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Abstract

We show that the algebraic local fundamental group of any Kawamata log terminal singularity as well as the algebraic fundamental group of the smooth locus of any log Fano variety are finite.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. The study of the local topology of singularities has a long history. In the surface case, Mumford proved that a point in a normal surface has a trivial local fundamental group if and only if it is smooth (cf. [Mum61]). Since then the investigation of the local topology of a singularity has been one of the most important tools to study singularities. It is agreed that there are three basic objects to study. Given a singularity, the *link* $L(0 \in X)$ should carry essentially all the local topological information of the singularity. It has a continuous map to a topological space whose deformation retract is the *simple normal crossing variety* E defined as the preimage of 0 for a log resolution $Y \to (X, 0)$. The combinatorial gluing data of E is then captured in the dual complex $\mathcal{D}(E) = \mathcal{DR}(0 \in X)$. See [Kol12] for more background.

Recently, examples (cf. [KK11, Kol11, Kol12]) have been constructed to show that for a general singularity, the dual complex can be as complicated as possible. When $0 \in X$ is log canonical, Kollár also constructed three-dimensional examples that have more complicated local topology than people expected. For instance, the local fundamental group of such a singularity can be the fundamental group of any connected two-dimensional manifold. Kollár indeed asked whether there is any nontrivial restriction of $\pi_1(E)$ (cf. [Kol11, Question 25]).

In this paper, we aim to show that the local topology of a Kawamata log terminal (klt) singularity should be much simpler than the log canonical case. In fact, the following conjecture is proposed by Kollár.

Conjecture 1 [Kol11, Question 26]. Let $0 \in (X, \Delta)$ be a klt singularity. Then the local fundamental group $\pi_1^{\mathrm{loc}}(X, 0) := \pi_1(L(0 \in X))$ is finite.

In this direction, Kollár and Takayama proved that $\pi_1(E)$ is trivial (cf. [Kol93, Tak03]). However, this is not enough to conclude that $\pi_1(L(0 \in X))$ is finite (e.g. consider a surface rational singularity that is not a quotient singularity).

We can similarly define the local algebraic fundamental group $\hat{\pi}_1^{\mathrm{loc}}(X, 0)$ which is just the pro-finite completion of $\pi_1^{\mathrm{loc}}(X, 0)$. Our first theorem says that Kollár's conjecture is true at least for $\hat{\pi}_1^{\mathrm{loc}}(X, 0)$.

Theorem 1. *Let $0 \in (X, \Delta)$ be an algebraic klt singularity. Then the algebraic local fundamental group $\hat{\pi}_1^{\mathrm{loc}}(X, 0)$ is finite.*

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We note that the main result of [KK11] implies that there exists an algebraic singularity $(X, 0)$ with $\hat{\pi}_1(X, 0) = \{e\}$ but $\pi_1(X, 0)$ is an infinite group.

The corresponding global result is the following.

Theorem 2. *Let $(X, \Delta + \Delta')$ be a projective klt pair with $\Delta' \geqslant 0$, and the coefficients of Δ are contained in $\{(m-1)/m | m \in \mathbb{N}\}$. Assume that $-(K_X + \Delta + \Delta')$ is ample. Denote by X^0 the maximal open locus where the restriction (X, Δ) is an orbifold. Then the algebraic orbifold fundamental group $\hat{\pi}_1^{\mathrm{orb}}(X^0, \Delta|_{X^0})$ is finite.*

We note that the question on the fundamental group of the smooth locus of a log Fano variety has attracted lots of interest. When the dimension is equal to 2 and there is no boundary divisor, we know that the topological fundamental groups are always finite (see [GZ95, KM99, MT84]) and we indeed have a classification of them (cf. [Xu09]).

Beyond only being an analog, we could really connect the above two theorems using the local-to-global induction, namely, we could show that the local result Theorem 1 implies the global result Theorem 2 in the same dimension,[1] and then the global result gives the local one for one dimension higher. However, to make the proof shorter, we present the proof of Theorem 2 using the boundedness theorem in [HMX12], and then establish Theorem 1.

2. Finiteness of algebraic fundamental groups

Notation and conventions. We follow the terminology in [KM98]. We call a finite morphism between two log pairs $f : (Y, \Delta_Y) \to (X, \Delta)$ *log étale in codimension* 1 if $f^*(K_X + \Delta) = K_Y + \Delta_Y$. We note that here Δ and Δ_Y are effective divisors. A projective log pair (X, Δ) is called *log Fano* if (X, Δ) has klt singularities and $-(K_X + \Delta)$ is ample. Given a point p on a log pair (X, Δ), we use $\mathrm{mld}(p, X, \Delta)$ to mean the *minimal log discrepancy* $\min_E a(E, X, \Delta) + 1$, where the minimum runs over all exceptional divisors E whose center on X is p.

Let $f : Y \to X$ be a morphism induced by a surjection $\hat{\pi}_1^{\mathrm{orb}}(X^0, \Delta|_{X^0}) \to G$ for some finite group G; then if we write $f^*(K_X + \Delta) = K_Y + \Delta_Y$, we have $\Delta_Y \geqslant 0$. Thus, if we write $f^*(K_X + \Delta + \Delta') = K_Y + \Delta'_Y$, it satisfies $\Delta'_Y \geqslant \Delta_Y \geqslant 0$. Therefore, Theorem 2 immediately follows from the following result.

Proposition 1. *Let (X, Δ) be a log Fano variety. Let $f : Y \to (X, \Delta)$ be a finite surjective morphism, such that if we write*

$$f^*(K_X + \Delta) = K_Y + \Delta_Y,$$

then Δ_Y is effective. Then the degree of f is bounded by a constant N only depending on (X, Δ).

Proof. Let $M \in \mathbb{N}$ be such that $M(K_X + \Delta)$ is Cartier; then $M(K_Y + \Delta_Y)$ is Cartier. It follows from [HMX12, Corollary 1.8] that

$$\mathrm{vol}(K_Y + \Delta_Y) = \deg(f) \cdot \mathrm{vol}(K_X + \Delta)$$

is bounded from above by a constant $C = C(M, n)$ which only depends on M and $n = \dim(X)$. Thus $\deg(f)$ is bounded from above by

$$\frac{C}{(-K_X - \Delta)^n}. \qquad \square$$

[1] After this paper was posted on arXiv, this approach was worked out by Greb, Kebekus, and Peternell in [GKP13].

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As we mentioned in the introduction, here we use the strong boundedness result in [HMX12]. We can argue more straightforwardly using Theorem 1 in the same dimension. Later we will see that in order to prove Theorem 1 we only need Proposition 1 in one dimension lower.

The next lemma associates to every klt singularity an exceptional log canonical place after adding a certain auxiliary divisor. Although the exceptional log canonical place depends on the choice of the auxiliary divisor, this construction proved to be useful for many questions (cf. [HX09, Kol07, LX11]).

Lemma 1. *Let $p \in (X,\Delta)$ be a klt point. There exist a $\mathbb{Q}$-divisor H on X and a birational morphism $f : Y \to X$ from a normal variety such that:*

(1) *Y has a prime divisor E such that $\mathrm{Center}_X(E) = p$, $-(K_Y + f_*^{-1}\Delta + E)$ and $-E$ are ample over X (in particular, $\mathrm{Ex}(f) = \mathrm{Supp}(E)$); and*
(2) *$(X, \Delta + H)$ is klt on $X \backslash \{p\}$, $\mathrm{mld}(p, X, \Delta + H) = 0$ and E is the unique divisor such that the discrepancy $a(E, X, \Delta + H) = -1$.*

Proof. We first choose an ample $\mathbb{Q}$-divisor L on X such that $(X, \Delta + L)$ is log canonical at p but klt at $X\backslash\{p\}$. Take a log resolution $g : Z \to (X, \Delta + L)$ such that $\mathrm{Ex}(g)$ supports a fixed relative ample divisor A over X. We write

$$g^*(K_X + \Delta + (t+\epsilon)L) \sim_{\mathbb{Q}} K_Z + g_*^{-1}(\Delta + tL) + \sum_{i=1}^{k} a_i E_i + (\epsilon g^* L + \delta A),$$

where $0 < \delta \ll \epsilon$ such that $\epsilon g^* L + \delta A \sim_{\mathbb{Q}} L'$ is a general ample $\mathbb{Q}$-divisor and each a_i depends on t and δA. Choosing L a general ample $\mathbb{Q}$-divisor with small coefficients passing through p, and using δA to perturb, we can assume that there exists a $t_0 > 0$ such that in the above formula, if we take $t = t_0$, there exist a unique a_i, say a_1, which is equal to 1, some other $a_i < 1$ $(i \geqslant 2)$ and the center of $E_Z := E_1$ on X is p.

Considering the pair $(Z, g_*^{-1}(\Delta + tL) + E_Z + \sum_{i=2}^{k} E_i + L')$, we have

$$K_Z + g_*^{-1}(\Delta + tL) + E_Z + \sum_{i=2}^{k} E_i + L' \sim_{X,\mathbb{Q}} \sum_{i=2}^{m} (1 - a_i) E_i.$$

We run a $(K_Z + g_*^{-1}(\Delta + tL) + E_Z + \sum_{i=2}^{k} E_i + L')$-MMP with scaling of L' over X. By [BCHM10], this minimal model program (MMP) will terminate with a good minimal model $h : W \to X$. As it contracts all the divisors whose supports are contained in the stable base locus, we know that $\phi : Z \dashrightarrow W$ precisely contracts all E_i for $i \geqslant 2$. Thus the divisorial part of $\mathrm{Ex}(h)$ is E_W, and on W we have

$$K_W + h_*^{-1}(\Delta + tL) + E_W + \phi_* L' \sim_{X,\mathbb{Q}} 0,$$

where E_W denotes the pushforward of E_Z on W. Since $(W, h_*^{-1}(\Delta + tL) + E_W + \phi_* L')$ is divisorial log terminal (dlt) with only one divisor of coefficient 1, it is indeed purely log terminal (plt). Furthermore, since it is an MMP with scaling of L', by the definition of MMP with scaling (cf. [BCHM10]), we know, for some sufficiently small $\sigma > 0$, that

$$K_W + h_*^{-1}(\Delta + tL) + E_W + (1+\sigma)\phi_* L'$$

is nef over X. We let Y be the log canonical model of $(W, h_*^{-1}(\Delta + tL) + E_W + (1+\sigma)\phi_* L')$, i.e.,

$$Y = \mathrm{Proj} \oplus_d h_* \mathcal{O}_W(d(K_W + h_*^{-1}(\Delta + tL) + E_W + (1+\sigma)\phi_* L')).$$

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As ϕ contracts E_i ($i \geqslant 2$), we know that $\phi_* A \sim -\lambda E_W$ for some $\lambda > 0$. Thus

$$K_W + h_*^{-1}(\Delta + tL) + E_W + (1+\sigma)\phi_* L' \sim_{X,\mathbb{Q}} \sigma\delta\phi_* A = -\sigma\delta\lambda E_W,$$

which is nef.

Define $H = tL + (h \circ \phi)_* L'$. We see that $W \to Y$ cannot contract E_W, thus it is a small morphism. As $(W, h_*^{-1}(\Delta + tL) + E_W + \phi_* L')$ is plt, so is $(Y, E + f_*^{-1}(H + \Delta))$ where E denotes the pushforward of E_W on Y and we know that $-E$ is f-ample.

Since (X, Δ) is klt, we know that

$$-(K_Y + f_*^{-1}\Delta + E) \sim_{X,\mathbb{Q}} -(1 + a(E, X, \Delta))E$$

is f-ample. □

Remark 1. In general, given a projective morphism $g : (X, \Delta) \to S$ from a klt pair to a normal variety such that $g_*(\mathcal{O}_X) = \mathcal{O}_S$ and $-(K_X + \Delta)$ is ample over $S \backslash \{p\}$ for some point $p \in S$, the same argument shows that we can find $f : Y \to S$ such that:

(i) X and Y are isomorphic over $S \backslash \{p\}$;
(ii) $f^{-1}(p)$ is an irreducible divisor E;
(iii) if we let Δ_Y be the birational transform of Δ on Y, then $-(K_Y + \Delta_Y + E)$ is f-ample.

In this construction, we call E a *Kollár component* of (X, Δ). As we mentioned, it depends on the auxiliary $\mathbb{Q}$-divisor H. If we write $(K_Y + f_*^{-1}\Delta + E)|_E = K_E + \Gamma$, where $\Gamma = \mathrm{Diff}_E f_*^{-1}\Delta$ as defined in [Kol+92, § 16], then the pair (E, Γ) is log Fano.

The Kollár component was first studied in [Kol07]. Later, in [LX11], it was interpreted as the only remaining exceptional divisor after an MMP sequence scaled by a carefully chosen ample divisor as above to make it log Fano.

Proof of Theorem 1. Let $0 \in (X, \Delta)$ be an algebraic singularity on a pair (X, Δ). Applying the construction in Lemma 1, we denote by $f : Y \to X$ a morphism which precisely extracts a Kollár component E.

Now let

$$\cdots \to (X_i, p_i) \to (X_{i-1}, p_{i-1}) \to \cdots \to (X_0, p_0) = (X, 0)$$

be a sequence of finite morphisms such that each one is finite and étale for the restriction $X_{i+1} \backslash \{p_{i+1}\} \to X_i \backslash \{p_i\}$ and Galois for $X_{i+1} \backslash \{p_{i+1}\} \to X \backslash \{0\}$. We want to show that it stabilizes for sufficiently large i.

For each i, we let Y_i be the normalization of the main component of $X_i \times_X Y$ with the morphism $f_i : Y_i \to X_i$. Thus there are commutative diagrams as follows.

$$\begin{tikzcd} E_{i+1} \subset Y_i \arrow[r, "\psi_i"] \arrow[d, "f_{i+1}"'] & E_i \subset Y_i \arrow[d, "f_i"] \\ p_{i+1} \in X_{i+1} \arrow[r] & p_i \in X_i \end{tikzcd}$$

Denote the pullback of Δ on X by Δ_i. Let $(K_{Y_i} + f_{i*}^{-1}\Delta_i + E_i)|_{E_i} = K_{E_i} + \Gamma_i$. Since $\psi_i^*(f_{i*}^{-1}\Delta_i) = f_{i+1*}^{-1}\Delta_{i+1}$, we conclude that

$$\psi_i^*(K_{Y_i} + f_{i*}^{-1}\Delta_i + E_i) = (K_{Y_{i+1}} + f_{i+1*}^{-1}\Delta_{i+1} + E_{i+1}).$$

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Restricting on the Kollár components, this implies that the induced morphism

$$\psi_i|_{E_{i+1}} : (E_{i+1}, \Gamma_{i+1}) \to (E_i, \Gamma_i)$$

is log étale in codimension 1.

It follows from Proposition 1 that there exists an $M \in \mathbb{N}$ such that $\psi_i|_{E_i}$ is an isomorphism for $i > M$. Thus, fixing such an $i > M$, ψ_i is a finite morphism, totally ramified over E_i. Let γ be the element in $\pi_1(L(p_i \in X_i))$ corresponding to the loop around a general point of E_i. We only need to verify that the order of γ is finite. Cutting Y_i to a surface S_i, and taking the Cartesian product, we have the following diagram.

$$\begin{tikzcd}
C_{i+1} \subset S_{i+1} \arrow[r] \arrow[d, "\phi_{i+1}"'] & C_i \subset S_i \arrow[d, "\phi_i"] \\
p_i \in T_{i+1} \arrow[r] & p_i \in T_i
\end{tikzcd}$$

As the corresponding ramified covering is trivial along $C_i = \mathrm{Ex}(\phi_i)$, we know that if we let the surjection $\hat{\pi}_1^{\mathrm{loc}}(T_i, p_i) \to G$ correspond to the covering, then G is a finite cyclic group, which is generated by the image of γ. Thus $\pi_1^{\mathrm{loc}}(T_i, p_i) = \pi_1(S_i \backslash C_i) \to G$ indeed factors through $H_1(S_i \backslash C_i)$. However, the homolog class $[\gamma]$ is in the kernel of

$$H_1(S_i \backslash C_i) \to H_1(S_i) = H_1(C_i).$$

By Mumford's calculation on the normal surface singularity (cf. [Mum61, p. 235]), we know that $[\gamma]$ is a torsion element. Thus for any $j \gg i$, $T_{j+1} \to T_j$ is an identity, and so is $X_{j+1} \to X_j$. $\square$

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