduce several derivations on $\mathcal{U}_{\mathcal{T}}(\mathcal{H}, \mathcal{E})$. This might be of interest in itself and also it allows us to formulate interpolation problems involving higher derivatives. The motivation for the definition comes from similar constructions for the algebra of upper triangular operators given in [10] and [14] (see [5] for more details and additional references).

Let $Z \in \mathcal{B}_N(\mathcal{E})$, $Z = \begin{bmatrix} Z_1 & \cdots & Z_N \end{bmatrix}$. Let $E_k^* = [\delta_{jk} I_{\mathcal{E}}]_{j=1}^N$. For $l \geq 1$, we introduce the lower triangular operators $F_k^{(l)} = [X_{ij}] \in \mathcal{L}(\bigoplus_{k=0}^{l} \mathcal{E}_k)$ with $X_{00} = Z_k^*$, $X_{10} = E_k$, $X_{i0} = 0$ for $i > 1$, and otherwise $X_{ij} = X_{i-1,j-1}^{\oplus N}$. Also define

$$U = [I_{\mathcal{E}} \quad \underbrace{0 \cdots 0}_{N+\cdots+N^l \text{ terms}} \quad]^*. \tag{4.1}$$

Then define $U_\infty^* = [F_\sigma^{(l)} U]_{|\sigma|=0}^{\infty}$ and notice that

$$U_\infty = \begin{bmatrix} L(Z)^* & [L(\sigma, Z)]^* \end{bmatrix}_{|\sigma|=1}^{l},$$

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where $L(\sigma, Z)$ are well-defined operators whose form follows from the previous relation. We define the partial derivatives of $T \in \mathcal{U}_{\mathcal{T}}(\mathcal{H}, \mathcal{E})$ at Z by the formula

$$D_\sigma T_Z = P_{\mathcal{E}} T L(\sigma, Z)^*, \quad |\sigma| = 1, \dots, l; \tag{4.2}$$

also set $D_\varnothing T_Z = T(Z)$.

This definition coincides with the usual differentiation in the case $N = 1$, $\mathcal{E} = \mathbb{C}$, up to a constant factor. However, we formulate the next version of the classical Carathéodory problem in a way that does not require that factor. This simplifies some calculations.

Problem 4.1 Given $Z \in \mathcal{B}_N(\mathcal{E})$ and l a positive integer, determine for which $B_k \in \mathcal{L}(\mathcal{E}_k, \mathcal{E})$, $0 \le k \le l$, there is a $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ such that $[D_\sigma T_Z]_{|\sigma|=k} = B_k$ for $k = 0, \dots, l$.

The solution of Problem 4.1 can be obtained by a construction similar to the one involved in the proof of Theorem 3.4. Thus, let $V = \begin{bmatrix} B_0 & \cdots & B_l \end{bmatrix}^*$ and set $G = \begin{bmatrix} U & V \end{bmatrix}$, where U is defined by (4.1). Then the unique solution of the displacement equation

$$A - \sum_{k=1}^{N} F_k^{(l)} A (F_k^{(l)})^* = GJG^*$$

is $A = U_\infty^* U_\infty - V_\infty^* V_\infty$, where we use the notation involved in the statement of Theorem 3.3. We obtain the following result.

Theorem 4.2 *Let $Z \in \mathcal{B}_N(\mathcal{E})$ and $B_k \in \mathcal{L}(\mathcal{E}_k, \mathcal{E})$, $0 \le k \le l$ be given. Then there is a $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ such that $[D_\sigma T_Z]_{|\sigma|=k} = B_k$, $k = 0, \dots, l$, if and only if the matrix $U_\infty^* U_\infty - V_\infty^* V_\infty$ is positive.*

Proof We first notice that the relation $V_\infty = T U_\infty$ for some $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ is equivalent to $[D_\sigma T_Z]_{|\sigma|=k} = B_k$, $k = 0, \dots, l$. Thus, we show by induction that

$$F_\sigma^{(l)} = \begin{bmatrix} P_0(\sigma) & 0 & 0 & \cdots & 0 \\ P_1(\sigma) & \big(P_0(\sigma)\big)^{\oplus N} & 0 & \cdots & 0 \\ P_2(\sigma) & \big(P_1(\sigma)\big)^{\oplus N} & \big(P_0(\sigma)\big)^{\oplus N^2} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ P_l(\sigma) & & & \cdots & \big(P_0(\sigma)\big)^{\oplus N^l} \end{bmatrix}, \tag{4.3}$$

where $P_k(\sigma)$ are column matrices with N^k entries given by polynomials in the variables $Z_1, Z_2, \dots, Z_N$. These polynomials can be obtained by using the recursions:

$$P_0(\sigma) = Z_\sigma^*, \quad \sigma \in \mathcal{F}_N^+,$$
$$P_j(\varnothing) = 0, \quad j \ge 1,$$

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and for $k = 1, \dots, N$, $\sigma \in \mathcal{F}_N^+$, and $j = 1, \dots, N$,

$$P_j(k\sigma) = E_k^{\oplus N^{j-1}} P_{j-1}(\sigma) + (Z_k^*)^{\oplus N^j} P_j(\sigma). \tag{4.4}$$

For $T = [T_{ij}]_{i,j=0}^{\infty} \in \mathcal{S}(\mathcal{E}, \mathcal{H})$ we set $T_{0j} = [C_\sigma]_{|\sigma|=j}$. Then the relation $V_\infty = TU_\infty$ means that

$$\sum_{|\tau|=0}^{\infty} P_k(\sigma\tau)C_\tau^* = \sum_{j=0}^{l} \big(P_{k-j}(\sigma)\big)^{\oplus N^j} B_j^*, \tag{4.5}$$

for all $\sigma \in \mathcal{F}_N^+$ and $k = 1, \dots, l$. In particular,

$$B_k^* = \sum_{|\sigma|=k}^{\infty} P_k(\sigma)C_\sigma^* = [D_\sigma T_Z]_{|\sigma|=k}^*, \quad k = 0, \dots, l,$$

so that $V_\infty = TU_\infty$ implies $[D_\sigma T_Z]_{|\sigma|=k} = B_k$, $k = 0, \dots, l$.

Conversely, assume $[D_\sigma T_Z]_{|\sigma|=k} = B_k$ for $k = 0, \dots, l$. This can be rewritten in the form $B_k^* = \sum_{|\sigma|=k}^{\infty} P_k(\sigma)C_\sigma^*$ for $k = 0, \dots, l$. We deduce from these relations that in order to have the relation $V_\infty = TU_\infty$, we must prove

$$P_k(\sigma\tau) = \sum_{j=0}^{k} \big(P_{k-j}(\sigma)\big)^{\oplus N^k} P_j(\tau). \tag{4.6}$$

But we notice that (4.6) is just a consequence of the relation $F_{\sigma\tau}^{(l)} = F_\sigma^{(l)} F_\tau^{(l)}$. Our claim is proved, and the proof of the theorem can be now concluded by an application of Theorem 3.3. ∎

We notice that for $Z = 0$ we deduce Corollary 5.2 in [13]. A more explicit form for $U_\infty^* U_\infty - V_\infty^* V_\infty$ can be obtained but that is only a notational matter.

We conclude this section by introducing another possible formulation of a Carathéodory type problem in this setting. Thus, we can introduce a total derivative of order k at Z by the formula:

$$D^k T_Z = \sum_{|\sigma|=k} D_\sigma T_Z, \quad k = 1, 2, \dots, \tag{4.7}$$

and by convention $D^0 T_Z = T(Z)$. A corresponding Carathéodory type problem can be formulated as follows.

Problem 4.3 Given $Z \in \mathcal{B}_N(\mathcal{E})$ and l a positive integer, determine for which $B_k \in \mathcal{L}(\mathcal{E})$, $0 \le k \le l$, there is a $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ such that $D^k T_Z = B_k$ for $k = 0, \dots, l$.

We can show that this problem can be also solved by using the displacement structure approach. Thus, let l be a positive integer and $Z \in \mathcal{B}_N(\mathcal{E})$, $Z = \begin{bmatrix} Z_1 & \cdots & Z_N \end{bmatrix}$. Then, for $1 \leq k \leq N$,

$$TF_k^{(l)} = \begin{bmatrix} Z_k^* & 0 & 0 & \cdots & \\ I_{\mathcal{E}} & Z_k^* & 0 & \cdots & \\ 0 & I_{\mathcal{E}} & Z_k^* & \cdots & \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & I_{\mathcal{E}} & Z_k^* \end{bmatrix}$$

and

$$U = [I_{\mathcal{E}} \quad \underbrace{0 \cdots 0}_{l \text{ terms}}]^*. \tag{4.8}$$

The associated $U_\infty^* = [TF_\sigma^{(l)} U]_{|\sigma|=0}^\infty$ can be written in the form

$$U_\infty = \begin{bmatrix} L(Z) & M_1(Z) & \cdots & M_l(Z) \end{bmatrix},$$

for some operators $M_k(Z)$, $k = 1, \ldots, l$.

Lemma 4.4

$$D^k T_Z = P_{\mathcal{E}} T M_k(Z).$$

Proof We prove by induction that

$$TF_\sigma^{(l)} = \begin{bmatrix} Q_0(\sigma) & 0 & 0 & \cdots & \\ Q_1(\sigma) & Q_0(\sigma) & 0 & \cdots & \\ Q_2(\sigma) & Q_1(\sigma) & Q_0(\sigma) & \cdots & \\ \vdots & & & \ddots & \\ Q_l(\sigma) & & \cdots & Q_1(\sigma) & Q_0(\sigma) \end{bmatrix},$$

where $Q_0(\sigma) = Z_\sigma^*$, $Q_j(\varnothing) = 0$ for $j \geq 1$, and for $k = 1, \ldots, N$, $\sigma \in \mathbb{F}_N^+$ and $j = 1, \ldots, l$, we have the recursion:

$$Q_j(k\sigma) = Q_{j-1}(\sigma) + Z_k^* Q_j(\sigma). \tag{4.9}$$

Now, each $P_j(\sigma)$ in (4.4) is a column matrix with N^j entries $P_j^s(\sigma)$. From (4.4) and (4.9) it follows that

$$\sum_{s=1}^{N^j} P_j^s(\sigma) = Q_j(\sigma)$$

and this implies the required formula. ∎

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This remark allows us to solve Problem 4.3 in the same way we solved Problem 4.1. Thus, define $V = \begin{bmatrix} B_0 & \cdots & B_l \end{bmatrix}^*$ and set $G = \begin{bmatrix} U & V \end{bmatrix}$, where U is defined by (4.8). Then the unique solution of the displacement equation

$$A - \sum_{k=1}^{N} TF_k^{(l)} A (TF_k^{(l)})^* = GJG^*$$

is $A = U_\infty^* U_\infty - V_\infty^* V_\infty$, where again we use the notation involved in the statement of Theorem 3.3. With the same proof as that of Theorem 4.2 we obtain the following result.

Theorem 4.5 *Let $Z \in \mathcal{B}_N(\mathcal{E})$ and $B_k \in \mathcal{L}(\mathcal{E})$, $0 \leq k \leq l$ be given. Then there is a $T \in \mathcal{S}(\mathcal{H}, \mathcal{E})$ such that $D^k T_Z = B_k$, $k = 0, \ldots, l$, if and only if the matrix $U_\infty^* U_\infty - V_\infty^* V_\infty$ is positive.*

Again, the explicit form of $U_\infty^* U_\infty - V_\infty^* V_\infty$ is just a matter of notation. Still, for $Z = 0$ we notice that Theorem 4.6 gives that the Toeplitz matrix

$$\begin{bmatrix} B_0 & B_1 & \cdots & B_l \\ 0 & B_0 & \cdots & B_{l-1} \\ & & \ddots & \\ 0 & 0 & & B_0 \end{bmatrix}$$

must be a contraction, which is the classical criterion for the solvability of a one-dimensional Carathéodory problem. One can check directly that indeed, both Problem 4.2 and Problem 4.4 for $Z = 0$ are equivalent to some one-dimensional, multivalued Carathéodory problems. However, it might be useful to know that these problems have a natural interpretation related to the tensor algebra.

References

[1] J. Agler and J. E. McCarthy, *Complete Nevanlinna-Pick kernels.* J. Funct. Anal. **175**(2000), 111–124.
[2] A. Arias and G. Popescu, *Noncommutative interpolation and Poisson transforms.* Israel J. Math. **115**(2000), 205–234.
[3] W. Arveson, *Subalgebras of C^*-algebras III: Multivariable operator theory.* Acta Math. **181**(1998), 476–514.
[4] J. A. Ball, I. Gohberg and M. A. Kaashoek, *Nevanlinna-Pick interpolation problem for time-varying input-output maps: The discrete case.* In: Operator Theory: Advances and Applications, Vol. **56**, Birkhäuser, 1992, 1–51.
[5] T. Constantinescu, *Schur Parameters, Factorization and Dilation Problems*. Birkhäuser, 1996.
[6] T. Constantinescu and J. L. Johnson, *Tensor algebras and displacement structure I. The Schur algorithm.* Preprint, 2001.
[7] T. Constantinescu, A. H. Sayed and T. Kailath, *Inverse scattering experiments, structured matrix inequalities, and tensor algebras.* Linear Alg. Appl., to appear.
[8] K. R. Davidson and D. R. Pitts, *The algebraic structure of non-commutative analytic Toeplitz algebras.* Math. Ann. **311**(1998), 275–303.
[9] K. R. Davidson and D. R. Pitts, *Nevanlinna-Pick interpolation for noncommutative analytic Toeplitz algebras.* Integral Equations Operator Theory **31**(1998), 321–337.
[10] P. Dewilde and H. Dym, *Interpolation for upper triangular operators.* In: Operator Theory: Advances and Applications, Vol. **56**, Birkhäuser, 1992, 153–260.

[11] T. Kailath and A. H. Sayed, *Displacement structure: theory and applications.* SIAM Rev. **37**(1995), 297–386.
[12] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*. Birkhäuser, 1992.
[13] G. Popescu, *Interpolation problems in several variables.* J. Math. Anal. Appl. **227**(1998), 227–250.
[14] A. H. Sayed, T. Constantinescu and T. Kailath, *Lattice structures for time variant interpolation problems.* In: Proc. IEEE Conf. Decision and Contr., Vol. **1**, Tucson, 1992, 116–121.

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