15
Twisted reduced models

There is an elegant alternative to the quenched Eguchi–Kawai model, described in the previous chapter, which also preserves the $U(1)^d$ symmetry. It was proposed by González-Arroyo and Okawa [GO83a, GO83b] on the basis of a twisting reduction prescription. The corresponding lattice version of the reduced model lives on a hypercube with twisted boundary conditions. The twisted reduced model for a scalar field was constructed by Eguchi and Nakayama [EN83].

The twisted reduced models reveal interesting mathematical structures associated with representations of the Heisenberg commutation relation (in the continuum) or its finite-dimensional approximation by unitary matrices (on the lattice). In contrast to the quenched reduced models which describe only planar graphs, the twisted reduced models make sense order by order in $1/N$ and even at finite N. In this case they are associated with gauge theories on noncommutative space, whose limit of large noncommutativity is given by planar graphs thereby reproducing a d-dimensional Yang–Mills theory at large N.

We begin this chapter with a description of the twisted reduced models first on the lattice and then in the continuum and show how they describe planar graphs of a d-dimensional theory.

15.1 Twisting prescription

We start by working on a lattice to make the results rigorous and then repeat them for the continuum.

The twisting reduction prescription is a version of the general reduction prescription described in Sect. 14.1. We again perform the unitary transformation (14.4), where the matrices $D(x)$ are now expressed via a

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set of d (unitary) $N \times N$ matrices Γ_μ by

$$D(x) \;=\; \Gamma_1^{x_1/a}\Gamma_2^{x_2/a}\cdots\Gamma_d^{x_d/a}\,, \tag{15.1}$$

and the coordinates of the (lattice) vector x_μ are measured in the lattice units so that all the exponents are integral.

The matrices Γ_μ obey the Weyl–'t Hooft commutation relation

$$\Gamma_\mu\Gamma_\nu \;=\; Z_{\mu\nu}\Gamma_\nu\Gamma_\mu \tag{15.2}$$

with $Z_{\mu\nu} = Z_{\nu\mu}^\dagger$ being elements of $Z(N)$ and d is assumed to be even. These matrices Γ_μ, which are called *twist eaters*, will be constructed explicitly in a moment.

Substituting (14.4) with $D(x)$ given by Eq. (15.1) into Eq. (14.1), we obtain the following partition function of the *twisted reduced model* [EN83]

$$Z_{\mathrm{TRM}} \;=\; \int \mathrm{d}\tilde{\varphi}\, \mathrm{e}^{-S_{\mathrm{TRM}}[\tilde{\varphi}]} \tag{15.3}$$

with the action

$$S_{\mathrm{TRM}}[\tilde{\varphi}] \;=\; -N\sum_\mu \operatorname{tr}\Gamma_\mu\tilde{\varphi}\Gamma_\mu^\dagger\tilde{\varphi} + N\operatorname{tr}\widetilde{V}(\tilde{\varphi})\,. \tag{15.4}$$

The "derivation" is again modulo the volume factor in the action.

Correspondingly, an analog of Eq. (14.9) for the twisting reduction prescription is given by

$$\Big\langle F[\varphi_x]\Big\rangle \;\stackrel{\text{red.}}{=}\; \Big\langle F\Big[D^\dagger(x)\tilde{\varphi}D(x)\Big]\Big\rangle_{\mathrm{TRM}}\,, \tag{15.5}$$

where the average on the RHS is calculated for the twisted reduced model:

$$\Big\langle F[\tilde{\varphi}]\Big\rangle_{\mathrm{TRM}} \;=\; Z_{\mathrm{TRM}}^{-1}\int \mathrm{d}\tilde{\varphi}\, \mathrm{e}^{-S_{\mathrm{TRM}}[\tilde{\varphi}]}F[\tilde{\varphi}]\,. \tag{15.6}$$

The equality of the LHS of Eq. (15.5) (calculated for the d-dimensional theory (14.1)) and the RHS (calculated for the twisted reduced model) takes place in the planar limit, i.e. for $N = \infty$, owing to the explicit form of $D(x)$ given by Eq. (15.1).

Problem 15.1 Demonstrate that the order of Γ_μ in Eq. (15.1) is inessential.

Solution Let us define a more general path-dependent factor

$$D(C_{x0}) \;=\; \boldsymbol{P}\prod_i \Gamma_{\mu_i}\,, \tag{15.7}$$

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where the path-ordered product runs over all links forming a path C_{x0} from the origin to the point x. Owing to Eq. (15.2), a change of the path multiplies $D(x)$ by the Abelian factor

$$Z(C) = \prod_{p\in S:\partial S=C} Z^*_{\mu\nu}\,, \tag{15.8}$$

where (μ,ν) is the orientation of the plaquette p. The product runs over any surface spanned by the closed loop C, which is obtained by passing the original path forward and the new path backward. Owing to the Bianchi identity

$$\prod_{p\in\mathrm{cube}} Z_{\mu\nu} = 1\,, \tag{15.9}$$

where the product goes over six plaquettes forming a three-dimensional cube on the lattice, the product on the RHS of Eq. (15.8) does not depend on the form of the surface S and is a functional solely of the loop C.

It is now easy to see that under this change of the path we obtain

$$D^\dagger_{ij}(x)D_{kl}(x) \longrightarrow |Z(C)|^2\, D^\dagger_{ij}(x)D_{kl}(x) \tag{15.10}$$

and the path-dependence is canceled because $|Z(C)|^2 = 1$. This is a general property which holds for the twisting reduction prescription of any even representation of $SU(N)$ (i.e. invariant under transformations from the center $Z(N)$).

Let us now explicitly construct the matrices Γ_μ which obey Eq. (15.2). We begin with the case of $d = 2$, where Γ_1 and Γ_2 can be chosen to coincide with the $L\times L$ Weyl "clock" and "shift" matrices [Wey31]:

$$\mathcal{Q} = \mathrm{diag}\left(1,\omega,\omega^2,\ldots,\omega^{L-1}\right) \tag{15.11}$$

and

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{15.12}$$

which are unitary and obey

$$\mathcal{Q}^L = 1 = \mathcal{P}^L\,, \tag{15.13}$$

$$\mathcal{P}\,\mathcal{Q} = \omega\,\mathcal{Q}\,\mathcal{P} \tag{15.14}$$

with $\omega \in Z(L)$. A solution to Eq. (15.2) in $d = 2$ is obviously given by $\Gamma_1 = \mathcal{P}$, $\Gamma_2 = \mathcal{Q}$ providing $L = N$ and $\omega = Z_{12} = \mathrm{e}^{2\pi\mathrm{i}/L}$.

For even $d > 2$, the factor of $Z_{\mu\nu}$ can always be written as

$$Z_{\mu\nu} = \mathrm{e}^{2\pi\mathrm{i}n_{\mu\nu}/N} \in Z(N) \qquad (n_{\mu\nu} = -n_{\nu\mu} \in \mathbb{Z}_N)\,, \tag{15.15}$$

where $n_{\mu\nu}$ can be represented in a canonical skew-diagonal form

$$n_{\mu\nu} = \begin{pmatrix} 0 & +n_1 & & & \\ -n_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & +n_{d/2} \\ & & & -n_{d/2} & 0 \end{pmatrix}. \tag{15.16}$$

Although a solution to Eq. (15.2) is known for an arbitrary set of $\{n_1, \ldots, n_{d/2}\}$ (it is described in Problem 15.3), it is enough for our purposes to consider the simplest case of $n_1 = n_2 = n$ in $d = 4$ dimensions.

The idea is now to combine $\Gamma_1, \ldots, \Gamma_4$ into two pairs: Γ_1, Γ_2 and Γ_3, Γ_4, so the commutator of the two matrices from the same pair is similar to that in two dimensions, while the matrices from different pairs commute. These rules are prescribed by an explicit form of $n_{\mu\nu}$ given for this simplest twist by

$$n_{\mu\nu} = \begin{pmatrix} 0 & +n & & \\ -n & 0 & & \\ & & 0 & +n \\ & & -n & 0 \end{pmatrix}. \tag{15.17}$$

The solution to Eq. (15.2) can then be represented by a direct product of the $L \times L$ Weyl matrices (15.11) and (15.12):

$$\left.\begin{aligned} \Gamma_1 &= \mathcal{P} \otimes \mathbb{I}\,, & \Gamma_2 &= \mathcal{Q} \otimes \mathbb{I}\,, \\ \Gamma_3 &= \mathbb{I} \otimes \mathcal{P}\,, & \Gamma_4 &= \mathbb{I} \otimes \mathcal{Q}\,. \end{aligned}\right\} \tag{15.18}$$

In other words, the solution is given on a subgroup $SU(L) \otimes SU(L)$ of the group $SU(N)$, which is possible only if $N = L^2$ and $n = L$. Once again, this solution is not the most general one, but it is enough for our purposes. Note that $\Gamma_\mu^L = 1$ for this simplest solution.

Problem 15.2 Extend the solution (15.18) to d dimensions assuming that all $n_i = L^{d/2-1}$ in Eq. (15.16).

Solution For such $n_{\mu\nu}$ the solution may be given on a subgroup $\prod_1^{d/2} \otimes SU(L)$ of $SU(N)$ so that $N = L^{d/2}$ and Γ_i, Γ_{i+1} (odd $i = 1, 3, \ldots, d-1$) can be chosen as a direct product of the Weyl matrices for the ith of $SU(L)$s and the unit matrices for the others. Again $\Gamma_\mu^L = 1$ for this simplest twist.

Problem 15.3 Find a solution to Eq. (15.2) for a general matrix $n_{\mu\nu}$.

Solution We proceed similarly to the previous Problem. Given N and the $d/2$ numbers $n_i \in \mathbb{Z}_N$, we introduce the integers $p_i = N/\gcd(n_i, N)$. A solution to

Eq. (15.2) exists if the product $p_1 \cdots p_{d/2}$, which plays the role of the dimension of irreducible representation of the algebra, divides N. In that case we write

$$N = p_0 \prod_{i=1}^{d/2} p_i \qquad (p_0 \in \mathbb{Z}) \tag{15.19}$$

and the solution may be given on the subgroup $SU(p_0) \otimes SU(p_1) \otimes \cdots \otimes SU(p_{d/2})$ of $SU(N)$ such that Γ_i, Γ_{i+1} are constructed as a direct product of the Weyl matrices on $SU(p_i)$ and unit matrices for the others. The subgroup of $GL(p, \mathbb{C})$ consisting of matrices which commute with the twist eaters Γ_μ is then $GL(p_0, \mathbb{C})$.

This solution was found in [BG86, LP86], where it was shown that Eq. (15.19) is a necessary and sufficient condition for the existence of solutions to Eq. (15.2). The simplest solution from the previous Problem is associated with $p_0 = 1$, $p_1 = \cdots = p_{d/2} = L$. Another simple example is $N = p_0 L^{d/2}$, $n_i = p_0 L^{d/2-1}$, $p_1 = \cdots = p_{d/2} = L$ when $\Gamma_\mu^L = 1$.

15.2 Twisted reduced model for scalars

We shall now demonstrate how the twisted reduced model, which is defined for a scalar field by Eqs. (15.3) and (15.6), reproduces [EN83] at large N the planar graphs of the d-dimensional theory.

In principle, we may try to simply repeat the perturbative analysis of Sect. 14.1, representing the propagator of $\tilde{\varphi}_{ij}$ via the Γ_μ and expecting that momentum integrals will be recovered after summing over indices circulating in closed loops owing to the explicit form of the twist eaters. This is indeed the case.

It is more instructive, however, to proceed in a slightly different way explicitly introducing the momentum variable via a sort of a Fourier transformation on $gl(N; \mathbb{C})$ (general complex $N \times N$ matrices).

A convenient Weyl basis on $gl(L; \mathbb{C})$ is given [Hoo78, Hoo81] by (symmetric) products of the "clock" and "shift" matrices (15.11) and (15.12).

Let us introduce L^2 matrices

$$J_{m_1,m_2} = \mathcal{P}^{m_1} \mathcal{Q}^{m_2} \omega^{-m_1 m_2/2}, \tag{15.20}$$

where $m_1, m_2 \in \mathbb{Z}_L$. An arbitrary $L \times L$ matrix M can be expanded in this basis:

$$M^{ij} = \frac{1}{N^2} \sum_{m_1,m_2=1}^{L} J^{ij}_{m_1,m_2} M(m_1, m_2), \tag{15.21}$$

where $M(m_1, m_2)$ are certain expansion coefficients.

We see that a pair of integers m_1 and m_2, forming a two-dimensional vector, $m = \{m_1, m_2\} \in \mathbb{Z}^2_L$, is naturally associated with this construction. As we shall see in a moment, these integers label momenta on an $L \times L$ periodic lattice.

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An extension of this construction to arbitrary (even) d dimensions is obvious for the simplest twist given by the matrix (15.16) with $n_i = L^{d/2-1}$ for all $i = 1, \ldots, d/2$, which is considered in Problem 15.2 on p. 354. We introduce the basis on $gl(N;\mathbb{C})$:

$$J_m \;=\; \Gamma_1^{m_1} \cdots \Gamma_d^{m_d}\, \mathrm{e}^{-\pi \mathrm{i} \sum_{\mu<\nu} m_\mu n_{\mu\nu} m_\nu / N}, \tag{15.22}$$

where $m = \{m_1, \ldots, m_d\} \in \mathbb{Z}_L^d$ is a d-dimensional vector (remember that $N = L^{d/2}$). The last factor* again makes the product symmetric.

There exist $L^d = N^2$ independent orthogonal generators J_m which obey

$$J_m^\dagger \;=\; J_{-m} \quad (\mathrm{mod}\ L)\,, \tag{15.23}$$

$$\frac{1}{N}\operatorname{tr} J_m J_n^\dagger \;=\; \delta_{mn}\,, \tag{15.24}$$

$$\sum_{m\in\mathbb{Z}_L^d} J_m^{ij} J_m^{\dagger kl} \;=\; N\delta^{il}\delta^{kj}\,, \tag{15.25}$$

$$J_m J_n \;=\; J_{m+n}\, \mathrm{e}^{\pi \mathrm{i} \sum_{\mu,\nu} m_\mu n_{\mu\nu} n_\nu / N} \quad (\mathrm{mod}\ L)\,. \tag{15.26}$$

Equations (15.24) and (15.25) represent, respectively, orthogonality and completeness of the generators. The product of two generators is given explicitly by Eq. (15.26).

An arbitrary $N \times N$ complex matrix M^{ij} can be expanded as

$$M^{ij} \;=\; \frac{1}{N^2} \sum_{m\in\mathbb{Z}_L^d} J_m^{ij}\, M(m) \tag{15.27}$$

and $M(-m) = M^*(m)$ if M is Hermitian as a consequence of Eq. (15.23). Using Eq. (15.24), the coefficient $M(m)$ is given by

$$M(m) \;=\; N \operatorname{tr}\left(M\, J_m^\dagger\right). \tag{15.28}$$

Equation (15.27) simply extends Eq. (15.21) to the multidimensional case.

A mapping of the twisted reduced model into a d-dimensional field theory can be established as follows.

We expand the matrix $\tilde{\varphi}$ in the basis of J_m:

$$\tilde{\varphi} \;=\; \frac{1}{N^2} \sum_{m\in\mathbb{Z}_L^d} J_m\, \varphi(m)\,. \tag{15.29}$$

* Strictly speaking, we assume that $n_{\mu\nu}$ is even and L is odd for the J_m to obey a periodicity property $J_{\ldots,m_i+L,\ldots} = J_{\ldots,m_i,\ldots}$. This is necessary only for finite N since this periodicity is lost as $N \to \infty$.

The measure (13.2) for the averaging over the matrices $\tilde{\varphi}$ is then represented by

$$\mathrm{d}\tilde{\varphi} \;=\; \prod_{m\in\mathbb{Z}_L^d} \mathrm{d}\varphi(m)\,. \tag{15.30}$$

The substitution of the expansion (15.29) into the kinetic part of the action of the twisted reduced model yields

$$\begin{aligned} S^{(2)}_{\mathrm{TRM}} &\equiv N\,\mathrm{tr}\left(\frac{M}{2}\tilde{\varphi}^2 - \sum_\mu \Gamma_\mu\tilde{\varphi}\Gamma_\mu^\dagger\tilde{\varphi}\right) \\ &= \frac{1}{N^2}\sum_{m\in\mathbb{Z}_L^d}\left(\frac{M}{2} - \sum_\mu \cos\frac{2\pi\sum_\nu n_{\mu\nu}m_\nu}{N}\right)\varphi(-m)\varphi(m) \end{aligned} \tag{15.31}$$

which coincides with the kinetic part of the action of a single-component scalar field on a d-dimensional lattice of spatial extent $L^d = N^2$ with periodic boundary conditions.

In the latter case, we substitute the Fourier expansion

$$\tilde{\varphi}_x \;=\; \frac{1}{L^d}\sum_{m\in\mathbb{Z}_L^d} \mathrm{e}^{2\pi\mathrm{i}\sum_{\mu,\nu} x_\mu n_{\mu\nu}m_\nu/aN}\varphi(m)\,, \tag{15.32}$$

which yields

$$\begin{aligned} S^{(2)} &\equiv \sum_x\left(\frac{M}{2}\tilde{\varphi}_x^2 - \sum_\mu \tilde{\varphi}_x\tilde{\varphi}_{x+a\hat{\mu}}\right) \\ &= \frac{1}{L^d}\sum_{m\in\mathbb{Z}_L^d}\left(\frac{M}{2} - \sum_\mu \cos\frac{2\pi\sum_\nu n_{\mu\nu}m_\nu}{N}\right)\varphi(-m)\varphi(m)\,. \end{aligned} \tag{15.33}$$

The number of degrees of freedom is the same in both cases: there are N^2 elements of the matrix $\tilde{\varphi}$ in the twisted reduced model which matches the $L^d = N^2$ sites of the lattice. The expansion coefficients in Eqs. (15.29) and (15.32) are just the same! Correspondingly, the measure for path integration over φ_x is

$$\prod_x \mathrm{d}\varphi_x \;=\; \prod_{m\in\mathbb{Z}_L^d} \mathrm{d}\varphi(m) \tag{15.34}$$

which coincide with the measure (15.30).

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In other words, the degrees of freedom described by the matrix (15.29) or the field (15.32) are the same.

The coincidence of the actions of the two theories at finite N is only for the kinetic part of the actions which is quadratic in fields. This is no longer the case for interaction terms. For the simplest cubic interaction, we have, using Eq. (15.26),

$$\begin{aligned} N\,\mathrm{tr}\,\tilde{\varphi}^3 &= \frac{1}{N^6}\sum_{m_1,m_2,m_3}\varphi(m_1)\varphi(m_2)\varphi(m_3)N\,\mathrm{tr}\,J_{m_1}J_{m_2}J_{m_3} \\ &= \frac{1}{N^4}\sum_{m,n}\varphi(-m-n)\varphi(m)\varphi(n)\,\mathrm{e}^{\pi\mathrm{i}\sum_{\mu,\nu}m_\mu n_{\mu\nu}n_\nu/N} \end{aligned} \tag{15.35}$$

which has an extra phase in contrast to the cubic interaction of a single-component scalar field outlined in Sect. 2.3.*

The presence of this factor is crucial in showing that the twisted reduced model at $N=\infty$ correctly reproduces planar graphs of the d-dimensional theory. This happens because of a remarkable *theorem* proven in [EN83, GO83b] which states that

(1) the phases cancel out in planar graphs,
(2) the phases remain in nonplanar graphs suppressing their contribution as $N\to\infty$.

In order to sketch a proof of the theorem, we introduce the momentum variables $p_\mu \equiv 2\pi\sum_\nu n_{\mu\nu}m_\nu/aN$ and $q_\mu \equiv 2\pi\sum_\nu n_{\mu\nu}n_\nu/aN$, which become continuous momenta from the first Brillouin zone $(-\pi/a,\pi/a]$ as $N\to\infty$, and pass to the momenta p_i, p_j, and p_k associated with the single lines carrying the indices i, j, and k as in Eq. (14.17).

The phase factor in Eq. (15.35) can then be rewritten in the form

$$\mathrm{e}^{\pi\mathrm{i}\sum_{\mu,\nu}m_\mu n_{\mu\nu}n_\nu/N} = \mathrm{e}^{-\mathrm{i}p\theta q/2} = \mathrm{e}^{-\mathrm{i}(p_i\theta p_j+p_j\theta p_k+p_k\theta p_i)/2}, \tag{15.36}$$

where we have used the rules of matrix multiplication of the Lorentz indices so that

$$p_i\theta p_j = \sum_{\mu,\nu}p_i^\mu\theta_{\mu\nu}p_j^\nu \tag{15.37}$$

and

$$\theta_{\mu\nu} = \frac{a^2N}{2\pi}n_{\mu\nu}^{-1}. \tag{15.38}$$

* We shall demonstrate in the next chapter that the twisted reduced model at finite N is precisely equivalent to a theory on a noncommutative lattice.

A proof of the theorem simplifies [IIK00] after rewriting the phase factor according to Eq. (15.36). Now each factor of $\exp(-\mathrm{i}p_i\theta p_j/2)$ can be assigned to each of the three propagators which meet at a vertex. The overall phase of a graph can be computed by summing up the phases associated with both ends of each of the propagator lines. Since the two ends of an internal double line are oriented in an opposite way for a planar graph, the contributions of the two ends mutually cancel. Therefore, the overall phase of a planar graph involves only external momenta and is the same to all orders of perturbation theory. For example, there is no such phase for vacuum graphs, while the RHS of Eq. (15.36) is reproduced for each planar graph contributing to the three-point vertex. This phase depending on external momenta is simply related to the mapping (15.5) of observables.

The cancellation of the phases does not occur for crossing lines which are inevitably present for nonplanar graphs. For example, the nonplanar graph depicted in Fig. 11.3 has the extra factor of $\exp(\mathrm{i}p\theta q)$ where p and q are momenta associated with the two lines which cross over each other. This is because if these two lines were to form a four-gluon vertex, instead of crossing, it would produce the additional phase factor $\exp(-\mathrm{i}p\theta q)$ and the graph would then be planar.

A nonplanar graph possesses such an extra momentum-dependent phase factor in the integrand, whose rapid oscillations suppress the integral over internal momenta. Note that $\theta_{\mu\nu}$ given by Eq. (15.38) is $\sim L$ so that the integral for a nonplanar diagram of genus h is suppressed by

$$\left(p^{2d}\det_{\mu\nu}\theta_{\mu\nu}\right)^{-h} \sim L^{-hd} \sim N^{-2h} \tag{15.39}$$

in accord with the topological expansion in $1/N$. Here p^2 is typical external momentum associated with the diagram.

If $N \to \infty$ then only planar diagrams survive in the twisted reduced model, thereby reproducing the planar limit of the d-dimensional theory.

Remark on twisted versus quenched models at large but finite N

The size of the lattice associated with the twisted reduced model at large but finite N is $L = N^{2/d}$. This is to be compared with its counterpart for the quenched reduced model where $L = N^{1/d}$ as discussed in the Remark on p. 329. For a given N, the value of L for the twisted reduced model is much larger than for the quenched reduced model. The former therefore provides a more economic approach to the limit of infinite volume.

We shall see in the next chapter that yet another continuum limit, associated with noncommutative theories, can be obtained in the twisted reduced models at the distances $\sim \sqrt{|\theta|} \sim aN^{1/d}$. The corresponding momenta are $p^2 \sim 1/|\theta|$ so that the dimensionless parameter on the LHS of Eq. (15.39) is ~ 1 and each term of the genus expansion is of order one in this continuum limit.

Remark on mapping between matrices and fields

The transition from matrices to functions on a periodic L^d lattice can be formalized by introducing the matrix-valued function [Bar90]

$$\Delta^{ij}(x) = \frac{1}{N^2} \sum_{m \in \mathbb{Z}_L^d} j_m^*(x)\, J_m^{ij}\,, \tag{15.40}$$

where the functions

$$j_m(x) = \mathrm{e}^{2\pi \mathrm{i} \sum_{\mu,\nu} x_\mu n_{\mu\nu} m_\nu / aN} \tag{15.41}$$

form a Fourier basis.

Thus defined $\Delta^{ij}(x)$ possesses the properties

$$\Delta^\dagger(x) = \Delta(x)\,, \tag{15.42}$$

$$N\,\mathrm{tr}\Big[J_m \Delta(x)\Big] = j_m(x)\,, \tag{15.43}$$

$$\sum_x \Delta^{ij}(x)\,\Delta^{kl}(x) = \frac{1}{N}\delta^{il}\delta^{kj}, \tag{15.44}$$

$$\Gamma_\mu\,\Delta(x)\,\Gamma_\mu^\dagger = \Delta(x - a\hat{\mu})\,, \tag{15.45}$$

$$N\,\mathrm{tr}\Big[\Delta(x)\Delta(y)\Big] = \delta_{xy}\,. \tag{15.46}$$

Equation (15.46) represents completeness of the Fourier basis (15.41) in the space of functions on a discrete torus.

Given the definitions (15.29), (15.32), and (15.40), we can directly relate matrices with functions of x by

$$\tilde{\varphi} = \sum_x \Delta(x)\varphi(x) \tag{15.47}$$

and vice versa

$$\varphi(x) = N\,\mathrm{tr}\Big[\tilde{\varphi}\,\Delta(x)\Big]\,. \tag{15.48}$$

In particular, the equivalence of the actions (15.31) and (15.33) is a consequence of the general formula

$$N\,\mathrm{tr}\,\widetilde{\mathcal{L}} = \sum_x \mathcal{L}(x)\,, \tag{15.49}$$

where $\widetilde{\mathcal{L}}$ is arbitrary and $\mathcal{L}(x)$ is related to it by Eqs. (15.47) and (15.48).

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As $N \to \infty$, we approach the limit of an infinite lattice since $aL \to \infty$. Then the discrete variable m_μ is to be substituted by a continuum momentum variable from the first Brillouin zone:

$$k_\mu = \frac{2\pi \sum_\nu n_{\mu\nu} m_\nu}{aN} \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right]. \tag{15.50}$$

The summation over m_μ is to be substituted in all the formulas above by an integration over k_μ:

$$\frac{1}{N^2} \sum_{m \in \mathbb{Z}_L^d} \cdots \longrightarrow \int \prod_{\mu=1}^{d} \frac{\mathrm{d}k_\mu}{2\pi} \cdots . \tag{15.51}$$

For *smooth* configurations when only modes with $|k_\mu| \ll 1/a$ are essential, we can additionally substitute the summation over the lattice sites x by an integration over the continuum variable $x \in \mathbb{R}^d$ (d-dimensional Euclidean space):

$$a^d \sum_x \cdots \Longrightarrow \int \mathrm{d}^d x \cdots . \tag{15.52}$$

Then Eq. (15.49) becomes

$$a^d N \operatorname{tr} \widetilde{\mathcal{L}} \Longrightarrow \int \mathrm{d}^d x \, \mathcal{L}(x) . \tag{15.53}$$

In particular, we have

$$a^d N \operatorname{tr} \mathbb{I} \Longrightarrow \int \mathrm{d}^d x = V \tag{15.54}$$

which relates the (infinite) trace of a unit matrix with the (infinite) volume.

We shall return to the relation between infinite-dimensional matrices (= operators) and functions on $\mathbb{R}^d$ in Sect. 15.4 when discussing a continuum limit of the twisted reduced models.

Remark on $SU(\infty)$ and symplectic diffeomorphisms in $d = 2$

The group $SU(N)$ can be approximated at large N by the group $SL(2;\mathbb{R})$ of *area-preserving* or *symplectic diffeomorphisms* (SDiff) in two dimensions.

It follows from Eq. (15.26) that

$$[J_m \, , J_n] = 2\mathrm{i} \sin \left[\frac{\pi}{N}(m_\mu \varepsilon^{\mu\nu} n_\nu)\right] J_{m+n} \, , \tag{15.55}$$

where $m_\mu \varepsilon^{\mu\nu} n_\nu = m_1 n_2 - m_2 n_1$.

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At large N and $m_\mu \varepsilon^{\mu\nu} n_\nu \ll N$ the sin can be expanded, which yields

$$[J_m\,, J_n] \;\approx\; \mathrm{i}\frac{2\pi}{N}(m_\mu \varepsilon_{\mu\nu} n_\nu) J_{m+n}\,. \tag{15.56}$$

Equation (15.56) is to be compared with the Poisson bracket

$$\begin{aligned} \{j_m, j_n\}_{\mathrm{PB}} &\equiv \varepsilon^{\mu\nu}\partial_\mu j_m \partial_\nu j_n \\ &\propto (m_\mu \varepsilon^{\mu\nu} n_\nu)\, j_{m+n} \end{aligned} \tag{15.57}$$

of the basis functions j_m given by Eq. (15.41). The group $SL(2;\mathbb{R})$ of symplectic diffeomorphisms arose since it is a symmetry of the Poisson structure.

For *smooth* matrices $\tilde{\varphi}^{ij}$ when the low modes dominate in Eq. (15.29), the commutator can be substituted by the Poisson bracket

$$[\,\cdot\,,\,\cdot\,] \;\Longrightarrow\; \mathrm{i}\,\{\,\cdot\,,\,\cdot\,\}_{\mathrm{PB}}\,. \tag{15.58}$$

This looks like a semiclassical approximation of commutators in quantum mechanics by the Poisson brackets. It is now justified by the large value of N.

This clarifies the relation between the group $SL(2;\mathbb{R})$ of symplectic diffeomorphisms and the group $SU(\infty)$ for smooth fields.

There is a vast literature on this issue starting from unpublished works by J. Goldstone and J. Hoppe (see [Hop89]) in the early 1980s who approximated symplectic diffeomorphisms of a spherical membrane by $SU(N)$. This relation was applied [FIT89] to classical Yang–Mills theory and formulated [FFZ89, FZ89] in an elegant way for a torus. These two cases describe two *different* $N = \infty$ limits of $SU(N)$ [PS89]. It was conjectured [Bar90] that strings could appear from the reduced models owing to this mechanism, which also seems to explain early results [Cre84] on an $SL(2;\mathbb{R})$ invariance of the $SU(\infty)$ Yang–Mills theory in two dimensions.

More on the relation between symplectic diffeomorphisms and $SU(\infty)$ can be found in the review [Ran92].

15.3 Twisted Eguchi–Kawai model

In order to construct the twisted Eguchi–Kawai model (TEK), we proceed quite similarly to Sect. 14.3 by substituting $D(x)$ given by Eq. (15.1). Equation (14.31) remains the same but the difference is that now

$$D_\mu \;\equiv\; D(x + a\hat{\mu})\, D^\dagger(x) \;=\; \Gamma_\mu \tag{15.59}$$

and, hence, the D_μ do not commute.

Reordering the matrices in $D(x)$ produces an Abelian factor which depends on the ordering prescription. It is possible to use a symmetric ordering (15.22) instead of the normal ordering (15.1) when

$$D(x) \;=\; J_{x/a}\,, \tag{15.60}$$

and

$$D_\mu \;=\; \Gamma_\mu \prod_{\nu=1}^{d} Z_{\mu\nu}^{x_\nu/2a}\,. \tag{15.61}$$

This extra Abelian factor cancels in most of the formulas.

The substitution of (14.31) with $D(x)$ given by Eq. (15.1) (or Eq. (15.60)) into the Wilson action (6.18) results in

$$\begin{aligned} S_{\text{TEK}} \;&=\; \frac{1}{2}\sum_{\mu\neq\nu}\left\{1-\frac{1}{N}\operatorname{tr}\left[\widetilde{U}_\nu^\dagger\Gamma_\nu^\dagger\widetilde{U}_\mu^\dagger\Gamma_\nu\Gamma_\mu^\dagger\widetilde{U}_\nu\Gamma_\mu\widetilde{U}_\mu\right]\right\} \\ &=\; \frac{1}{2}\sum_{\mu\neq\nu}\left\{1-Z_{\mu\nu}\frac{1}{N}\operatorname{tr}\left[\left(\widetilde{U}_\nu^\dagger\Gamma_\nu^\dagger\right)\left(\widetilde{U}_\mu^\dagger\Gamma_\mu^\dagger\right)\left(\Gamma_\nu\widetilde{U}_\nu\right)\left(\Gamma_\mu\widetilde{U}_\mu\right)\right]\right\}, \end{aligned} \tag{15.62}$$

where the factor of $Z_{\mu\nu}$ emerged because of the commutation relation (15.2).

Introducing the new variable

$$U_\mu \;=\; \Gamma_\mu \widetilde{U}_\mu \tag{15.63}$$

as in Eq. (14.37), we finally obtain

$$S_{\text{TEK}}[U] \;=\; \frac{1}{2}\sum_{\mu\neq\nu}\left(1-Z_{\mu\nu}\frac{1}{N}\operatorname{tr} U_\nu^\dagger U_\mu^\dagger U_\nu U_\mu\right) \tag{15.64}$$

for the action of the twisted Eguchi–Kawai model.

Noting that the Haar measure $\mathrm{d}\widetilde{U}_\mu = \mathrm{d}U_\mu$ is not changed* under the (left) multiplication (15.63) by a unitary matrix Γ_μ, we arrive at the partition function of the twisted Eguchi–Kawai model

$$Z_{\text{TEK}} \;=\; \int \prod_\mu \mathrm{d}U_\mu\, \mathrm{e}^{-NS_{\text{TEK}}[U]/g^2}. \tag{15.65}$$

The only difference from the original Eguchi–Kawai model (14.40) resides in the twisting factor of $Z_{\mu\nu}$ in the action (15.64).

* We shall see in Sect. 16.6 some examples when this is not the case.

The twisted Eguchi–Kawai model possesses the gauge symmetry (14.39) and the $U(1)^d$ symmetry (14.41). The second one is not broken for all values of the coupling g^2N owing to the presence of the twisting factor as will be demonstrated in a moment. For this reason, the twisted Eguchi–Kawai model is equivalent at large N to the planar limit of d-dimensional Yang–Mills theory for all values of the coupling g^2N.

The vacuum state of the twisted Eguchi–Kawai model is given modulo a gauge transformation by

$$U_\mu^{\rm cl} \;=\; \Gamma_\mu\,, \tag{15.66}$$

where the twist eaters Γ_μ were constructed explicitly in Sect. 15.1. The value of the action (15.64) of the twisted Eguchi–Kawai model is 0 for this configuration which is smaller, say, than the value of $\sum_{\mu\nu}(1-\,{\rm Re}\,Z_{\mu\nu})$ of the action for a configuration given by diagonal matrices.

An analog of Eq. (14.42) for the twisted Eguchi–Kawai model is given by

$$\left\langle F[U_\mu(x)]\right\rangle \;\stackrel{\rm red.}{=}\; \left\langle F\Big[D^\dagger(x+a\hat\mu)\,U_\mu D(x)\Big]\right\rangle_{\rm TEK}. \tag{15.67}$$

But the fact that D_μ no longer commute results in subtleties in representing the averages, in particular the Wilson loops, in the language of the twisted Eguchi–Kawai model.

The Wilson loop averages in the twisted Eguchi–Kawai model are defined by

$$W_{\rm TEK}(C) \;=\; \left\langle \frac{1}{N}\,{\rm tr}\,D^\dagger(C)\,\frac{1}{N}\,{\rm tr}\,\boldsymbol{P}\prod_i U_{\mu_i}\right\rangle_{\rm TEK}, \tag{15.68}$$

where

$$D(C) \;=\; \boldsymbol{P}\prod_i \Gamma_{\mu_i} \tag{15.69}$$

and the product runs over links forming the contour C. This is an analog of Eq. (14.45).

Note that $W_{\rm TEK}(C)=1$ for the vacuum configuration (15.66) when C is closed.

For closed loops $D(C)\in Z(N)$ which can be proven using the commutation relation (15.2). For instance,

$$D(\partial p) \;=\; \Gamma_\nu^\dagger\Gamma_\mu^\dagger\Gamma_\nu\Gamma_\mu \;=\; Z_{\mu\nu}^*\,. \tag{15.70}$$

The value of $D(C)$ for a closed loop is the same as that prescribed by Eq. (15.67).

The first trace on the RHS of Eq. (15.68) vanishes for open loops, thereby providing the vanishing of the open Wilson loop averages themselves. Strictly speaking, this vanishing does not hold, say, for the loops in the form of a straight line consisting of L links for the Γ_μ given by Eq. (15.18) when $\Gamma_\mu^L = 1$. But this will be inessential for the purposes of this section since such loops are infinitely long as $N \to \infty$. We shall return to this point below when considering the twisted Eguchi–Kawai model at finite N and associating it with a theory on a finite lattice.

Because the averages of the open Wilson loops vanish in the twisted Eguchi–Kawai model as $N \to \infty$ by construction, all that was said in Sect. 14.3 concerning the equivalence with the d-dimensional lattice gauge theory is applicable for the twisted Eguchi–Kawai model as well.

Problem 15.4 Extend Eq. (15.70) to arbitrary closed contours.

Solution The calculation is similar to that in Problem 15.1 on p. 352. The result is

$$D(C) \;=\; \prod_{p\in S:\partial S=C} Z_{\mu\nu}^{*}\,, \tag{15.71}$$

where (μ,ν) is the orientation of the plaquette p. The product runs over any surface spanned by the closed loop C and is surface-independent owing to the Bianchi identity (15.9).

Problem 15.5 Derive the loop equation for the twisted Eguchi–Kawai model.

Solution The derivation is quite similar to Problem 14.2 on p. 336. Performing the shift (14.49) in the action (15.64), we obtain an extra factor of $Z_{\mu\nu}^{*}$ which is absorbed by $D(C)$ in the definition (15.68) of the Wilson loop averages in the twisted Eguchi–Kawai model owing to Eq. (15.70):

$$D^{\dagger}(C)\,Z_{\mu\nu} \;=\; D^{\dagger}(C\,\partial p) \tag{15.72}$$

and similarly

$$D^{\dagger}(C)\,Z_{\mu\nu}^{*} \;=\; D^{\dagger}\big(C\,\partial p^{-1}\big) \tag{15.73}$$

for the Hermitian conjugate term in the action.

The resulting loop equation reads [GO83b]

$$\sum_{p}\left[W_{\mathrm{TEK}}(C\,\partial p) - W_{\mathrm{TEK}}\big(C\,\partial p^{-1}\big)\right] \\ =\; g^2 N \sum_{l\in C} \tau_\nu(l)\, W_{\mathrm{TEK}}(C_{yx})\, W_{\mathrm{TEK}}(C_{xy})\,. \tag{15.74}$$

The Kronecker symbol δ_{xy} is again restored on the RHS of Eq. (14.50) since the averages of the open Wilson loops vanish:

$$W_{\mathrm{TEK}}(C_{xy}) \;=\; \delta_{xy} W_{\mathrm{TEK}}(C_{xx})\,. \tag{15.75}$$

This reproduces at $N = \infty$ the loop equation (12.65) of the d-dimensional lattice gauge theory which proves the equivalence.

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Remark on twisted boundary conditions

When gauge theory is defined in a box, the boundary conditions are not necessarily periodic. The values of the gauge field at opposite sides of the box can rather coincide modulo an $SU(N)$ gauge transformation:

$$\mathcal{A}_\mu(x+\ell_\nu) \;=\; \Omega_\nu(x)\,\mathcal{A}_\mu(x)\,\Omega_\nu^\dagger(x) + \mathrm{i}\,\Omega_\nu(x)\,\partial_\mu\,\Omega_\nu^\dagger(x)\,. \tag{15.76}$$

Here ℓ_ν denotes the spatial extent of the box in direction ν. This equation represents a *twisted boundary condition*.

A box with periodic boundary conditions looks geometrically like a torus $\mathbb{T}^d$ with the period matrix $\ell_{\mu\nu} = \ell_\nu \delta_{\mu\nu}$. Similarly, a box with the twisted boundary conditions (15.76) is often called a *twisted torus*.

The transition matrices Ω_ν in Eq. (15.76) obey the consistency condition [Hoo79]

$$\Omega_\mu(x+\ell_\nu)\,\Omega_\nu(x) \;=\; Z_{\mu\nu}\,\Omega_\nu(x+\ell_\mu)\,\Omega_\mu(x)\,, \tag{15.77}$$

where $Z_{\mu\nu} \in Z_N$.

This factor of $Z_{\mu\nu}$ cannot be removed in a pure Yang–Mills theory since the gauge group is actually $SU(N)/Z(N)$. Therefore, there are N distinct choices of boundary conditions per plane of a box, which are not related by a gauge transformation. The factor of $Z_{\mu\nu}$ is associated with an additive flux known as the *'t Hooft flux*, which is a feature of non-Abelian field configurations.

A lattice counterpart of Eq. (15.76) reads

$$U_\mu(x+\ell_\nu) \;=\; \Omega_\nu(x+a\hat\mu)\,U_\mu(x)\,\Omega_\nu^\dagger(x)\,. \tag{15.78}$$

Correspondingly, a periodic lattice of finite size L^d is called a *discrete* torus $\mathbb{T}^d_L$.

The twisted Eguchi–Kawai model is of the same type as Wilson's lattice gauge theory on a unit hypercube with the twisted boundary condition and $\Omega_\mu = \Gamma^\dagger_\mu$. This explains the terminology.

Problem 15.6 Show the equivalence between the twisted Eguchi–Kawai model and Wilson's lattice gauge theory on a unit hypercube with the twisted boundary condition.

Solution The twisted boundary condition (15.78) for a hypercube with the corner at $x=0$ is given generically as

$$U_\mu(a\hat\nu) \;=\; \Omega_\nu(a\hat\mu)\,U_\mu(0)\,\Omega_\nu^\dagger(0)\,, \tag{15.79}$$

or

$$U_\mu(a\hat\nu) \;=\; \Gamma_\nu^\dagger U_\mu(0)\Gamma_\nu \tag{15.80}$$

for $\Omega_\nu(0) = \Omega_\nu(a\hat\mu) = \Gamma^\dagger_\nu$.

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The action of Wilson's lattice gauge theory on a unit hypercube with the twisted boundary condition can be transformed using Eq. (15.80) to the form

$$
\begin{aligned}
&\frac{1}{2}\sum_{\mu\neq\nu}\left\{1-\frac{1}{N}\,\mathrm{tr}\left[U_\nu^\dagger(0)U_\mu^\dagger(a\hat\nu)U_\nu(a\hat\mu)U_\mu(0)\right]\right\}\\
&\quad = \frac{1}{2}\sum_{\mu\neq\nu}\left\{1-\frac{1}{N}\,\mathrm{tr}\left[U_\nu^\dagger(0)\left(\Gamma_\nu^\dagger U_\mu^\dagger(0)\Gamma_\nu\right)\left(\Gamma_\mu^\dagger U_\nu(0)\Gamma_\mu\right)U_\mu(0)\right]\right\}\\
&\quad = \frac{1}{2}\sum_{\mu\neq\nu}\left\{1-Z_{\mu\nu}\frac{1}{N}\,\mathrm{tr}\left[\left(U_\nu^\dagger(0)\Gamma_\nu^\dagger\right)\left(U_\mu^\dagger(0)\Gamma_\mu^\dagger\right)\left(\Gamma_\nu U_\nu(0)\right)\left(\Gamma_\mu U_\mu(0)\right)\right]\right\},
\end{aligned}
\tag{15.81}
$$

where we have used Eq. (15.2). Introducing the variable $U_\mu = \Gamma_\mu U_\mu(0)$, we arrive at the action (15.64) and the partition function (15.65) of the twisted Eguchi–Kawai model.

The consideration of this Problem was the original motivation of [GO83b] for introducing the twisting factor of $Z_{\mu\nu}$ in the action of the naive Eguchi–Kawai model.

Remark on $U(N)$ gauge fields

The consistency condition for the $U(N)$ gauge group is simply

$$
\Omega_\mu(x+\ell_\nu)\,\Omega_\nu(x) \;=\; \Omega_\nu(x+\ell_\mu)\,\Omega_\mu(x) \qquad \boxed{U(N)\text{ matrices}} \tag{15.82}
$$

in order for a field in the fundamental representation to be single-valued on a twisted torus.

But now field configurations are characterized by the first Chern class

$$
q_{\mu\nu} \;=\; \frac{1}{2\pi}\int \mathrm{d}x_\mu \mathrm{d}x_\nu \frac{1}{N}\,\mathrm{tr}\,\mathcal{F}_{\mu\nu} \qquad \boxed{\text{no sum over }\mu,\nu} \tag{15.83}
$$

which is nothing but the (magnetic) $U(1)$ flux through the (μ,ν)-plane of the torus. It is quantized since the homotopy group $\pi_1(U(N))=\mathbb{Z}$.

Given a $U(N)$ field configuration with a constant $U(1)$ flux and subtracting it, we arrive at an $SU(N)$ part of the gauge field:

$$
\mathcal{A}_\mu^{SU(N)} \;=\; \mathcal{A}_\mu + \frac{\pi q_{\mu\nu}x_\nu}{\ell^2 N}\,, \tag{15.84}
$$

which obeys Eq. (15.76) with

$$
\Omega_\mu^{SU(N)} \;=\; \mathrm{e}^{-\pi\mathrm{i}\,q_{\mu\nu}x_\nu/\ell N}\,\Omega_\mu \tag{15.85}
$$

satisfying Eq. (15.77) with $Z_{\mu\nu}=\mathrm{e}^{-2\pi\mathrm{i}\,q_{\mu\nu}/N}$. Therefore, the $U(1)$ (magnetic) flux induces [LPR89] the 't Hooft flux for the $SU(N)$ part of the $U(N)$ gauge group.

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15.4 Twisting prescription in the continuum

The twisting reduction prescription can be formulated directly for the continuum theory [GK83] by substituting

$$\tilde{\varphi} \;\to\; a^{d/2-1}\tilde{\varphi}\,, \qquad \Gamma_\mu \;=\; \mathrm{e}^{-\mathrm{i}aP_\mu}\,, \tag{15.86}$$

with the lattice spacing $a \to 0$ and $N \to \infty$. The $N \times N$ Hermitian matrices $\tilde{\varphi}$ and P_μ become Hermitian operators $\tilde{\boldsymbol{\varphi}}$ and $\boldsymbol{P}_\mu$ as $N \to \infty$.

While the Γ_μ in Eq. (15.86) look like Eq. (14.34), $\boldsymbol{P}_\mu$ are no longer diagonal and do not commute. As a consequence of the Weyl–'t Hooft relation (15.2), they obey the Heisenberg commutation relation

$$[\boldsymbol{P}_\mu, \boldsymbol{P}_\nu] \;=\; -\,\mathrm{i}B_{\mu\nu}\,\mathbf{1}\,, \tag{15.87}$$

where

$$B_{\mu\nu} = \frac{2\pi n_{\mu\nu}}{Na^2} \tag{15.88}$$

from the matrix approximation. However, we shall not refer to the matrix approximation during most of this section and consider $B_{\mu\nu}$ as an independent variable.

The commutator (15.87) is similar to that for the coordinate and momentum operators in quantum mechanics. For this reason, the formulation of the continuum twisted reduced model uses operator calculus of quantum mechanics.

Let us mention once again that a solution to Eq. (15.87) for $\boldsymbol{P}_\mu$ exists only for infinite-dimensional Hermitian matrices (representing operators). This is a well-known property of the Heisenberg commutation relation. It can be seen by taking the trace of both sides of Eq. (15.87). If $\boldsymbol{P}_\mu$ were finite-dimensional matrices, the trace of the LHS would vanish owing to the cyclic property of the trace, while that of the RHS would not. In contrast, Eq. (15.2) which is written for unitary matrices possesses a solution for finite N.

A continuum (operator) analog of Eq. (15.1) is

$$\boldsymbol{D}(x) \;=\; \prod_{\mu=1}^{d} \mathrm{e}^{-\mathrm{i}\boldsymbol{P}_\mu x_\mu} \tag{15.89}$$

and similarly for Eq. (14.4):

$$\varphi^{ij}(x) \;\stackrel{N\to\infty}{\longrightarrow}\; \boldsymbol{D}^\dagger(x)\tilde{\boldsymbol{\varphi}}\boldsymbol{D}(x)\,. \tag{15.90}$$

We can change the order of operators in the product on the RHS of Eq. (15.89) by introducing a more general path-dependent operator

$$\boldsymbol{D}(C_{x0}) \;=\; \boldsymbol{P}\,\mathrm{e}^{-\mathrm{i}\int_{C_{x0}} \mathrm{d}\xi^\mu \boldsymbol{P}_\mu}\,, \tag{15.91}$$

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where the integration contour C_{x0} connects the origin and the point x, but is arbitrary otherwise. Changing the shape of the contour results in an extra factor

$$\boldsymbol{P}\,\mathrm{e}^{-\mathrm{i}\oint \mathrm{d}\xi^\mu \boldsymbol{P}_\mu} \;=\; \mathrm{e}^{-\mathrm{i}B_{\mu\nu}\int \mathrm{d}\sigma^{\mu\nu}} \tag{15.92}$$

which is a c-number and cancels in the reduction formula (15.90). This is quite similar to the consideration in Problem 15.1 on p. 352.

In particular, we can always pass in Eq. (15.89) from the normal ordering of the operators to a symmetric ordering:

$$\boldsymbol{D}(x) \;=\; \mathrm{e}^{-\mathrm{i}\sum_{\mu=1}^{d} \boldsymbol{P}_\mu x_\mu}. \tag{15.93}$$

This is an operator analog of Eq. (15.60).

The action of the continuum twisted reduced model is given by the same formula (14.21) as for the continuum quenched reduced model except that $\boldsymbol{P}_\mu$ obey the commutation relation (15.87) rather than commuting as in the quenched reduced model. A "volume element" v is again given for the lattice regularization by Eq. (14.23). Just as in the case of the quenched reduced model, the very formulation of the continuum twisted reduced model implies a regularization.

What remains to be defined are two related issues: how to understand the trace in Eq. (14.21) and how to introduce a regularization directly within the continuum theory.

We begin with a two-dimensional case where $B_{\mu\nu} = B\varepsilon_{\mu\nu}$. The operators $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are then similar to the position and momentum operators in one-dimensional quantum mechanics, with B playing the role of Planck's constant.

A Hilbert space is spanned either by $|\,p_1\rangle$ or $|\,p_2\rangle$ states which are the eigenstates of either $\boldsymbol{P}_1$ or $\boldsymbol{P}_2$:

$$\boldsymbol{P}_1\,|\,p_1\rangle \;=\; p_1\,|\,p_1\rangle\,, \qquad \boldsymbol{P}_2\,|\,p_2\rangle \;=\; p_2\,|\,p_2\rangle\,, \tag{15.94}$$

and are normalized to $\langle p'|p\rangle = \delta^{(1)}(p-p')$.

In either basis the trace of an operator $\boldsymbol{O}$ on the Hilbert space is defined via its (diagonal) matrix elements by

$$\mathrm{tr}_{\mathcal{H}}\,\boldsymbol{O} \;=\; \int dp\,\langle p\,|\,\boldsymbol{O}\,|\,p\rangle\,. \tag{15.95}$$

The matrix element can be easily calculated, representing $\boldsymbol{O}$ by the use of the commutator (15.87) in a normal form, where all $\boldsymbol{P}_1$ are to the left of $\boldsymbol{P}_2$, and

$$\mathrm{e}^{-\mathrm{i}k_1\boldsymbol{P}_2/B}\,|\,p_1\rangle \;=\; |\,p_1-k_1\rangle\,, \qquad \mathrm{e}^{\mathrm{i}k_2\boldsymbol{P}_1/B}\,|\,p_2\rangle \;=\; |\,p_2-k_2\rangle\,. \tag{15.96}$$

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There exists a simple operator representation of the $N = \infty$ limit of the basis elements J_m introduced in Sect. 15.2. Substituting the operator limit (15.86) of $\Gamma_1 = \mathcal{P}$ and $\Gamma_2 = \mathcal{Q}$ into Eq. (15.20), we obtain

$$J_m^{ij} \;\longrightarrow\; \mathrm{e}^{-\mathrm{i}am_1\boldsymbol{P}_1}\,\mathrm{e}^{-\mathrm{i}am_2\boldsymbol{P}_2}\,\mathrm{e}^{-\mathrm{i}\pi m_1 m_2/L} \;\equiv\; \boldsymbol{J}_m\,. \tag{15.97}$$

The order of operators on the RHS of Eq. (15.97) is normal. Applying the Baker–Campbell–Hausdorff formula

$$\mathrm{e}^{A}\,\mathrm{e}^{B}\,\mathrm{e}^{-\frac{1}{2}[A,B]} \;=\; \mathrm{e}^{A+B}, \tag{15.98}$$

which is exact when the commutator $[A, B]$ is a c-number as in our case, it can be represented conveniently in a symmetric- or Weyl-ordered form

$$\boldsymbol{J}_m \;=\; \mathrm{e}^{-\mathrm{i}a(m_1\boldsymbol{P}_1+m_2\boldsymbol{P}_2)}. \tag{15.99}$$

The continuum operator counterparts of the formulas of Sect. 15.2 are obvious.

Introducing the continuum momentum variable $k_\mu = 2\pi\varepsilon_{\mu\nu}m_\nu/aL$ which is a $d = 2$ version of Eq. (15.50) and using the substitution (15.51), we have

$$\boldsymbol{f} \;=\; \int\prod_\mu \frac{\mathrm{d}k_\mu}{2\pi}\boldsymbol{J}_k\, f(k) \tag{15.100}$$

which is quite analogous to the Fourier transform of a function

$$f(x) \;=\; \int\prod_\mu \frac{\mathrm{d}k_\mu}{2\pi}\,\mathrm{e}^{\mathrm{i}kx}\, f(k)\,. \tag{15.101}$$

Here

$$\boldsymbol{J}_k \;=\; \mathrm{e}^{\mathrm{i}(k_2\boldsymbol{P}_1-k_1\boldsymbol{P}_2)/B} \;=\; \mathrm{e}^{\mathrm{i}k_2\boldsymbol{P}_1/B}\,\mathrm{e}^{-\mathrm{i}k_1\boldsymbol{P}_2/B}\,\mathrm{e}^{-\mathrm{i}k_1k_2/2B} \tag{15.102}$$

as follows from Eq. (15.97).

The coefficients $f(k)$ on the RHSs of Eqs. (15.100) and (15.101) are the same. Therefore, these equations relate operators in Hilbert space and functions to each other. This relation is often called the *Weyl transform*.*

Given Eqs. (15.100) and (15.101) and using Eqs. (15.94) and (15.96), we can alternatively write down the Weyl transform via the matrix element

$$f(k_1,k_2) \;=\; \frac{2\pi}{B}\int \mathrm{d}p_1\,\mathrm{e}^{-\mathrm{i}k_2p_1/B}\,\left\langle p_1+\tfrac{1}{2}k_1\,\middle|\,\boldsymbol{f}\,\middle|\,p_1-\tfrac{1}{2}k_1\right\rangle. \tag{15.103}$$

An extension to d dimensions is straightforward. Say, k and x in Eqs. (15.100) and (15.101) were to simply become d-dimensional vectors. Similarly, the integration as well as the matrix element are

* More rigorous mathematical definitions can be found in the book [Won98].

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taken in Eq. (15.103) with respect to half of the momentum variables: $p_1, p_3, \ldots, p_{d-1}$.

The Weyl transform can, of course, be formulated without any reference to the discrete formulas of Sect. 15.2. We simply followed the spirit of Weyl's original book [Wey31].

However, an advantage of such an approach which starts from a lattice discretization is that it provides an ultraviolet cutoff, making the continuum twisted reduced model well-defined. The values of momenta are bounded by $|k_\mu| \leq \pi/a$, which introduces the cutoff. Instead of the lattice regularization, we can use a Lorentz-invariant regularization of [GK83] directly for the continuum theory restricting $k^2 \leq \Lambda^2$ in the integral over k_μ in Eqs. (15.100) and (15.101). This will both regularize perturbation theory and bound operators on the Hilbert space.

The action of the continuum twisted reduced model regularized in such a way can be represented in the form

$$S_{\rm TRM} \;=\; \frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})}\mathrm{tr}_{\mathcal{H}}\left\{-\frac{1}{2}\left[\boldsymbol{P}_\mu, \tilde{\boldsymbol{\varphi}}\right]^2 + \widetilde{V}\left(\tilde{\boldsymbol{\varphi}}\right)\right\}, \tag{15.104}$$

where we have substituted

$$vN \;=\; \frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})} \tag{15.105}$$

and $\mathrm{Pf}\,(B_{\mu\nu}) = \sqrt{\det_{\mu\nu} B_{\mu\nu}}$. This substitution is justified by the definition (15.88) of $B_{\mu\nu}$ and v is again a volume element given by Eq. (14.23) for the lattice regularization.

We have already met the factor (15.105) for $d = 2$ in Eq. (15.103). It appears whenever the trace over the Hilbert space is substituted by the integral over space as

$$\frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})}\,\mathrm{tr}_{\mathcal{H}}\,\boldsymbol{\mathcal{L}} \;=\; \int \mathrm{d}^d x\, \mathcal{L}(x)\,, \tag{15.106}$$

where $\mathcal{L}(x)$ is the Weyl transform of $\boldsymbol{\mathcal{L}}$. This formula is a counterpart of Eq. (15.53)

The proof of how the continuum twisted reduced model reproduces planar graphs is quite similar to that of Sect. 15.2 on the lattice. The integral over space is reproduced according to Eq. (15.106). Nonplanar graphs are again suppressed as $\theta_{\mu\nu} = B^{-1}_{\mu\nu} \to \infty$.

Remark on the number of states in Hilbert space

For the matrix approximation, the Hilbert space is spanned by N states. A question arises as to what is the number of states in the Hilbert space regularized in a Lorenz-invariant way.

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This can be easily understood from an analogy with the semiclassical limit of quantum mechanics when B, which plays the role of Planck's constant, is small. The volume occupied by the N states in a phase space is given semiclassically by the Bohr–Sommerfeld formula. It can be written in our notation as

$$\prod_{\mu} \frac{\Delta p_\mu}{2\pi} \quad = \quad N \frac{\mathrm{Pf}\,(B_{\mu\nu})}{(2\pi)^{d/2}} \,. \tag{15.107}$$

Dividing by N, we conclude that the factor on the RHS of Eq. (15.105) is related semiclassically to the inverse volume of a cell in the phase space.

Given a regularization which determines the LHS of Eq. (15.107) via the cutoff Λ, we can solve Eq. (15.107) for N which gives the number of states in the regularized Hilbert space.

Of course, all of these formulas become exact for the lattice regularization when $\mathrm{Pf}\,(B_{\mu\nu}) \sim 1/N \to 0$.

15.5 Continuum version of TEK

The continuum version of the twisted Eguchi–Kawai model can be constructed [GK83] from the lattice counterpart of Sect. 15.3 by substituting

$$U_\mu \;=\; \mathrm{e}^{\mathrm{i}aA_\mu}, \qquad \Gamma_\mu \;=\; \mathrm{e}^{-\mathrm{i}aP_\mu}, \tag{15.108}$$

when the lattice spacing $a \to 0$ and $N \to \infty$. Here A_μ and P_μ are $N \times N$ Hermitian matrices which become Hermitian operators when $N \to \infty$ as is described in the previous section. We shall imply, but not explicitly use the operator notation.

To derive the action of the continuum twisted Eguchi–Kawai model, we first obtain from Eqs. (15.108) and (15.87)

$$U_\nu^\dagger U_\mu^\dagger U_\nu U_\mu \;=\; \mathrm{e}^{a^2[A_\mu, A_\nu]}, \qquad Z_{\mu\nu} \;=\; \mathrm{e}^{\mathrm{i}a^2 B_{\mu\nu}} \tag{15.109}$$

to order a^2. Finally, we arrive at the following action of the continuum twisted Eguchi–Kawai model:

$$S_{\mathrm{TEK}}[A] \;=\; -\frac{v}{4g^2}\,\mathrm{tr}\,\left([A_\mu, A_\nu] + \mathrm{i}B_{\mu\nu}\right)^2, \tag{15.110}$$

where v is again a "volume element" given for the lattice regularization by Eq. (14.23). Just as in the case of the quenched Eguchi–Kawai model, the very formulation of the continuum twisted Eguchi–Kawai model implies a regularization.

It is worth mentioning here a subtlety associated with the fact that A_μ are Hermitian operators (infinite-dimensional matrices). The point is that

$$\mathrm{tr}\,[A_\mu, A_\nu] \;\neq\; 0 \tag{15.111}$$

in this case so that $B_{\mu\nu}$ cannot be omitted. This is a well-known property of operators obeying the Heisenberg commutation relation (15.87) as has already been pointed out.

Nevertheless, the presence of the $B_{\mu\nu}$ does not affect the classical equation of motion for the continuum twisted Eguchi–Kawai model which coincides with Eq. (14.71) since $B_{\mu\nu}$ is a c-number.

Owing to the presence of $B_{\mu\nu}$ in the action (15.110), the vacuum configuration of the continuum twisted Eguchi–Kawai model is given by

$$A_\mu^{\mathrm{cl}} \;=\; -P_\mu \tag{15.112}$$

modulo a gauge transformation $A_\mu^{\mathrm{cl}} \to \Omega A_\mu^{\mathrm{cl}} \Omega^\dagger$. The minimum of the action is reached when P_μ obey Eq. (15.87) rather than being diagonal matrices.

The continuum limit of Eqs. (15.68) and (15.69) determines the averages of Wilson loops in the continuum twisted Eguchi–Kawai model:

$$W_{\mathrm{TEK}}(C_{yx}) \;=\; \left\langle \frac{1}{N}\,\mathrm{tr}\, D^\dagger(C_{yx})\,\frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{yx}} \mathrm{d}\xi^\mu A_\mu} \right\rangle_{\mathrm{TEK}}, \tag{15.113}$$

where $D(C_{yx})$ is defined in Eq. (15.91). They are nontrivial since A_μ do not commute.

The trace of $D^\dagger(C_{yx})$ on the RHS of Eq. (15.113) vanishes for open loops. This provides the vanishing of the averages of open Wilson loops as is prescribed by the R^d symmetry (14.61) of the action (15.110).

For closed loops this factor does not vanish and represents the flux of the $B_{\mu\nu}$-field through a surface bounded by the contour C. It is needed to provide the equivalence with planar graphs of d-dimensional Yang–Mills theory, since the classical extrema of the continuum twisted Eguchi–Kawai model are given by Eq. (15.112) and perturbation theory is constructed by expanding around this classical solution. The equivalence can be demonstrated perturbatively using the theorem stated at the end of Sect. 15.2.

The proof of the equivalence between the large-N limit of d-dimensional Yang–Mills theory and the continuum Eguchi–Kawai model can be given using the continuum loop equation, for which the lattice regularization was considered in Problem 15.5 on p. 365. The loop equation for the continuum twisted Eguchi–Kawai model coincides with Eq. (14.65) for the continuum naive Eguchi–Kawai model. This is because the loop operator on the LHS of Eq. (14.65) is of first order (obeys the Leibnitz rule of the type of Eq. (12.96)). For this reason, the first trace in Eq. (15.113) produces

$$[P_\mu,[P_\mu,P_\nu]] \;=\; 0 \tag{15.114}$$

which vanishes since the commutator of P_μ with P_ν is a c-number. The manipulations with the result of acting with the loop operator on the second trace in Eq. (15.113) is exactly the same as for the naive Eguchi–Kawai model with an unbroken R^d symmetry, which are described in Sect. 14.4. Also the treatment of the averages of open Wilson loops according to Eqs. (14.68) and (14.69) remains the same. This shows, in particular, that the "volume factor" v for the twisted Eguchi–Kawai model is the same as for the quenched Eguchi–Kawai model if integrals over momentum are regularized in the same way.

Problem 15.7 Calculate $\mathrm{tr}_{\mathcal{H}} D^\dagger(C_{yx})$ for a straight line connecting x and y.

Solution Using the formulas of Sect. 15.4, we obtain in $d=2$

$$\begin{aligned}
&\mathrm{tr}_{\mathcal{H}}\, \mathrm{e}^{\mathrm{i}(y_1-x_1)\boldsymbol{P}_1+\mathrm{i}(y_2-x_2)\boldsymbol{P}_2} \\
&\quad= \int \mathrm{d}p_1\, \langle p_1 \,|\, \mathrm{e}^{\mathrm{i}(y_1-x_1)\boldsymbol{P}_1}\, \mathrm{e}^{\mathrm{i}(y_2-x_2)\boldsymbol{P}_2}\, \mathrm{e}^{-\mathrm{i}(x_1-y_1)(x_2-y_2)B/2} \,|\, p_1\rangle \\
&\quad= \frac{2\pi}{B}\delta^{(1)}(y_1-x_1)\,\delta^{(1)}(y_2-x_2)\,.
\end{aligned} \tag{15.115}$$

An extension to d dimensions is straightforward:

$$\mathrm{tr}_{\mathcal{H}}\, \mathrm{e}^{\mathrm{i}(y-x)\boldsymbol{P}} \;=\; \frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})}\delta^{(d)}(x-y)\,. \tag{15.116}$$

This demonstrates how the averages of open Wilson loops vanish in the continuum twisted Eguchi–Kawai model.

Remark on TEK with fundamental matter

As has already been discussed in Sect. 11.5, matter in the fundamental representation of the gauge group $SU(N)$ can survive the large-N limit of Yang–Mills theory only in the Veneziano limit when the number N_f of flavors is proportional to the number N of colors.

Such a limit with $N_\mathrm{f} = n_\mathrm{f} N$ can be described [Das83] for an integral n_f by the following generalization of the twisted Eguchi–Kawai model.

We begin for simplicity with a scalar field on the lattice, whose free action for Hermitian matrices is given by the first line in Eq. (15.31). An interaction with the gauge field is introduced by gauging the first of the two matrix indices of the general complex matrix $\tilde{\varphi}^{ij}$, i.e. by replacing the second Γ_μ by U_μ, which is essentially an exponential of the covariant derivative as has already been pointed out.

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The generalized action is given as

$$S = S_{\mathrm{TEK}} + N\,\mathrm{tr}\left[M\tilde{\varphi}^\dagger\tilde{\varphi} - \sum_\mu \left(\Gamma_\mu\tilde{\varphi}^\dagger U_\mu^\dagger\tilde{\varphi} + \Gamma_\mu^\dagger\tilde{\varphi}^\dagger U_\mu\tilde{\varphi}\right)\right], \qquad (15.117)$$

where S_{TEK} is the action (15.64) describing self-interactions of the gauge field. Repeating the analysis of Sect. 15.2, we see that this model reproduces planar graphs of the d-dimensional Yang–Mills theory with $N_\mathrm{f} = N$ species of scalars in the fundamental representation.

We can easily associate an extra index running from 1 to n_f to the matrix $\tilde{\varphi}$ in order to have a theory with $N_\mathrm{f} = n_\mathrm{f} N$ flavors.

A similar generalization of the twisted Eguchi–Kawai model can be made by incorporating fermions which belong to the fundamental representation, thereby describing the Veneziano limit of QCD. Introducing Grassmann-valued matrices $\widetilde{\psi}$ and $\bar{\widetilde{\psi}}$, we write down the action as

$$S = S_{\mathrm{TEK}} + N\,\mathrm{tr}\left[M\bar{\widetilde{\psi}}\widetilde{\psi} - \sum_\mu \left(\Gamma_\mu\bar{\widetilde{\psi}}P_\mu^- U_\mu^\dagger\widetilde{\psi} + \Gamma_\mu^\dagger\bar{\widetilde{\psi}}P_\mu^+ U_\mu\widetilde{\psi}\right)\right], \qquad (15.118)$$

where $P_\mu^\pm$ are the projectors for lattice fermions that are defined in Chapter 8.

The continuum counterparts of Eqs. (15.117) and (15.118) can be easily written down by noting that the interaction with the gauge field can be incorporated by the substitution

$$[P_\mu, \tilde{\varphi}] \;\to\; -A_\mu\tilde{\varphi} - \tilde{\varphi}P_\mu \qquad (15.119)$$

in the free actions (cf. Eq. (15.104) for Hermitian scalars), since A_μ is associated with the covariant derivative in the fundamental representation.

Finally for the action of the continuum twisted Eguchi–Kawai model with fundamental matter we find

$$S = S_{\mathrm{TEK}} + vN\,\mathrm{tr}\left[m^2\tilde{\varphi}^\dagger\tilde{\varphi} + \sum_\mu \left|A_\mu\tilde{\varphi} + \tilde{\varphi}P_\mu\right|^2\right] \qquad (15.120)$$

for scalars and

$$S = S_{\mathrm{TEK}} + vN\,\mathrm{tr}\left[m\bar{\widetilde{\psi}}\widetilde{\psi} - \mathrm{i}\sum_\mu \bar{\widetilde{\psi}}\,\gamma_\mu\left(A_\mu\tilde{\psi} + \tilde{\psi}P_\mu\right)\widetilde{\psi}\right] \qquad (15.121)$$

for fermions. Here S_{TEK} is given by Eq. (15.110).

When formulated in terms of operators on the Hilbert space, vN in Eqs. (15.120), (15.121) and (15.110) is to be substituted according to Eq. (15.105).

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