Hence for the whole circular strip through S

$$dW = 2\pi K i r \sin^2 a dr / r^3$$
$$= 2\pi K i \sin^2 a dr / r^2$$

Integrating from $r = R$ to $r = \infty$, we get for the potential energy of the circuit and the magnet

$$W = -\frac{2i\pi K \sin^2 a}{R} = -K\frac{2\pi i a^2}{R^3}$$

where $a$ is the radius of the circle.

Hence the field at the point P is

$$i\frac{2\pi a^2}{R^3}.$$

At the centre this becomes

$$i\frac{2\pi}{a}.$$

In this last example, it is assumed that the equivalence of circuits and magnets has been established experimentally. For this purpose the usual experiments are amply sufficient.

The experimental treatment of the subject of magnetic induction has been greatly improved in these later days, thanks chiefly to such men as Ewing and Hopkinson, following up along the lines of Faraday and Maxwell.

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## On a surface of the third order.

By R. E. ALLARDICE, M.A.

In the second part of Professor Chrystal's *Algebra* (Exer. V., No. 4) the following exercise is given :—

If $2xyz - x^2 - y^2 - z^2 + 1 = 0$, and $x, y, z$ are all real, then all, or none, of the quantities $x, y, z$ lie between $-1$ and $+1$.

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This result follows at once from the fact that, if D be the $x$-discriminant of the above function, then

$$D \equiv 4(y^2z^2 - y^2 - z^2 + 1) \equiv 4(y^2 - 1)(z^2 - 1).$$

It occurred to me to consider the geometrical interpretation of this theorem, and then to consider the surface itself which the above equation represents; and finally I made a rough model in clay of the finite sheet of the surface, got this model cast in plaster, and exhibited the cast to the *Society*, giving at the same time some account of the properties of the surface. The following is a brief account of some of these properties :—

The geometrical interpretation of the theorem quoted above is, that part of the surface lies within the cube formed by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$; but that no part of the surface lies within any one of the six infinite spaces that can be generated by the motion of a face of the cube in a direction perpendicular to that face. From this it follows, that, if the surface is continuous, certain of the vertices of the cube must be conical points on the surface. It is easily seen that the points $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ lie on the surface; and they are therefore conical points. It may easily be shown that the enveloping cones at the conical points are right circular cones with their vertical angles right angles.

The section made by each of the co-ordinate planes is a circle (with a straight line at infinity).

The section made by a plane that bisects the angle between two of the co-ordinate planes consists of a parabola and a straight line (which is a diagonal of one of the faces of the cube referred to above).

All the sections parallel to one of the co-ordinate planes are ellipses and hyperbolas inscribed in the same square.

There are nine real straight lines on the surface; of which six are at a finite distance, and consist of one of the diagonals of each of the six faces of the cube; and the other three are at an infinite distance in the direction of the co-ordinate planes.

As four is the maximum number of conical points that a cubic surface may have, the surface under consideration possesses the maximum number. A cubic surface contains, in general, twenty-seven straight lines; but when a straight line passes through a conical point it counts for two, and when it passes through two conical points, for four. Thus, in the present case, the six straight lines at a finite distance, which are the six edges of a tetrahedron whose vertices are the four conical points, count for twenty-four. The remaining three straight lines always lie in one plane; in the case of the surface we are now considering, this plane is the plane at infinity.

The projection on the $z$-plane of the section made by any plane containing the conical points $(1, 1, 1)$ and $(-1, -1, 1)$ is a rectangular hyperbola whose asymptotes are parallel to the axes of $x$ and $y$.

If we make the equation homogeneous, we get

$$2xyz - ax^2 - ay^2 - az^2 + a^3 = 0\,;$$

the finite sheet of which surface is contained by the planes $x = \pm a$, $y = \pm a$, $z = \pm a$. If we make $a$ zero, the surface degenerates into the three co-ordinate planes.

The equation may be further generalised in the form

$$2abcxyz - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2 + a^2b^2c^2 = 0.$$

## Illustration of the Composition of two Simple Harmonic Motions.

The composition of two simple harmonic motions, of equal amplitudes, in directions at right angles to one another, may be illustrated by means of the surface $2xyz - ax^2 - ay^2 - az^2 + a^3 = 0$, in the case in which the periods of the motions are very nearly equal.

### First Method.

We may write

$$\begin{aligned} x &= a\cos\omega t, \\ y &= a\cos(\omega' t - \alpha) \\ &= a\cos\{\omega t + (\omega' - \omega)t - \alpha\} = a\cos(\omega t + \theta), \end{aligned}$$

where $\omega' - \omega$ is small, and hence $\theta$, which is equal to $(\omega' - \omega)t - \alpha$, varies slowly with the time.

The resultant of these two simple harmonic motions for a small interval of time, is given by

$$x^2 + y^2 - 2xy\cos\theta = a^2\sin^2\theta. \qquad (1)$$

Multiplying this equation by $a$, and comparing with the equation to the surface, we have

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$$2xya\cos\theta - ax^2 - ay^2 + a^3\sin^2\theta = 0\ ;$$
$$2xyz - ax^2 - ay^2 - az^2 + a^3 = 0\ ;$$
$$\therefore \qquad z = a\cos\theta \text{ and } -az^2 + a^3 = a^3\sin^2\theta,$$

and these two equations are consistent.

If we look upon (1) as the equation to a family of curves, it will always be possible to obtain them as the projection of a family of curves on any surface; but it will not in general be possible to obtain $z$ as a function of the parameter $\theta$ alone.

SECOND METHOD.

In the equation

$$2xyz - ax^2 - ay^2 - az^2 + a^3 = 0,$$

put

$$x = a\cos\omega t, \quad y = a\cos(\omega t + \theta).$$

Then

$$2a^2z\cos\omega t\cos(\omega t + \theta) - a^3\cos^2\omega t - a^3\cos^2(\omega t + \theta) - az^2 + a^3 = 0\ ;$$

that is

$$z^2 - 2az\cos\omega t\cos(\omega t + \theta) + a^2\{\cos^2\omega t + \cos^2(\omega t + \theta) - 1\} = 0.$$

If D be the discriminant of this equation, we have

$$\begin{aligned} D/4a^2 &= \cos^2\omega t\cos^2(\omega t + \theta) - \{\cos^2\omega t - \cos^2(\omega t + \theta) + 1\} \\ &= (1 - \cos^2\omega t)(1 - \cos^2(\omega t + \theta)) = \sin^2\omega t\sin^2(\omega t + \theta). \end{aligned}$$

Hence

$$\begin{aligned} z &= \{2a\cos\omega t\cos(\omega t + \theta) \pm \sqrt{D}\}/2 \\ &= a\{\cos\omega t\cos(\omega t + \theta) \pm \sin\omega t\sin(\omega t + \theta)\} \\ &= a\cos\theta \quad \text{or} \quad a\cos(2\omega t + \theta). \end{aligned}$$

The second of these solutions corresponds to the curve traced out by the one end of a straight line parallel to the axis of $z$, when the other end traces out the curve given by the first solution.

In the first method of solution the representation is regarded rather as approximate and discontinuous, while in the second it is exact and continuous.

It should be noticed that $z = a\cos\theta = a\cos\{(\omega' - \omega)t - a\}$ is also a simple harmonic motion; and hence the spiral traced out on the surface is the resultant of three particular simple harmonic motions.

In the second method of looking at the problem, it is not necessary that $\omega$ and $\omega'$ should be nearly equal.

In the case of the second value of $z$, $z = a\cos(2\omega t + \theta)$, if $\omega' - \omega$ is small, the period for $z$ is just about half of that for $x$ or $y$.