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GROUP REPRESENTATIONS AND CARDINAL ALGEBRAS

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Dedicated to the memory of Josephine J. Mehlberg

0. Introduction. Our aim in this work is to show that the global theory of group representations (as presented in [**4**] for example) is actually and naturally a part of the theory of cardinal algebras. In this sense it is analogous to the work of Kaplansky [**2**] and Loomis [**3**] on operator rings. In his paper Loomis proposed an abstract scheme for representation theory; it appears, however, that the idea of abstracting the equivalence class of a representation is more suitable. It is the set of all these classes that forms the cardinal algebra which we study.

The connection between cardinal algebras and operator theory has been made explicit by Fillmore [**1**], where he worked out a dimension theory for a class of cardinal algebras satisfying certain conditions. In the next section we shall discuss the relation between the two systems; we could say that the main difference lies in our introducing such axioms as to make possible the study of type III cases.

We work out the case of representations in a separable Hilbert space. To cover the general case, a peculiar kind of algebra is necessary, where sums can extend over families of any cardinality. This, as pointed out by Loomis [**3**], involves set-theoretic complications but it seems that it can be carried out.

The book by Tarski [**5**] will be used frequently throughout the paper as a reference. We shall first state the axioms for a cardinal algebra that we shall use, next we shall show that group representations satisfy them, and then we shall proceed with the development of the theory.

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1. The axioms. We state below a set of axioms for cardinal algebras which, as shown by Fillmore [**1**], is equivalent to that of Tarski. This form, however, is much more convenient to verify for the case of group representations.

Let A be a set; we assume that to each sequence $a_1, a_2, \ldots$ of elements in A there corresponds a unique element of A which we shall denote by $\sum a_i$.

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Axiom I (*Commutative law*). If p is a permutation of the positive integers and $b_i = a_{p(i)}$, then $\sum b_i = \sum a_i$.

Axiom II (*Existence of identity*). There exists an element 0 of A such that if $a_1 = a$ and $a_2 = a_3 = \ldots = 0$, then $\sum a_i = a$.

We shall write $\sum_{i=1}^{n} a_i$ or $a_1 + a_2 + \ldots + a_n$ for the element $\sum a_i$ where $a_i = 0$ for $i > n$.

If I is a finite or countable set and f is a one-to-one map of I onto the appropriate segment of the cardinals, we shall write $\sum_{i \in I} a_i$ or just $\sum a_i$ for the element $\sum a_{f(i)}$ (by Axiom I this is consistent). If $a_i = a$ for all $i \in I$ and n is the cardinality of the index set I, we shall write na for the element $\sum a_i$.

Axiom III (*Associative law*). If $I_1, I_2, \ldots$ is a partition of the finite or countable set I, then $\sum_{i \in I} a_i = \sum(\sum_{i \in I_n} a_i)$.

Definition (1.1). We say that a is a part of b if $b = a + c$ for some element c; we write this as $a \leqq b$ or $b \geqq a$. If $a \leqq b$ and $a \neq b$, we write $a < b$ or $b > a$. If a is not a part of b, then we write $a \nleqq b$ or $b \ngeqq a$.

Axiom IV (*Splitting law*). If $b \leqq \sum a_i$, then $b = \sum b_i$, where $b_i \leqq a_i$ for all i.

Axiom V (*Remainder*). If $a_i = a_{i+1} + b_i$ for $i = 1, 2, \ldots,$ then there exists an element c such that $a_i = c + \sum_{k \geqq i} b_k$.

Axiom VI (*Refinement law*). If $a^1 + a^2 = a_1 + a_2$, then there exist four elements a_i^j such that $a_1^j + a_2^j = a^j$ and $a_i^1 + a_i^2 = a_i$.

Now to these six axioms we shall add two more which will allow the development of the complete theory.

Axiom VII (*Division law*). If $a \geqq b \neq 0$, then there exists a cardinal n and an element $c \ngeqq b$ such that $a = nb + c$.

Axiom VIII (*Separability*). Any well-ordered bounded subset of A contains at most countably many elements.

In [**1**] Fillmore deduced the structure of certain cardinal algebras satisfying Axioms I–VI plus the next two:

(i) Any two elements have a greatest lower bound.

(ii) Every element is the sum of finite elements (see Definition (5.1)).

As we shall derive results quite similar to (although somewhat more detailed than) Fillmore's, it is important to discuss these two properties. To begin with we observe that (i) is valid in our system, being a simple corollary of the comparability theorem (4.2). Fillmore deduced the comparability theorem from (i) and (ii), but it is very difficult to verify directly that (i) holds for any two equivalence classes of representations. It is thus clear that (i) is not suitable for the purpose of developing an abstract representation theory.

Furthermore, property (ii) above does not, in general, characterize elements of type $\neq$ III (Definition (7.1)), as the following example shows. Consider the set of all non-negative integers together with two elements x and y; a finite sum of integers is the usual sum; an infinite sum of integers is x; any sum in which x appears but not y is x; any sum in which y appears is y. In this cardinal algebra property (i) holds but not (ii); although y has finite type, it is not a sum of finite elements. In our system, because of Axiom VII, we can show (Theorem (9.3)) that (ii) characterizes elements of finite type, and we can further analyze elements of type III.

Incidentally, the above example shows that Axiom VII is independent of all the others (Axiom VIII included). It is not hard to verify that Axiom VIII is also independent of all the rest by looking at the cardinal algebra of all subsets of a given uncountable set, where addition is defined as union.

2. Verification. Let G be a group and A the set of all equivalence classes of unitary representations of G in a separable Hilbert space. For any sequence $a_1, a_2, \ldots$ in A we choose representations $R_i \in a_i$; the equivalence class of the direct sum of the R_i depends only on the sequence (a_i) and will be denoted by $\sum a_i$. Trivially, Axioms I and II hold, where 0 is the class of representations on a zero-dimensional space. Then the finite sum is the finite direct sum of the representations chosen, and it is clear that Axiom III holds.

To establish Axiom IV, let H be the direct sum of the spaces H_i on which the representations R_i act. Let R, acting on H, be the direct sum of the R_i. Then $b \leqq \sum a_i$ means that for some invariant subspace M of R the restriction $R|M$ is in the class b. Now consider families (M_i, K_i) such that the M_i are pairwise orthogonal invariant subspaces of M, the K_i are invariant subspaces of the H_i, while $R|M_i$ is equivalent to $R|K_i$. Such families with non-zero M_i exist: for if P_L is the projection on the subspace L, then $P_{H_i}P_M$ intertwines $R|H_i$ and $R|M$ and cannot be zero for all i. We order the set of all families described above by: $(M_i', K_i') \subset (M_i'', K_i'')$ if M_i' is a proper subspace of M_i'' and $K_i' \subseteq K_i''$. Clearly, if (M_i^α, K_i^α) is a linearly ordered set of such families, the family $(\overline{\bigcup_\alpha M_i^\alpha}, \overline{\bigcup_\alpha K_i^\alpha})$ is an upper bound. Thus there exists a maximal element say (M_i, K_i); maximality implies that $\bigoplus M_i = M$, for otherwise the argument above can be repeated in the orthocomplement of $\bigoplus M_i$ in M thereby producing a family $\supset (M_i, K_i)$. If we set b_i for the class of $R|M_i$, we have $b = \sum b_i$ and $b_i \leqq a_i$.

The argument for Axiom V is shorter: Let a_1 be the class of R acting on H_1; then there exists a sequence of invariant subspaces $H_2 \supseteq H_3 \supseteq \ldots$ such that $R|H_i$ is in the class a_i and R restricted to $H_i \cap H_{i+1}{}^\perp$ (where $\perp$ denotes orthogonal complement) belongs to b_i. The subspaces $H_i \cap H_{i+1}{}^\perp$ are pairwise orthogonal and the complement H relative to H_1 of their direct sum is contained in all H_i. If c is the class of $R|H$, then $a_i = c + \sum_{k \geqq i} b_k$.

Now for Axiom VI. We are given a space H, a representation R, and two invariant subspaces M, N such that $R|M \in a_1$, $R|M^\perp \in a_2$, $R|N \in a^1$, and

$R|N^\perp \in a^2$. Then we have $M = (M \cap N) \oplus M \cap (M \cap N)^\perp$ and also $M^\perp = (M^\perp \cap N^\perp) \oplus M^\perp \cap (M^\perp \cap N^\perp)^\perp$; all four subspaces in these two direct sums are invariant so that we can restrict R to $M \cap N$, $M \cap (M \cap N)^\perp$, $M^\perp \cap (M^\perp \cap N^\perp)^\perp$, and $M^\perp \cap N^\perp$ to obtain representations whose classes we denote (respectively) by a_1^1, a_1^2, a_2^1, and a_2^2; it is clear that $a_1^1 + a_1^2 = a_1$ and $a_2^1 + a_2^2 = a_2$. Now observe that $M^\perp \cap (M^\perp \cap N^\perp)^\perp = M^\perp \cap (M \cup N)$ and that the four spaces $N \cap (M \cap N)^\perp$, $N/(M \cap N)$, $(M \cup N)/M$, and $(M \cup N) \cap M^\perp$ are canonically isomorphic, which implies that R restricted to the first and the last yields equivalent representations; since

$$N = (N \cap M) \oplus N \cap (N \cap M)^\perp,$$

we have $a^1 = a_1^1 + a_2^1$. In a similar way we obtain the last equality.

Thus we have shown that A is a cardinal algebra. As Axiom VIII is rather obvious because of the at most countable dimension of the representation spaces, we shall verify only Axiom VII. Let a be the class of R acting on H and consider families of orthogonal subspaces H_i such that $R|H_i \in b$. The inclusion ordering provides us with a maximal such family, which means that $H \cap (\bigoplus H_i)^\perp$ contains no subspace on which R belongs to b. If c is the class of $R|H \cap (\bigoplus H_i)^\perp$, then $a = nb + c$ and $c \not\geqq b$.

3. Types. We shall define all terms used, although many are defined in [**5**].

Definition (3.1). The elements a and b are disjoint if they have no non-zero common part; write $a \perp b$. The type of the element a is the set of all elements disjoint from a; write $\bar{a}$. Also, write $\emptyset$ for the type of 0 and T for the set of all types.

We shall use on T the partial order $\leqq$ which is reverse to inclusion. Thus for $s, t \in T$ we shall say that $s \leqq t$ if and only if $t \subseteq s$ as sets, or equivalently that, if $s = \bar{a}$ and $t = \bar{b}$, then $c \perp b$ implies $c \perp a$.

Definition (3.2). The types s and t are disjoint if there is no type $r \neq \emptyset$ such that $r \leqq s$ and $r \leqq t$; write $s \perp t$.

If $s = \bar{a}$ and $t = \bar{b}$, then $s \perp t$ means $a \perp b$. For let $a \perp b$; then $\bar{c} \leqq \bar{a}$ implies $b \perp c$ and if further $\bar{c} \leqq \bar{b}$, we have $c \perp c$ or $c = 0$. Conversely, let $\bar{a} \perp \bar{b}$; if $c \leqq a$ and $c \leqq b$, then $\bar{c} \leqq \bar{a}$ and $\bar{c} \leqq \bar{b}$ hence $\bar{c} = \emptyset$ and $c = 0$.

LEMMA (3.1). *If the elements a_i are pairwise disjoint and $a_i \leqq a$ for all i, then $\sum a_i \leqq a$.*

Proof. First we prove this for two elements $a_1, a_2 \leqq a$: we have $a = a_1 + b$, and since $a_2 \leqq a$ we also have $a_2 = c + d$, where $c \leqq a_1$ and $d \leqq b$; as $a_1 \perp a_2$ we obtain $c = 0$ and thus $a_2 = d \leqq b$, hence $a \geqq a_1 + a_2$. Using again the splitting axiom we see that $a_{k+1} \perp \sum_{i=1}^k a_i$ and so by induction we have $\sum_{i=1}^k a_i \leqq a$ for all k. Thus by [**5**, Theorem 2.21] we obtain $\sum a_i \leqq a$.

LEMMA (3.2). *Let $s_i = \bar{a}_i$ and $s = \bar{a}$, where $a = \sum a_i$; then s is the supremum of the family (s_i).*

Proof. Since $a_i \leqq a$, every $b \perp a$ is also disjoint from a_i and thus $s_i \leqq s$. Let $s_i \leqq \bar{c}$ for all i; if $b \perp c$, then $b \perp a_i$ for all i, hence $b \perp \sum a_i = a$. Because if d is a part of b and of $\sum a_i$, then $d = \sum d_i$ where $d_i \leqq a_i$ and this implies that $d_i = 0$ or $d = 0$. But then we have $s \leqq \bar{c}$.

We shall write $\bigcup s_i$ or, for a finite family, $s_1 \cup s_2 \cup \ldots \cup s_n$ for the supremum.

We can now prove the main theorem on types.

THEOREM (3.1). *Let $a \in A$ and $s \in T$; then there exist unique $b, c \in A$ such that $a = b + c$, $\bar{b} \leqq s$ and $\bar{c} \perp s$. This element b is the largest part of a of type $\leqq s$.*

Proof. We consider (finite or countable) families of pairs (n_i, b_i) such that the n_i are cardinals, the elements b_i are distinct and non-zero, $\bar{b}_i \leqq s$, and $\sum n_i b_i \leqq a$. For any two such families (n_{1i}, b_{1i}) and (n_{2j}, b_{2j}) we say that the first precedes the second if each b_{1i} is among the b_{2j} while for $b_{1i} = b_{2j}$ we have $n_{1i} \leqq n_{2j}$; thus $\sum n_{1i} b_{1i} \leqq \sum n_{2j} b_{2j}$. Since these sums are bounded by a, any well-ordered subset of such families has at most countably many distinct sums. The bs appearing in these sums are at most countable, and so we can write them as $b_1, b_2, \ldots$, index the families by some ordered set K, and represent each family by (n_{ik}, b_i), where $n_{ik} = 0$ for those b_i which do not occur in the family; the set K is countable and $n_{ik_1} \leqq n_{ik_2}$ for all i if $k_1 < k_2$. Now let $n_i = \sup_{k \in K} n_{ik}$; we wish to show that (n_i, b_i) is a bound for the well-ordered set of families we started with, so that by Zorn's lemma we shall obtain a maximal family. It suffices to show that $\sum n_i b_i \leqq a$ since, by Lemma (3.2), the type of the sum is $\leqq s$. First we choose a sequence $k_1, k_2, \ldots$ cofinal in K, i.e. such that $n_i = \sup_m n_{ik_m}$ for all i. Then we have

$$n_i = n_{ik_1} + \sum_{m=1}^{\infty} (n_{ik_{m+1}} - n_{ik_m}),$$

where the expression $\infty - \infty$ is by convention set equal to zero. Thus we obtain

$$\sum n_i b_i = \sum n_{ik_1} b_i + \sum_i \sum_{m=1}^{\infty} (n_{ik_{m+1}} - n_{ik_m}) b_i$$
$$= \sum n_{ik_1} b_i + \sum_{m=1}^{\infty} \sum_i (n_{ik_{m+1}} - n_{ik_m}) b_i.$$

But we see that if we extend the sum in this last term (over the index m) from 1 to j we obtain $\sum_i n_{ik_j} b_i$ which is $\leqq a$ by hypothesis, and hence the whole sum from 1 to ∞ will be $\leqq a$ by [**5**, Theorem 2.21]. Thus there exists a maximal family (n_i, b_i) and we write $a = \sum n_i b_i + c$. We shall now show that if $d \leqq a$ and $\bar{d} \leqq s$, then $d \leqq \sum n_i b_i$. We split: $d = d_1 + d_2$, where $d_1 \leqq c$ and $d_2 \leqq \sum n_i b_i$. Since $d_1 + \sum n_i b_i \leqq a$ and $\bar{d}_1 \leqq s$, d_1 must necessarily (if not zero) be one of the b_i, say b_{i_0}, or else we could adjoin $(1, d_1)$ to the maximal family (n_i, b_i); then $n_{i_0} = \infty$, otherwise we could replace (n_{i_0}, b_{i_0}) by the pair $(n_{i_0} + 1, b_{i_0})$. Thus

$$\sum n_i b_i = \sum n_i b_i + d_1 \geqq d_2 + d_1 = d.$$

Finally we shall show that c can be chosen to have type disjoint from s. We shall use Axiom VII: write $c = m_1 b_1 + c_1$ where $c_1 \ngeqq b_1$; this allows us to write $a = \sum n_i b_i + c_1$, since n_1 must equal $n_1 + m_1$ by the maximality of the family (n_i, b_i). Let $a = \sum n_i b_i + c_k$, where $c_k \ngeqq b_1, b_2, \ldots, b_k$ and write $c_k = m_{k+1} b_{k+1} + c_{k+1}$, where $c_{k+1} \ngeqq b_{k+1}$ (and, of course, $\ngeqq b_i$ for $i \leqq k$). Then, as before, we have $a = \sum n_i b_i + c_{k+1}$. By Axiom V we have $c_k = c_0 + \sum_i m_{k+i} b_{k+i}$ for some c_0, and clearly c_0 contains no b_i. Since $n_i = n_i + m_i$, we obtain $a = \sum n_i b_i + c_0$, and by the argument given previously we see that c_0 can contain no non-zero element of type $\leqq s$, as it would then have to contain some b_i. Thus $\bar{c}_0 \perp s$. Uniqueness is immediate from the refinement law.

We can now proceed to study the structure of the set T of all types and show that the partial order makes it a generalized Boolean σ-algebra (i.e. a Boolean σ-ring without identity).

Lemma (3.3). *Let $s \leqq t$; then there exists a unique $r \perp s$ such that $t = s \cup r$.*

Proof. Let $t = \bar{a}$, and write $a = b + c$ as in Theorem (3.1). If $\bar{b} \neq s$, there must exist an element d of type $\leqq s$ which is disjoint from b, because otherwise $s \leqq \bar{b}$ and hence $s = \bar{b}$. Since $\bar{d} \leqq s \leqq \bar{a}$ there exists a non-zero element $e \leqq a$ which is also $\leqq d$. Then $e \perp b$ so that $b < b + e$ while $b + e \leqq a$; as $\overline{b + e} \leqq s$ we have found an element strictly larger than b of type $\leqq s$ which is a part of a; this contradicts Theorem (3.1). Hence $\bar{b} = s$ and so $t = s \cup \bar{c}$ with $\bar{c} \perp s$. Uniqueness is implied by Theorem (3.1).

The relative complement of s in t, whose existence we have just proved, will be written as $t \cap s'$.

Lemma (3.4). *If $r \leqq s \leqq t$, then $t \cap s' \leqq t \cap r'$.*

Proof. Let $t = \bar{a}$ with $a = b + c$, where $\bar{b} = s$ and $\bar{c} = t \cap s'$. Since $r \leqq s$ we have $b = d + e$ with $\bar{d} = r$ and $e \perp d$. Then $a = d + c + e$ and as both c, e are disjoint from d we have $\overline{c + e} = t \cap r'$; but $c \leqq c + e$, hence the conclusion.

Lemma (3.5). *Let $s_i \leqq t$ and $r = \bigcup (t \cap s_i')$; then $t \cap r'$ is the infimum of the s_i.*

Proof. Since $t \cap s_i' \leqq r \leqq t$, we have $t \cap r' \leqq t \cap (t \cap s_i')' = s_i$ by the uniqueness of relative complements. Let $u \leqq s_i$ for all i so that $t \cap s_i' \leqq t \cap u'$; hence $r \leqq t \cap u'$ or $u \leqq t \cap r'$.

For arbitrary s_i we shall write $\bigcap s_i$ for their infimum. For any s, t we shall write $t \cap s'$ for the type $t \cap (t \cap s)'$. To complete the argument we only need the distributive law which follows from the following lemma.

LEMMA (3.6). *For any s, t we have $s = (s \cap t) \cup (s \cap t')$.*

Proof. It is clear that $s \geqq (s \cap t) \cup (s \cap t')$, and so consider an r disjoint from both $s \cap t$ and $s \cap t'$. We shall show that $r \perp s$. Let $u \leqq r, u \leqq s$ so that $u \perp s \cap t$. If $v \leqq u \cap t$, we have $v \leqq s \cap t$ and also $v \perp s \cap t$, and thus $v = \emptyset$; but then $u \perp t$ and since $u \leqq s$ we have $u \leqq s \cap t'$. On the other hand, $u \perp s \cap t'$ because $u \leqq r \perp s \cap t'$. Therefore $u = \emptyset$ and $r \perp s$.

4. Restriction to a type.

Definition (4.1). The largest part of a with type $\leqq s$ is the restriction of a to s; write $a|s$.

THEOREM (4.1). *For any a, b, a_i and s, t, s_i we have:*
(i) *$a|s = a$ if and only if $\bar{a} \leqq s$,*
(ii) *$a \leqq b$ implies $a|s \leqq b|s$, and $s \leqq t$ implies $a|s \leqq a|t$,*
(iii) *$(\sum a_i)|s = \sum (a_i|s)$,*
(iv) *$(a|s)|t = a|(s \cap t)$,*
(v) *$\overline{a|s} = \bar{a} \cap s$,*
(vi) *If the s_i are pairwise disjoint, then $a|\bigcup s_i = \sum (a|s_i)$.*

Proof. There is no need to give an argument for (i) and (ii). We prove (iii). Since $a_i|s \leqq a_i$, we have $\sum (a_i|s) \leqq \sum a_i$, and since the type of the left side is $\leqq s$, we have $\sum (a_i|s) \leqq (\sum a_i)|s$. Let $b \leqq \sum a_i$, $\bar{b} \leqq s$; we have $b = \sum b_i$ with $b_i \leqq a_i$ and since $\bar{b}_i \leqq s$ we also have $b_i \leqq a_i|s$, which implies that $b \leqq \sum (a_i|s)$. Thus $\sum (a_i|s)$ is the largest element of type $\leqq s$ which is a part of $\sum a_i$. For (iv), we consider any $b \leqq a|s$ with type $\leqq t$; then $\bar{b} \leqq s \cap t$ and thus $b \leqq a|(s \cap t)$. This means that $(a|s)|t \leqq a|(s \cap t)$. The reverse is immediate. For (v), we write $a = (a|s) + c$ with $\bar{c} \perp s$; take types of both sides and intersect with s to obtain $\bar{a} \cap s = \overline{a|s}$ since the type of $a|s$ is $\leqq s$. For (vi), we can restrict our attention to the case $\bar{a} = \bigcup s_i$, so that $a = \sum a_i$ with $\bar{a}_i = s_i$; then $a|s_j = \sum (a_i|s_j) = a_j|s_j = a_j$.

The main result is the following.

THEOREM (4.2) (*Comparison*). *For any a, b there exist types s, t such that $s \cap t = \emptyset$, $a|s \leqq b|s$, $a|t \geqq b|t$, and $a|u = b|u = 0$ for any $u \perp s \cup t$.*

Proof. Working within the Boolean σ-algebra of all types $\leqq \bar{a} \cup \bar{b}$ (which satisfies the countable chain condition because of Axiom VIII), we consider sets L, M of types such that the elements of $L \cup M$ are pairwise disjoint, while $s \in L$, $t \in M$ imply $a|s \leqq b|s$ and $a|t \geqq b|t$. We can then find L_0, M_0 maximal with respect to inclusion; each will contain, by the chain condition, at most countably many elements, and we can take the supremum of L_0 for s, and the supremum of M_0 for t.

Although the type s' does not exist (since T has no unity) we shall abbreviate the conclusion of the theorem to $a|s \leqq b|s$, $a|s' \geqq b|s'$.

5. Finite and infinite elements.

Definition (5.1). An element a is finite if $a + b = a$ implies $b = 0$; an element is infinite if it is not finite. A type s is finite if there exists a finite element a such that $s = \bar{a}$.

THEOREM (5.1). *If $a \geqq b$ and a is finite, so is b. If a, b are finite, so is $a + b$. If the a_i are pairwise disjoint and finite, so is $\sum a_i$. If a is finite and $a + b = a + c$, then $b = c$.*

The proof can be found in [**5**, Theorems 4.14, 4.16, 4.18, 4.19].

LEMMA (5.1). *For any a there exists a largest type f such that $a|f$ is finite.*

Proof. We consider within the Boolean σ-algebra of all types $\leqq \bar{a}$ families of types t such that $a|t$ is finite. We can obtain a maximal such family of pairwise disjoint types which will be countable; its supremum is f.

Definition (5.2). The finite part of a is the element $a|f$. An element a is purely infinite if its finite part is 0.

LEMMA (5.2). *An element a is infinite if and only if it contains some non-zero element ∞b.*

Proof. If a is not finite, then $a = a + b$ for some $b \neq 0$, hence $a \geqq \infty b$ by [**5**, Theorem 2.21]. The converse is obvious.

Thus, if we have $n_k b \leqq a$ with a finite and n_k tending to infinity, then $b = 0$.

THEOREM (5.2). *An element $b \neq 0$ is purely infinite if and only if $b = \infty b$.*

Proof. If $b = \infty b$, then $b = b + b$ and, for any type s, $b|s = b|s + b|s$; thus $b|s$ cannot be finite (if non-zero) and so b is purely infinite. Now suppose that b is purely infinite. By Lemma (5.2) we have $b \geqq \infty d$ for some $d \neq 0$ and thus $b = \infty d + c$, where $c \ngeqq d$. Write $a = \infty d$ so that $b = a + c$ with $a = \infty a$ and $c \ngeqq a$. Now compare a and c: $a|s \geqq c|s$, $a|s' \leqq c|s'$; then $b|s = a|s + c|s = a|s$ since $a|s + c|s \leqq 2(a|s) = a|s$. If $a|s = 0$, then $c|s = 0$ and $c|s' = c$ while $a|s' = a$ and so $c \geqq a$ which is not the case. Thus the set S of all types $s \leqq \bar{b}$ such that $b|s = \infty a$ for some $a \neq 0$ is not empty. By the usual procedure we find a maximal element s_0 of S. But this must be $\bar{b}$, for otherwise we consider $b|\bar{b} \cap s_0' \neq 0$ and repeat the argument to increase s_0. Thus $b = \infty a$ for some a and hence $b = \infty b$.

We shall now give an alternate characterization of types.

THEOREM (5.3). *For a, b purely infinite we have $a \leqq b$ if and only if $\bar{a} \leqq \bar{b}$.*

Proof. First let $\bar{a} = \bar{b}$; by comparison we may assume that $a \leqq b$. As in the proof of the previous theorem we can find a maximal type s such that $b|s = a|s \neq 0$. If $\bar{a} \cap s' \neq 0$, we restrict ourselves to it and repeat the

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construction to contradict the maximality of s. Thus $\bar{a} = s$ and since $\bar{b} = \bar{a}$ we have $a = b$. Now, if $\bar{a} \leqq \bar{b}$ we have a and $b|\bar{a}$ having the same type, so that they are equal and $a \leqq b$.

Since for any a we have $\bar{a} = \overline{\infty a}$, we see that the set of all types is in a one-to-one correspondence with the set of all purely infinite elements (idem-multiple in Tarski's terminology) which preserves the ordering. It is easy to see that if s corresponds to ∞b, then $a|s$ is just the infimum of a and ∞b: we have $s = \bar{b}$ and if $c \leqq a$, $\bar{c} \leqq s$, then $c \leqq \infty b$ so that $a|s$ is $\leqq a$ and also $\leqq \infty b$; on the other hand any element $\leqq a$ and $\leqq \infty b$ has type $\leqq s$ and is thus $\leqq a|s$. It is in exactly this way that Fillmore [**1**] bypassed the introduction of types by assuming instead that any two elements have an infimum.

The next theorem will be very useful in structure theory.

THEOREM (5.4). *If $a \in A$ and n is any cardinal, then there exist b, c such that $a = nb + c$ and c contains no non-zero element nd.*

Proof. First the case where a is finite. If $n = \infty$, take $b = 0$ and $c = a$. Let $n \neq \infty$ and suppose that $a = nb + c$; if $c \ngeqq nd$ for any d, our proof is complete, otherwise write $b = b_1$, $c = c_1$ and suppose that for all ordinals $\tau < \rho$ we have $a = nb_\tau + c_\tau$ with $b_{\tau+1} > b_\tau$ and all c_τ containing some non-zero nd_τ. If $\rho = \sigma + 1$, let $c_\sigma = nd_\sigma + c_{\sigma+1}$ so that $a = n(b_\sigma + d_\sigma) + c_{\sigma+1}$; then the element $b_{\sigma+1} = b_\sigma + d_\sigma > b_\sigma$ since $d_\sigma \neq 0$ and $b_\sigma \leqq a$ is finite. If ρ is a limit ordinal, it has to be countable (because of Axiom VIII) and so there exists a cofinal sequence τ_i. By Axiom V we obtain $a = nb' + c'$, where $b' > b_\tau$ for all τ; if c' contains no non-zero nd, our proof is complete, otherwise we write $c' = nd + c_\rho$, $b_\rho = b' + d$ to obtain $a = nb_\rho + c_\rho$ with $b_\rho > b_\tau$ for all $\tau < \rho$. But this process cannot continue since the set of all countable ρs is not countable and Axiom VIII is assumed. Thus we have the result in the case where a is finite. For the general case, write $a = a_1 + a_2$ with a_1 purely infinite and a_2 finite. If the given cardinal $n = \infty$, this is the desired decomposition, since $a_1 = \infty a_1$. If n is finite, we have $a_2 = nb + c$ with c containing no non-zero nd, and then $a = n(a_1 + b) + c$.

6. Multiplicity free elements.

Definition (6.1). An element is multiplicity free if it has no non-zero part of the form $2b$.

LEMMA (6.1). *The element a is multiplicty free if and only if $a = b + c$ implies $b \perp c$. If a is multiplicity free and $b \leqq a$, then b is also. Every multiplicity free element is finite. The sum of pairwise disjoint multiplicity free elements is multiplicity free.*

Proof. Only the last assertion may need a proof. If $a_i \perp a_j$ and $\sum a_i \geqq 2b$, then $a_i \geqq 2(b|\bar{a}_i)$; hence $b|\bar{a}_i = 0$ for all i and $b = b|\bigcup \bar{a}_i = \sum (b|\bar{a}_i) = 0$.

THEOREM (6.1). *Let a, b be multiplicity free. If $\bar{a} = \bar{b}$, then $a = b$; if $\bar{a} \leqq \bar{b}$, then $a \leqq b$.*

Proof. Since the types of $a|s$ and $b|s$ will be the same in case $\bar{a} = \bar{b}$, we may by comparison restrict ourselves to the case $a \leqq b$. Thus $b = a + c$ and hence $\bar{c} \leqq \bar{b} = \bar{a}$; but b is multiplicity free and thus $a \perp c$, or $c = 0$. For the second compare a, b: $a|s \leqq b|s$, $a|s' \geqq b|s'$. This implies that $\bar{a} \cap s' \geqq \bar{b} \cap s'$, hence $\bar{a} \cap s' = \bar{b} \cap s'$, or $\overline{a|s'} = \overline{b|s'}$; but these elements are multiplicity free, and so $a|s' = b|s'$. Since $a|s \leqq b|s$, we have $a \leqq b$.

THEOREM (6.2). *If $\bar{a} \leqq \bar{c}$ and $b \leqq a$ with b multiplicity free, then $b \leqq c$.*

Proof. We compare a to c and b to c: $a|s \geqq c|s$, $a|s' \leqq c|s'$, and $b|t \leqq c|t$, $b|t' \geqq c|t'$. Since $b \leqq a$, we have $b|(t' \cap s') = c|(s' \cap t')$. If $r = s \cap t'$, then $c|r \leqq b|r$ so that $c|r$ is multiplicity free; on the other hand, we have

$$\overline{c|r} = \bar{c} \cap r \geqq \bar{a} \cap r \geqq \bar{b} \cap r = \overline{b|r}$$

so that $c|r = b|r$. Therefore $b|t' = c|t'$ and so $b \leqq c$.

COROLLARY. *If $\bar{a} = \bar{b}$ and b is multiplicity free, then $b \leqq a$.*

LEMMA (6.2). *If $c \leqq b$ and b is multiplicity free, then $c = b|\bar{c}$.*

Proof. If $d \leqq b$ and $\bar{d} \leqq \bar{c}$, then d is multiplicity free and hence $d \leqq c$; thus c is the largest part of b of type $\bar{c}$.

Definition (6.2). An element is irreducible if it covers 0. A type is primary if it covers $\emptyset$. An element is primary if its type is primary.

The following are typical theorems in representation theory.

THEOREM (6.3). *An element is irreducible if and only if it is primary and multiplicity free.*

Proof. Let a be irreducible and $a = 2b + c$. Then either $2b = 0$ or $c = 0$; but if $c = 0$, then $a = b + b$ so that again $b = 0$. Thus a is multiplicity free. Suppose that $\bar{b} \leqq \bar{a}$ so that $a|\bar{b} \leqq a$, and hence either $a|\bar{b} = 0$ or $a|\bar{b} = a$; in the first case we have $\bar{b} = \emptyset$ and in the second $\bar{b} = \bar{a}$. Thus a is primary. Conversely, suppose that a is primary and multiplicity free. Let $a = b + c$; then $b \perp c$ and $\bar{a} = \bar{b} \cup \bar{c}$ so that either $\bar{b}$ or $\bar{c}$ is $\emptyset$. But then either $b = 0$ or $b = a$, and a is irreducible.

THEOREM (6.4). *If a is primary, b is irreducible, and $b \leqq a$, then $a = nb$ for some cardinal n.*

Proof. We have $a = nb + c$, where $c \ngeqq b$. Then $\bar{c} \leqq \bar{a}$ so that either $\bar{c} = \emptyset$ or $\bar{c} = \bar{a}$. In the first case, our proof is complete. In the second, since $\bar{b} = \bar{a}$ (because $\bar{b} \leqq \bar{a}$ and $b \neq 0$), we have $\bar{c} = \bar{b}$ and by the corollary above $b \leqq c$ which is impossible. Then $c = 0$ in this case also.

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Corollary. *If a is primary and b and c are irreducible parts of a, then $b = c$.*

Proof. We have $a = nb$, $a = mc$, hence $\bar{b} = \bar{c}$ and thus $b = c$.

7. Classification of types.

Definition (7.1). A type s is called discrete (write $s \leqq \mathrm{I}$) if $s = \bar{a}$ for some multiplicity free element a. A type t is called continuous if $t \perp s$ for all $s \leqq \mathrm{I}$. For a continuous finite type we write $s \leqq \mathrm{II}$; for a continuous type t disjoint from all finite types we write $t \leqq \mathrm{III}$. If $\bar{a} \leqq j$ ($j = \mathrm{I}, \mathrm{II}, \mathrm{III}$), we say that a has type j or is of type j.

Theorem (7.1). *Any type s is the disjoint union of types s_{I}, s_{II}, s_{III} where $s_j \leqq j$; this decomposition is unique. Any element a is the disjoint sum of elements a_{I}, a_{II}, a_{III} where a_j has type j; this decomposition is also unique.*

Proof. Using Lemma (6.1) we find the largest $s_{\mathrm{I}} \leqq s$ which is discrete. Then $s \cap s_{\mathrm{I}}'$ is continuous. Using the results of § 5 we find the largest $s_{\mathrm{II}} \leqq s \cap s_{\mathrm{I}}'$ which is finite. Then $s_{\mathrm{III}} = s \cap s_{\mathrm{I}}' \cap s_{\mathrm{II}}'$ is $\leqq \mathrm{III}$ and the requirements are fulfilled. Uniqueness is obvious. Given a we write $\bar{a}$ as a disjoint union of types $\leqq \mathrm{I}$, II, III and restrict a accordingly.

8. Structure of type I elements. This is completely determined by the following.

Theorem (8.1). *If a has type $\leqq \mathrm{I}$, then $a = \sum n_i b_i$, where the n_i are distinct cardinals and the b_i are pairwise disjoint multiplicity free elements; this form is unique.*

Proof. There exists some multiplicity free element b such that $\bar{b} = \bar{a}$, and then $b \leqq a$ by § 6. Thus $a = nb + c$ with $c \ngeqq b$. If $s = \bar{a} \cap \bar{c}'$, then $a|s = n(b|s)$, and observe that $s \neq \emptyset$ for otherwise $\bar{c} \geqq \bar{a}$ and this by Theorem (6.2) would imply that $c \geqq b$ which is not the case. We can then obtain a maximal $s \leqq \bar{a}$ such that $a|s = \sum n_i b_i$ with n_i, b_i as required by the theorem. Since $a|s'$ will also be of type I, we see that $s = \bar{a}$. The proof of uniqueness is straightforward.

We shall reformulate this theorem to make it similar to the corresponding theorems for types II, III.

Theorem (8.2). *Let b have type $\leqq \bar{a}$ where a is multiplicity free. Then $b = \sum n_i(a|s_i)$, where the n_i are distinct and the s_i pairwise disjoint. Given a, then n_i, s_i are unique.*

9. Structure of type II elements.

Lemma (9.1). *If a has continuous type and $2 \leqq n < \infty$, then there exists a unique b such that $a = nb$.*

Uniqueness is proved in [**5**, Theorem 2.34], and the proof, although complicated, is valid for any cardinal algebra. A short proof can be given using Theorem (4.2). Existence for finite elements follows immediately from Theorem (5.4).

For rational $p > 0$, $p = m/n$, we shall write pa for the element mb, where $nb = a$; this is of course unique, and for purely infinite a we have $pa = a$.

LEMMA (9.2). *Let q, r, p_k, and $p = \sum p_k$ be rationals >0, and a, a_i have continuous types. Then $q(ra) = (qr)a$, $q(\sum a_i) = \sum (qa_i)$, $pa = \sum (p_k a)$, and $p(\sum a_i) = \sum_{k,i} p_k a_i$.*

Proof. The only part that may need proof is the third relation for an infinite series. Since by the remainder axiom we have $\sum_{k=1}^{\infty} (p_k a) \leqq pa$, and also for any m we have

$$pa \leqq \sum_{k=1}^{n} (p_k a) + \frac{1}{m} a$$

for large enough n, we obtain

$$pa \leqq \sum (p_k a) + \frac{1}{m} a$$

for all m, and hence $pa \leqq \sum (p_k a)$ for finite a. For purely infinite a all relations are trivial.

Now assume that $\bar{b} \leqq \bar{a}$, where a has type II and consider first the case of finite b.

For p a non-negative rational let s_p be the largest type $s \leqq \bar{a}$ such that $b|s \leqq p(a|s)$. We then have the following.

LEMMA (9.3). *For any p, $s_p = \bigcup_{p<q} s_q$.*

The proof is straightforward. This lemma shows that if we assign to each interval $(p, q]$ the type $s_q \cap s_p'$, we can extend this map to a spectral measure in the sense of the following definition.

Definition (9.1). A spectral measure is a map μ from the Borel sets of the real line into the Boolean σ-algebra of all types $\leqq \bar{a}$ such that:

(i) if the sets E_i are pairwise disjoint, then $\mu(E_i) \perp \mu(E_j)$ and $\mu(\bigcup E_i) = \bigcup \mu(E_i)$,

(ii) $\mu(E) = 0$ if $E \subseteq (-\infty, 0)$.

Observe that such measures correspond to Borel measurable non-negative functions on any measurable space whose ring of measurable sets modulo null sets is isomorphic to the Boolean algebra of all types $\leqq \bar{a}$.

We shall use spectral measures to integrate elements as follows. Let $a_0 = 0$ and let a_n be the element

$$\sum_{k=1}^{n(2^n)} \frac{k}{2^n} a \left| \mu\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right] \right. .$$

Then $a_n \leqq a_{n+1}$ and we set b_{n+1} for the unique b such that $a_{n+1} = a_n + b$. The sum $\sum b_i$ will be written as $\int pa|d\mu(p)$.

LEMMA (9.4). *If a is finite, then so is the integral $\int pa|d\mu(p)$.*

Proof. By considering the sum defining a_n we see that $b_n \leqq c_n + (1/2^n)a$, where c_n is finite of type $\leqq \mu(n + 1/2^n, n + 1 + 1/2^{n+1}]$. Thus

$$\sum b_i \leqq \sum c_i + a$$

and as the c_i are pairwise disjoint and finite, so is their sum.

We can now prove the following structure theorem.

THEOREM (9.1). *Let b be finite, and $\bar{b} \leqq \bar{a}$, where a is finite of type* II. *Then $b = \int pa|d\mu(p)$, where μ is the spectral measure determined by the family s_p. This representation is unique.*

Proof. By using the facts $\mu(p, q] \subseteq s_q$ and $\mu(p, q] \perp s_p$ we obtain easily that $a_n \leqq b$; since $\int pa|d\mu(p) = \sum b_i$ and $b_1 + b_2 + \ldots + b_n = a_n$, we have $\int pa|d\mu(p) \leqq b$. On the other hand, the definition of the integral shows that $b \leqq \int pa|d\mu(p) + (1/2^m)a$ for all m, and since a is finite, equality follows. Now if $b = \int pa|d\nu(p)$ for some spectral measure ν, then for any type $s \leqq \nu(-\infty, p]$ we have $b|s \leqq p(a|s)$ hence $\nu(-\infty, p] \leqq s_p$. Also,

$$t \perp \nu(-\infty, q]$$

implies $b|t \geqq q(a|t)$ and thus $s_p \leqq \nu(-\infty, q]$ for all $p < q$. Since ν is a spectral measure we obtain $s_p \leqq \nu(-\infty, p]$, or $\mu = \nu$.

Now the case of purely infinite b.

THEOREM (9.2). *Let b be purely infinite, $\bar{b} \leqq \bar{a}$ with a finite. Then $b = \infty(a|\bar{b})$.*

Proof. We compare b and na, where n is a finite cardinal: $b|s_n \geqq na|s_n$, $b|s_n' \leqq na|s_n'$. Since b is purely infinite and na is finite, we have $b|s_n' = 0$, or $b = b|s_n$. Therefore $b \geqq n(a|s_n) \geqq na|\bar{b}$ for all n, and hence $b \geqq \infty(a|\bar{b})$. Now apply Axiom VII to obtain an element c such that $b = \infty(a|\bar{b}) + c$, where $c \ngeqq \infty(a|\bar{b})$. Since $\bar{b} \leqq \bar{a}$ we have $\bar{c} \leqq \bar{a}$, and we wr te $c = c_1 + c_2$ with c_1 finite and c_2 purely infinite. We claim that $\infty(a|\bar{b}) + c_1 = \infty(a|\bar{b})$. By comparison we can consider the cases $a|\bar{b} \leqq c_1$, $c_1 \leqq a|\bar{b}$ without loss of generality. In the second, our claim is obvious; in the first, the above argument where $\infty(a|\bar{b})$ plays the role of b and c_1 the role of a yields $c_1 \leqq \infty(a|\bar{b})$ which again proves our claim. Thus $b = \infty(a|\bar{b}) + c$, where c is purely infinite $\ngeqq a|\bar{b}$. Now, if the type $\bar{a} \cap \bar{b} \cap \bar{c}' = s = \emptyset$, then $\bar{c} = \overline{a|\bar{b}}$ and by the first part of the argument we obtain $c \geqq \infty(a|\bar{b}) \geqq a|\bar{b}$ which is not the case. Thus $s \neq \emptyset$ and hence $b|s = \infty(a|s) \neq 0$. The maximal such type s must be $\bar{b}$, for otherwise we repeat the argument on $\bar{b} \cap s$ to obtain a contradiction to the maximality of s. Thus $b = \infty(a|\bar{b})$.

We shall end this section by characterizing the elements of finite type.

THEOREM (9.3). *The type of a is finite if and only if a is the sum of finite elements.*

Proof. Since the set of finite types is a filter closed under any disjoint unions, it is closed under any unions, so that the type of $\sum a_i$ is finite if all a_i are finite. Conversely, if $\bar{a}$ is finite and we write $a = a_1 + a_2$ with a_1 purely infinite and a_2 finite, we see that $a_1 = \infty c$ for some finite c by Theorem (9.2), and thus a is the sum of finite elements.

10. Structure of type III elements. This is quite simple.

THEOREM (10.1). *Let $\bar{b} \leqq \bar{a}$ where a is of type* III*; then $b = a|\bar{b}$.*

Since we have $\bar{b} = \bar{b} \cap \bar{a} = \overline{a|\bar{b}}$, and as b and $a|\bar{b}$ are both purely infinite, we see that they are equal by Theorem (5.3).

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