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CONDITIONS FOR THE SOLVABILITY OF SINGULAR BOUNDARY VALUE PROBLEMS

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Consider the singular boundary value problem $(r(x'))' + f(t, x) = 0$, $0 < t < 1$. We give necessary and sufficient conditions for this problem to have solutions. In addition, a uniqueness result is obtained.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the solvability of the following singular boundary value problem of the second order differential equation:

$$-(r(x'(t)))' = f(t, x(t)), \quad t \in (0, 1) \tag{1.1}$$

with x satisfying the so-called Dirichlet boundary condition

(i) $x(0) = x(1) = 0$

or the mixed boundary condition (see [1]):

(ii) $x(0) = \delta x(1) + x'(1) = 0, \quad \delta \geqslant 0.$

Equations of the above form occur in many mathematical models such as fluid theory and turbulent flow of gas. Equation (1.1) is singular at $t = 0, 1$ and $x = 0$, for example:

$$f(t, x) = t^{-\alpha}(1-t)^{-\beta}\left(x^{-\gamma} + x^{\theta}\right), \quad \alpha, \beta, \gamma > 0. \tag{1.2}$$

We call α, β, γ the order of the singularities at $t = 0, 1$ and $x = 0$.

When $r(x) = x$, it has been shown that for $\alpha < 1$, $\beta < 1$, $\gamma > 0$, problem (1.1) and (i) has solutions, see [2]. Recently O'Regan [3] proved that when $\alpha, \beta < 1$, $\gamma = 0$, problem (1.1) with (i) or (ii) has solutions. So a natural question is what it is the greatest order of the singularities for equation (1.1) to be solvable? In this paper, we give an answer to this question. Our results show that this greatest order is $\alpha = 2$, $\beta = 2$ $(r(x) = x)$. In addition, a uniqueness result is given.

In the following, we call x a solution to (1.1) with (i) if $x \in C[0, 1] \cap C^2(0, 1)$ with $x(t) > 0$ for $t \in (0, 1)$ and satisfies (1.1) and (i), while for (1.1) and (ii) we need $x \in C[0, 1] \cap C^1(0, 1] \cap C^2(0, 1)$ and $x(t) > 0$ for $t \in (0, 1)$.

We always assume $r(x)$ is odd, strictly increasing and $r \in C^1(R^1)$ with $r(\infty) = \infty$. In addition, $f\colon (0, 1) \times (0, \infty) \to (0, \infty)$ is continuous. Our main results are the next four theorems.

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Theorem 1.1. *Suppose the following conditions are satisfied:*

(N) $f(t, x) \geqslant a(t)g(x)$, $t \in (0,1)$, $x \in (0, \infty)$, *with* $a\colon (0,1) \to (0,\infty)$ *and* $g\colon (0,\infty) \to (0,\infty)$ *continuous while* $\inf\{g(x)\colon x \in (0,1)\} > 0$.

(H_0) $r(xy) \geqslant m r(x) r(y)$ *for* $x, y \geqslant M$, *where* M, m *are positive constants.*

Then the necessary condition for problem (1.1) and (i) to have solutions is

$$\int_0^1 r^{-1}(|A(t)|)\,dt < \infty, \text{ where } A(t) = \int_{1/2}^t a(s)\,ds \tag{1.3}$$

and for problem (1.1) and (ii) the necessary conditions are

$$\int_{1/2}^1 a(s)\,ds < \infty, \int_0^1 r^{-1}(A_1(t))\,dt < \infty, \text{ where } A_1(t) = \int_t^1 a(s)\,ds. \tag{1.4}$$

Theorem 1.2. *Suppose* $f(t, x)$ *is decreasing with respect to* x, *then problem (1.1) with (i) or (ii) has at most one solution.*

Theorem 1.3. *Suppose* $(H_0)(H_1)(H_2)(H_3)(H_4)(H_5)$ *are satisfied. Then problem (1.1) with (i) has solutions, where the conditions are:*

(H_1) $f(t, x) \leqslant b(t)h(x)$, $t \in (0,1)$, $x \in (0,\infty)$ *where* $b\colon (0,1) \to (0,\infty)$, $h\colon (0,\infty) \to (0,\infty)$ *are continuous.*

(H_2) $\int_0^1 r^{-1}(|B(t)|)\,dt < \infty$, *where* $B(t) = \int_{1/2}^t b(s)\,ds$.

(H_3) $r^{-1}(h(x))/x \to 0$ *as* $x \to +\infty$.

(H_4) $\int_0^1 (dx)/\left(r^{-1}(h(x))\right) < \infty$.

(H_5) *For any* $H > 0$ *there exists* $\psi(t) \in C[0,1]$ *such that* $f(t,x) \geqslant \psi(t) \geqslant 0$, $t \in (0,1)$, $x \in (0, H]$, *and*

$$0 < \int_0^\varepsilon \psi(s)\,ds < \infty, \quad 0 < \int_{1-\varepsilon}^1 \psi(s)\,ds < \infty$$

for $\varepsilon \in (0,1)$.

Theorem 1.4. *Suppose* $(H_0)(H_1)(H_2')(H_3)(H_4)(H_5)$ *are satisfied. Then problem (1.1) with (ii) has solutions, where* (H_2') *is:*

$$\int_{1/2}^1 b(s)\,ds < \infty, \text{ and } \int_0^1 r^{-1}(B_1(t))\,dt < \infty, \text{ where } B_1(t) = \int_t^1 g(s)\,ds.$$

Corollary 1.5. *Consider the equation*

$$-(r(x'))' = b(t)h(x) \tag{1.5}$$

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where $(H_0)(H_3)(H_4)$ *are satisfied,* b *and* h *are positive and continuous and* $\inf\{h(x): x \in (0, 1)\} > 0$. *Then (1.5) is solvable if and only if:*

$$\int_0^1 r^{-1}\left(\left|\int_{1/2}^t b(s)\, ds\right|\right) dt < \infty, \text{ for boundary condition (i);}$$

$$\int_{1/2}^1 b(s)\, ds < \infty, \int_0^1 r^{-1}\left(\int_t^1 b(s)\, ds\right) dt < \infty, \text{ for boundary condition (ii).}$$

EXAMPLE 1.6. *Equation (1.6)*

$$\left(|x'|^{\beta-1} x'\right)' + b(t)x^{-\gamma} = 0, \quad x(0) = x(1) = 0 \tag{1.6}$$

where $\beta > 0$, $\gamma > 0$ *has a unique solution if and only if*

$$\int_0^1 \left(\left|\int_{1/2}^t b(s) ds\right|^{1/\beta}\right) dt < \infty.$$

REMARK. If $b(t) = t^{-\alpha}(1-t)^{-\alpha}$ and $\beta = 1$, then the above condition becomes $\alpha < 2$.

PROOF OF THEOREM 1.1 AND THEOREM 1.2

We shall prove Theorem 1.1 in two steps. First we consider problem (1.1) and (i). Choose $t_0 \in (0, 1)$ so that $x(t_0) = \max\{x(t): t \in (0, 1)\}$, hence $x'(t_0) = 0$, and $x'(t) < 0$, for $t \in (t_0, 1)$ whereas $x'(t) > 0$ as $t \in (0, t_0)$. From condition (N) we have $c > 0$ with $cg(x(t)) \geqslant 1$, $t \in (0, 1)$. Then integration yields

$$A(t) \leqslant -c\big(r(x'(t)) - r(x'(1/2))\big).$$

We can assume without loss of generality that $x'(t) \to -\infty$ as $t \to 1$.

Hence $r(x'(t)) \to -\infty$ as $t \to 1$ and for some $t_1 > t_0$

$$A(t) \leqslant 2cr(-x'(t)), \ t \in (t_1, 1).$$

By (H_0) we get

$$A(t) \leqslant r\big(-r^{-1}(2c/m)x'(t)\big).$$

Hence

$$\int_{t_1}^1 r^{-1}(|(A(t))|)\, dt \leqslant r^{-1}(2c/m)x(t_0).$$

The estimate on $(0, t_0)$ is just the same, and hence (1.3) is true.

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Next we consider (1.1) with the boundary condition (ii). Let $x \in C[0, 1] \cap C^1(0, 1] \cap C^2(0, 1)$ satisfies (1.1) and (ii). Similarly to the first step we get $c > 0$ with

$$r(x'(t_1)) - r(x'(t_2)) \geqslant (1/c) \int_{t_1}^{t_2} a(s)\, ds.$$

So by letting $t_2 \to 1$ we get

$$\int_{t_1}^{1} a(s)\, ds < \infty, \quad t_1 \in (0, 1).$$

And

$$A_1(t) \leqslant cr(x'(t)) + cr(\delta x(1)).$$

Assume $x'(t) \to +\infty$ as $t \to 0$ without loss of generality, then we get $t_3 \in (0, 1)$ with $A_1(t) \leqslant 2c(r(x'(t)))$. The same reasoning as in step one will yield (1.4). So the proof is complete. □

Next we turn to prove Theorem 1.2. Let x and y be solutions to (1.1) and (i) with $\max z(t) > 0$, where $z(t) = x(t) - y(t)$. Thus we can choose $t_0 \in (0, 1)$ such that $z(t_0) = \max\{z(t) \colon t \in (0, 1)\}$ and $t_0 \in (t_1, t_2)$ with $z(t) > 0$ for $t \in (t_1, t_2)$. Evidently $z'(t_0) = 0$ and so we have for $t \in (t_1, t_0)$ that $r(x'(t)) \leqslant r(y'(t))$, hence $z'(t) \leqslant 0$. As a result we can let $t_1 = 0$, in contradiction to the boundary condition (i).

Finally we consider boundary condition (ii). In this case $t_0 \in (0, 1]$. If $\delta = 0$ we have $z'(t_0) = 0$. Thus we can get the desired contradiction. If $\delta > 0$ and $t_0 = 1$, then $z'(1) \geqslant 0$ in contradiction to $z'(1) = -\delta z(1) = -\delta z(t_0) < 0$. Thus $t_0 \in (0, 1)$. The rest of the proof is similar to the case of boundary condition (i). □

3. Approximate problems

In order to prove Theorem 1.2 and 1.3, we need to consider the following approximate problems:

$$(3.1) \qquad (rx'(t))' + \lambda f_n(t, x(t)) = 0, \quad 0 < t < 1,\ \lambda \in [0, 1]$$

where $f_n(t, x) = f(t, \max\{x, 1/n\})$. Define the functional $H(e, \lambda, x)$ by:

$$H(e, \lambda, x) = \int_0^1 r^{-1}(e - \lambda F_n(s, x))\, ds$$

where $e \in R^1$, $x \in C[0, 1]$ and

$$F_n(s, x) = \int_{1/2}^{s} f_n(t, x(t))\, dt.$$

We shall work in the space $C[0, 1]$. Let $P = \{x \in C[0, 1]: x(t) \geqslant 0, t \in [0, 1]\}$. Then from $(H_0)(H_1)(H_2)$ we have:

$$|F_n(s, x)| \leqslant C\,|B(s)| \tag{3.2}$$

where $C = \max\{h(x): x \in [1/n, x]\}$. From (H_0) we have

$$r^{-1}(xy) \leqslant m_1 r^{-1}(x) r^{-1}(y), \quad x, y \geqslant M_1,$$

where M_1, m_1 are positive constants. Hence,

(3.3)
$$r^{-1}(xy) \leqslant C\big(1 + r^{-1}(x) + r^{-1}(y) + r^{-1}(x)r^{-1}(y)\big), \quad x, y \geqslant 0, C = \text{ constant.}$$

$$r^{-1}(x + y) \leqslant C\big(1 + r^{-1}(x) + r^{-1}(y)\big), \quad x, y \geqslant 0, C = \text{ constang.} \tag{3.4}$$

And from (3.2) we know

$$\begin{aligned} \left|r^{-1}(e - \lambda F_n(s, x))\right| &\leqslant r^{-1}(|e| + C\,|B(s)|) \\ &\leqslant C\big(1 + r^{-1}(|e|) + r^{-1}(|B(s)|)\big) \end{aligned} \tag{3.5}$$

where C is dependent only on n and the bound of $\|x\|$, and may vary when it appears at different places. Hence H is well defined.

Lemma 3.1. *$H(e, \lambda, x)$ is continuous in all its variables and strictly increasing with respect to e for $\lambda \in (0, 1]$.*

Proof: Suppose $e \to e_0$, $\lambda \to \lambda_0$, $x \to x_0$. Then we have

$$\begin{aligned} H(e, \lambda, x) - H(e_0, \lambda_0, x_0) &= \int_0^1 \big(r^{-1}(e - \lambda F_n(s, x)) - r^{-1}(e_0 - \lambda_0 F_n(s, x_0))\big)\, ds \\ &= \int_0^{\varepsilon} + \int_{\varepsilon}^{1-\varepsilon} + \int_{1-\varepsilon}^{1}. \end{aligned}$$

And from (3.5) we can deduce

$$\left|\int_0^{\varepsilon}\right| \leqslant 2C \int_0^{\varepsilon} \big(1 + r^{-1}(|e|) + r^{-1}\big(|e_0| + r^{-1}(|B(s)|)\big)\big) ds.$$

Hence from (H_2) we know

$$\int_0^{\varepsilon} \to 0, \int_{1-\varepsilon}^{\varepsilon} \to 0, \text{ as } \varepsilon \to 0.$$

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But on the interval $[\varepsilon, 1-\varepsilon]$ F_n is continuous so we get

$$\int_{\varepsilon}^{1-\varepsilon} \to 0, \text{ as } e \to e_0, \; \lambda \to \lambda_0, \; x \to x_0$$

and the results follows. ∎

Lemma 3.2. *$H(e, \lambda, x) = 0$ has a unique root $e = e(\lambda, x)$.*

Proof: It is obvious that $F_n(s, x)$ is strictly increasing with respect to s, hence $H(e, \lambda, x) \to +\infty$ as $e \to +\infty$ provided λF_n is bounded from above. Otherwise $\lambda F_n(s, x) \to +\infty$ as $s \to 1$. Choose $e = \lambda F_n(t, x)$. Hence $e \to +\infty$ as $t \to 1$ and we can assume $t > 1/2$. Se we have

$$\begin{aligned}\int_0^t r^{-1}(e - \lambda F_n(s, x))\,ds &\geqslant \int_0^{1/2} r^{-1}(e - \lambda F_n(s, x))\,ds \\ &\geqslant \int_0^{1/2} r^{-1}\big(\lambda F_n(1/2, x) - \lambda F_n(s, x)\big)\,ds > 0.\end{aligned}$$

But since $e > 0$ we also have

$$\begin{aligned}\left|\int_t^1 r^{-1}(e - \lambda F_n(s, x))\,ds\right| &= \int_t^1 r^{-1}(\lambda F_n(s, x) - e)\,ds \\ &\leqslant \int_t^1 r^{-1}(\lambda F_n(s, x))\,ds \leqslant C\int_t^1 r^{-1}(|B(s)|)\,ds \to 0.\end{aligned}$$

So we get $H(e, \lambda, x) > 0$ for e large. Similarly $H(e, \lambda, x) < 0$ for $-e$ large enough, and the proof is complete. ∎

Lemma 3.3. *Suppose $(H_0)(H_1)(H_2')$ are satisfied. Then the functional $H_1(e, \lambda, x)$ is continuous in (e, λ, x), strictly increasing in e, and $H_1(e, \lambda, x) = 0$ has a unique root $e = e_1(\lambda, x)$, where*

$$H_1(e, \lambda, x) = r^{-1}(e) + \delta \int_0^1 r^{-1}\left(e + \lambda \int_s^1 f_n(t, x(t))\,dt\right)ds.$$

Proof: We only list the following inequalities and omit the details.

$$\int_s^1 f_n(t, x(t))\,dt \leqslant CB_1(s)$$

$$r^{-1}\left(e + \lambda \int_s^1 f_n(t, x(t))\,dt\right) \leqslant C\big(1 + r^{-1}(|e|) + r^{-1}(B_1(s))\big)$$

$$|H_1(e, \lambda, x)| \leqslant r^{-1}(e) + \delta \int_0^1 r^{-1}(C\lambda B_1(s))\,ds, \text{ for } e < 0.$$

∎

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REMARK 3.4. From the proof of Lemma 3.1 and Lemma 3.2 we know that the continuity of H and H_1 is uniform for $\lambda \in [0, 1]$, e and x in bounded sets.

LEMMA 3.5. *$e\colon [0, 1] \times C[0, 1] \to R^1$ is bounded and uniformly continuous for $\lambda \in [0, 1]$ and x in a bounded set.*

PROOF: First we shall show that e is bounded. Otherwise we would have $e_m = e(\lambda_m, x_m) \to +\infty$ while x_m is bounded. Since F_n is increasing with respect to s and

$$\int_0^1 r^{-1}(e_m - \lambda_m F_n(s, x_m))\, ds = 0$$

there exists $t_m \in (0, 1)$ such that $e_m = \lambda_m F_n(t_m, x_m)$ and $t_m \to 1$. Let $e_m > 0$, hence from (3.5)

$$\begin{aligned}\int_0^{t_m} r^{-1}(e_m - \lambda_m F_n(s, x_m))\, ds &= \int_{t_m}^1 r^{-1}(\lambda_m F_n(s, x_m) - e_m)\, ds \\ &< \int_{t_m}^1 r^{-1}(\lambda_m F_n(s, x_m))\, ds \to 0 \text{ as } m \to \infty.\end{aligned}$$

But when $t_m > 1/2$ we have

$$\begin{aligned}\int_0^{t_m} r^{-1}(e_m - \lambda_m F_n(s, x_m))\, ds &\geqslant \int_0^{1/2} r^{-1}(e_m - \lambda_m F_n(s, x_m))\, ds \\ &\geqslant \int_{1/4}^{1/2} r^{-1}(e_m - C\,|B(s)|)\, ds \to +\infty,\end{aligned}$$

Hence we get a contradiction. Next we shall show that e is uniformly continuous. First let $|e| \leqslant R$, $0 \leqslant \lambda \leqslant 1$, $\varepsilon > 0$, $\|x\| \leqslant R$, where $R > 0$ is a constant. Then

$$H(e+\varepsilon, \lambda, x) - H(e, \lambda, x) \geqslant \int_{1/4}^{1/2} \big(r^{-1}(e + \varepsilon - \lambda F_n) - r^{-1}(e - \lambda F_n)\big) ds.$$

But we have from (3.2)

$$|e - \lambda F_n| \leqslant R + C\,|B(s)| \leqslant C_1$$

where C_1 is dependent only on n and R. Hence

$$H(e+\varepsilon, \lambda, x) - H(e, \lambda, x) \geqslant \beta \tag{3.6}$$

where $4\beta = \min\{r^{-1}(s+\varepsilon) - r^{-1}(s)\colon s \in [-C_1, C_1]\} > 0$ and depends only on R, ε. Let $\varepsilon > 0$ and $\lambda \in [0, 1]$, $|x| \leqslant R$. From the first part of the proof we have $|e(\lambda, x)| \leqslant K =$ constant. Hence from (3.6)

$$H(e(\lambda, x) + \varepsilon, x) \geqslant \beta > 0, \tag{3.7}$$

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where $\beta = \beta(R, \varepsilon)$. From Remark 3.4 we get $\eta > 0$ such that

$$H(e(\lambda, x) + \varepsilon, \lambda_0, x_0) > 0, \ |\lambda - \lambda_0| < \eta, \ \|x - x_0\| < \eta.$$

Similarly we can obtain

$$H(e(\lambda, x) - \varepsilon, \lambda_0, x_0) < 0, \ |\lambda - \lambda_0| < \eta, \ \|x - x_0\| < \eta.$$

Hence $|e(\lambda, x) - e(\lambda_0, x_0)| < \varepsilon$. The proof is complete. ∎

Now we come back to (3.1) with boundary condition (i). We always suppose $(H_0)(H_1)(H_2)(H_3)(H_4)$ are satisfied. Define the integral operator N by:

$$N(\lambda, x)(t) = \int_0^t r^{-1}(e(\lambda, x) - \lambda F_n(s, x))\, ds. \tag{3.8}$$

Denote $A_n(x) = N(1, x)$, where $x \in P$, $e(\lambda, x)$ is determined by Lemma 3.2. Obviously $N(\lambda, x)(0) = N(\lambda, x)(1) = 0$. Let $\lambda \in (0, 1]$, then from the definition of $e(\lambda, x)$ there exists $t_0 \in (0, 1)$ such that

$$e(\lambda, x) = \lambda \int_{1/2}^{t_0} f_n(t, x(t))\, dt.$$

And for $t \in (t_0, 1)$ we know

$$e(\lambda, x) < \lambda \int_{1/2}^{t_0} f_n(t, x(t))\, dt.$$

Hence

$$N(\lambda, x)(1) = 0 < \int_0^t r^{-1}(e(\lambda, x) - \lambda F_n(s, x))\, ds = N(\lambda, x)(t).$$

Similarly we can prove $N(\lambda, x)(t) > 0$ for $t \in (0, t_0)$. Hence $N\colon [0, 1] \times P \to P$ and for $\lambda \in (0, 1]$, $t \in (0, 1)$, $N(\lambda, x)(t) > 0$. It is easy to show that N is continuous and bounded. And if x belongs to some bounded set, then from (3.6) we have

$$|N(\lambda, x)(t_2) - N(\lambda, x)(t_1)| \leqslant C \int_{t_1}^{t_2} \left(1 + r^{-1}(|e(\lambda, x)|) + r^{-1}(|B(s)|)\right) ds.$$

Hence from Arzela's theorem, N is compact.

Lemma 3.6. *Suppose $(H_0)(H_1)(H_2)(H_3)(H_4)(H_5)$ are satisfied. Then the approximate problem (3.1) and (i) has solutions for any $n \in N$.*

Proof: Let $\lambda \in (0, 1]$ and $x \in P$ with $x = N(\lambda, x)$. Then x satisfies (3.1) and (i) while $x(t) > 0$ for $x \in (0, 1)$. Let $x(t)$ assume its maximum at $t_0 \in (0, 1)$. Then for $t \in (0, t_0)$ we have from (3.1) by integration that

$$r(x'(t)) \leqslant \int_t^{t_0} b(s) h(\max\{1/n, x(s)\})\, ds. \tag{3.9}$$

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We can assume without loss of generality that $t_0 \leqslant 1/2$. In the following we use C to represent any constant independent of λ, x and ε. From (H_3) we have $M > 0$ such that $h(x) \leqslant r(\varepsilon x)$ for $x \geqslant M$. Hence there exists $C(\varepsilon) > 0$ such that

$$H(x) \leqslant C(\varepsilon) + r(\varepsilon x), \text{ for } x \geqslant 1/n. \tag{3.10}$$

So from (3.9), (3.3), (3.4) we get:

$$\begin{aligned}
r(x'(t)) &\leqslant \int_t^{1/2} b(s)(C(\varepsilon) + r(\varepsilon + \varepsilon\,\|x\|))\,ds \\
x'(t) &\leqslant r^{-1}\left(|B(t)|\,(C(\varepsilon) + r(\varepsilon + \varepsilon\,\|x\|))\right) \\
x'(t) &\leqslant C\left(1 + r^{-1}(|B(t)|) + r^{-1}(C(\varepsilon) + r(\varepsilon + \varepsilon\,\|x\|))\right. \\
&\qquad \left. + r^{-1}\left(|B(t)|\right)r^{-1}(C(\varepsilon) + r(\varepsilon + \varepsilon\,\|x\|))\right) \\
\|x\| &\leqslant C\left(1 + C + r^{-1}(C(\varepsilon) + r(\varepsilon + \varepsilon\,\|x\|))\right. \\
&\qquad \left. + Cr^{-1}(C(\varepsilon) + r(\varepsilon + \varepsilon\,\|x\|))\right) \\
&\leqslant C\big(1 + r^{-1}(C(\varepsilon) + r(\varepsilon + \varepsilon\,\|x\|))\big) \\
&\leqslant C\big(1 + r^{-1}(C(\varepsilon)) + \varepsilon + \varepsilon\,\|x\|\big).
\end{aligned}$$

By letting $\varepsilon C < 1$ we have $\|x\| \leqslant K$ where K is a constant independent of x and λ. If we choose $R > K$ and put $B_R = \{x \in C[0, 1]\colon \|x\| < R\}$, then $x \neq N(\lambda, x)$ for $\lambda \in [0, 1]$, $\|x\| = R$. Therefore

$$\begin{aligned}
i(N(\lambda, x), P \cap B_R, P) &= i(A_n, R \cap B_R, P) \\
&= i(N(0, x), P \cap B_R, P) = 1
\end{aligned}$$

where i denote the fixed point index on P. Hence A_n has fixed points. The proof is now complete. ∎

In the following we turn to problem (3.1) with boundary condition (ii), that is, $x(0) = 0$, $\delta x(1) + x'(1) = 0$. We shall assume $(H_0)(H_1)(H_2')(H_3)(H_4)(H_5)$ are satisfied throughout. Instead of the operator A_n and N, we shall use the following:

$$N_1(\lambda, x)(t) = \int_0^t r^{-1}\left(e_1(\lambda, x) + \lambda \int_s^1 F_n(s, x)\right) ds. \tag{3.11}$$

Then $B_n x = N_1(1, x)$, where e_1 is determined by Lemma 3.3. It is straight forward to show that the integral is well defined in (3.10). Hence for $x \in C[0, 1]$, $N_1(\lambda, x) \in C[0, 1]$ and satisfies the boundary condition (ii). Write $y = N_1(\lambda, x)$. Then for $\lambda \in (0, 1]$ and $t \in (0, 1)$, $(r(y'))' < 0$, and as a consequence $y'(t) > 0$ provided $\delta = 0$, and furthermore $y(t) > 0$. When $\delta > 0$, the boundary condition yields $y'(1) = -\delta y(1)$. If $y(1) = 0$, then $y'(t) > 0$ for $t \in (0, 1)$. If $y(1) > 0$, then $y'(1) < 0$, but $r(y')$ is strictly decreasing, so we know y is concave and $y(t) > 0$ for $t \in (0, 1)$. The next Lemma is just parallel to Lemma 3.3.

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Lemma 3.7. *The functional $e_1(\lambda, x)$ is continuous and bounded.*

Lemma 3.8. *Suppose $(H_0)(H_1)(H_2')(H_3)(H_4)(H_5)$ are satisfied. Then problem (3.1) with (ii) has solutions for any integer n.*

Proof: From Lemma 3.3 and Arzela's theorem, N_1 is compact. Let $\lambda \in (0, 1)$, $x \in P$ and $x = N_1(\lambda, x)$, then x satisfies (3.1) and (ii). When $\delta = 0$, then integration of (3.1) on $[t, 1]$ and (3.10) will yield:

$$(3.12)\qquad \begin{aligned} r(x'(t)) &\leqslant \int_t^1 b(s)h(\max\{1/n, x\})\,ds \\ &\leqslant [C(\varepsilon) + r(\varepsilon + \|x\|)]B_1(t) \\ x'(t) &\leqslant C\left(1 + r^{-1}(B_1(t)) + r^{-1}(C(\varepsilon) + r(\varepsilon + \varepsilon\|x\|))\right. \\ &\quad \left. + r^{-1}(B_1(t))r^{-1}(C(\varepsilon) + r(\varepsilon + \varepsilon\|x\|))\right) \\ \|x\| &\leqslant C\big(1 + r^{-1}(C(\varepsilon) + r(\varepsilon + \varepsilon\|x\|))\big) \\ &\leqslant C\big(1 + r^{-1}(C(\varepsilon)) + \varepsilon + \varepsilon\|x\|\big). \end{aligned}$$

When $\delta > 0$, $x(t)$ will assume its maximum at $t_0 \in (0, 1)$. Integration on $[t, t_0]$, then on $[0, t_0]$ will also yield (3.12). The remainder of the proof is just the same as in the proof of Lemma 3.6. ∎

4. Proof of Theorem 1.3 and 1.4

In this section we shall complete the proof of Theorems 1.3 and 1.4.

Lemma 4.1. *Let $(H_1)(H_1)(H_2)(H_3)$ be satisfied. Then there exists a constant $R > 2$ independent of n such that for any solution x to problem (3.1) and (i) the following estimate holds:*

$$0 \leqslant x(t) \leqslant R - 1, \quad t \in [0, 1].$$

Proof: Let x be a solution which assumes its maximum at $t_0 \in (0, 1)$. For simplicity we assume $x(t_0) = \|x\| > 1$. Choose $t_1 \in (0, t_0)$ such that $x(t_1) = 1$ and $x(t) \geqslant 1$ for $t \in [t_1, t_0]$. From (H_3) we have $C(\varepsilon) > 0$ independent of n such that

$$(4.1)\qquad H(x) \leqslant C(\varepsilon) + r(\varepsilon, x), \text{ for } x \geqslant 1.$$

Upon integration of equation (3.1) on $[t, t_0]$ we have: (where C denotes some constant independent of n and ε and may vary at different appearances)

$$\begin{aligned} r(x'(t)) &\leqslant \int_t^{t_0} b(s)h(x(s))\,ds \\ &\leqslant B(s)(C(\varepsilon) + r(\varepsilon\|x\|)) \end{aligned}$$

provided $t_0 \leqslant 1/2$. If $t_0 \geqslant 1/2$, we can integrate on $[t_0, t]$ and obtain

$$-r(x'(t)) \leqslant |B(s)|\,(C(\varepsilon) + r(\varepsilon\,\|x\|)).$$

In both cases we get

$$\|x\| \leqslant C\big(1 + r^{-1}(C(\varepsilon)) + \|x\|\big).$$

Choose $\varepsilon C < 1$ and Lemma 4.1 is proved. ∎

Now let $R > 2$ be determined by Lemma 4.1. Replace $h(x)$ in (H_1) by $h(x) + 1$ if necessary and we can assume $h(x) \geqslant 1$ for $x \in (0, \infty)$. Define

$$D(x) = \sup\{h(u) \colon x \leqslant u \leqslant R\}$$
$$T(x) = \int_0^x \frac{du}{r^{-1}(D(u))}.$$

It is straight forward that $D \in (0, R]$, $h(x) \leqslant D(x)$ and D is decreasing. In addition $T \in C[0, R]$ by (H_4). Let ψ be determined by (H_5) for $H = R$, and suppose x is a solution to (3.1) and (i) with $x(t_0) = \|x\|$. Then we deduce

$$r(x'(t)) \geqslant \int_t^{t_0} \psi(s)\,ds$$

$$x(t) \geqslant \int_0^t r^{-1}\left(\int_u^{t_0} \psi(s)\,ds\right) du, \quad \text{for } t \in (0, t_0) \tag{4.2}$$

$$x(t) \geqslant \int_t^1 r^{-1}\left(\int_{t_0}^u \psi(s)\,ds\right) du, \quad \text{for } t \in (t_0, 1). \tag{4.3}$$

LEMMA 4.2. *Suppose $(H_0)(H_1)(H_2)(H_3)(H_4)(H_5)$ are satisfied, and let x be a solution of (3.1) with boundary condition (i), which assumes its maximum at t_x^0, that is, $x(t_x^0) = \|x\|$. Then there is a constant η independent of n such that $\eta \leqslant t_x^0 \leqslant 1 - \eta$.*

PROOF: Assume the contrary we would have a sequence of solutions x_n with corresponding maximum points t_n^0 satisfying $t_n^0 \to 0$ or 1. For convenience we suppose $t_n^0 \to 0$ and $t_n^0 < 1/2$. Thus from (4.3) and (H_5) we know

$$\|x_n\| = x(t_n^0) \geqslant \int_{1/2}^1 r^{-1}\left(\int_{1/2}^u \psi(s)\,ds\right) du > 0.$$

But $\|x_n\|$ is bounded from Lemma 4.1 and as a result we can set

$$\|x_n\| \to c_0 > 0, \text{ as } n \to \infty. \tag{4.4}$$

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Since x_n increases as $t \in (0, t_n^0)$, we obtain for $t \in (0, t_n^0)$ that

$$0 < r(x_n'(t)) \leqslant \int_t^{t_n^0} b(s) D(\max\{1/n, x_n(s)\})\, ds$$
$$\leqslant D(x_\infty(t)) \int_t^{t_n^0} b(x)\, ds \leqslant D(x_n(t))\, |B(t)|$$
$$0 < x_n'(t) \leqslant C\left(1 + r^{-1}(D(x_n(t))) + r^{-1}(|B(t)|) + r^{-1}(D(x_n(t))) r^{-1}(|B(t)|)\right).$$

Define $z_n(t) = T(x_n(t))$, then $z_n(0) = 0$ and evidently $D(x_n(t)) \geqslant D(R) > 0$. Hence $r^{-1}(D(x_n)) \geqslant r^{-1}(D(R)) > 0$ and

$$0 < z_n'(t) \leqslant C\left(1 + r^{-1}(|B(t)|)\right) \tag{4.5}$$

where C is independent of x_n. Thus integration on $(0, t_n^0)$ yields

$$z_n(t_n^0) \leqslant C \int_0^{t_n^0} \left(1 + r^{-1}(|B(t)|)\right) dt \to 0, \quad \text{as } n \to \infty.$$

Hence $T(c_0) = 0$, a contradiction to (4.4). The proof of lemma is complete. □

Proof of Theorem 1.3: We now complete the proof of theorem 1.3. Let x_n be an approximate solution to (3.1) and (i) and put $z_n(t) = T(x_n(t))$. Thus z_n is bounded in $C[0, 1]$. We again use t_n^0 to denote the maximum point of x_n. Hence for $t \in (0, t_n^0)$

$$0 < r(x_n'(t)) \leqslant D(x_n(t))\left(|B(t)| + \int_\eta^{1-\eta} b(s)\, ds\right) \leqslant CD(x_n(t))(1 + |B(t)|)$$
$$x_n'(t) \leqslant C\left\{1 + r^{-1}(x_n(t)) + r^{-1}(1 + |B(t)|) + r^{-1}(x_n(t)) r^{-1}(1 + |B(t)|)\right\}$$
$$\leqslant C\left\{1 + r^{-1}(x_n(t)) + r^{-1}(|B(t)|) + r^{-1}(x_n(t)) r^{-1}(|B(t)|)\right\}.$$

As a result we have for $t \in (0, t_n^0)$ that

$$0 < |z_n'(t)| \leqslant C\{1 + r^{-1}(|B(t)|)\}. \tag{4.6}$$

(Note that the above C's may be different.) Similarly we can prove (4.6) is true for $t \in (t_n^0, 1)$. Thus using the well known technique and Arzela's theorem, z_n has $C[0, 1]$ convergent subsequences. For simplicity we set $z_n \to z$ and $x = T^{-1}(z)$. Therefore $x_n \to x$ and $x(0) = x(1) = 0$. From (4.2) and (4.3), for $t \in (0, \eta)$:

$$x_n(t) \geqslant \int_0^t r^{-1}\left(\int_u^n \psi(s)\, ds\right) du.$$

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But $x_n(t) \geqslant x_n(\eta)$ for $t \in (\eta, t_0)$. If we define x^* by:

$$x^*(t) = \int_0^t r^{-1}\Big(\int_u^\eta \psi(s)\,ds\Big)du, \text{ for } t \in (0, \eta)$$

$$x^*(t) = \min\Big\{\int_0^\eta r^{-1}\Big(\int_u^\eta \psi(s)\,ds\Big)du, \int_{1-\eta}^1 r^{-1}\Big(\int_{1-\eta}^u (s)\,ds\Big)du\Big\}.$$

for $t \in [\eta, 1-\eta]$, and

$$x^*(t) = \int_t^1 r^{-1}\Big(\int_{1-\eta}^u \psi(s)\,ds\Big)du, \text{ for } t \in [1-\eta, 1].$$

Then $x_n(t) \geqslant x^*(t) > 0$; hence $x(t) > 0$ for $t \in (0, 1)$. From (4.6) $z_n'(1/2)$ is bounded; hence $x_n'(1/2)$ is bounded and we can assume $x_n'(1/2) \to \mu$. Then (3.1) yields:

$$x_n(t) - x_n(1/2) = \int_{1/2}^t r^{-1}[r(x_n'(1/2)) - \int_{1/2}^t f_n(s, x_n)\,ds]\,dt$$

$$x(t) - x(1/2) = \int_{1/2}^t r^{-1}[r(\mu) - \int_{1/2}^t f(s, x)\,ds]\,dt.$$

Thus $(r(x'))' = -f(t, x)$ for $t \in (0, 1)$. The proof is complete. □

Finally we turn to the proof of Theorem 1.4. Since it is essentially the same as the above steps, we only list the following lemma and omit the details.

LEMMA 4.3. *Suppose $(H_0)(H_1)(H_2')(H_3)(H_4)(H_5)$ are satisfied. Then there exists a constant $\eta \in (0, 1)$ independent of n such that $t_x^0 \geqslant \eta$, where t_x^0 is the maximum point of x, the solution of (3.1) with (ii).*

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