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THE MAP $SJ \to SF$ DOES NOT DELOOP MOD 2

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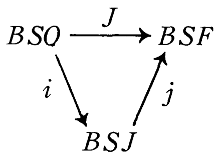
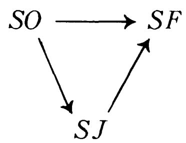
It has been widely conjectured that there exists a homotopy commutative diagram

(A)



where J is the stable Whitehead J-homomorphism and BSJ is the space constructed in [**3**]. In [**4**], Stasheff and the author proved that this conjecture is false. However, Quillen's proof of the Adams conjecture in [**7**] has as a corollary the existence of the homotopy commutative diagram

(B)



where $SJ = \Omega BSJ$. Indeed, Sullivan has proved that there is a space, called Coker(J), such that

$$SF = SJ \times \text{Coker}(J).$$

This suggests the possibility that BSJ is simply the wrong classifying space for SJ: It is conceivable that SJ can be delooped in another way so as to give rise to diagram (A). In this paper, we show that this is not the case.

More precisely, let us assume that BSJ is any space whose homotopy groups conform to the table below:

n mod 8	0	1	2	3	4	5, 6, 7
$\pi_n(BSJ)$	$Z/2^k$	$Z/2$	$Z/2 \oplus Z/2$	$Z/2$	$Z/8$	0
generators	ρ_n	$\rho_{n-1}\eta$	$\rho_{n-2}\eta\eta$, μ_n	$\mu_{n-1}\eta = \mu_n$	ν_n	

In this table, 2^k is the Milnor-Kervaire number. In dimensions n congruent to 4 mod 8 we have the relation $4\nu_n = \mu_{n-1}\eta$. Dimensions 0, 1, and 2 are exceptional. We have $\pi_0(BSJ) = \pi_1(BSJ) = 0$. The group $\pi_2(BSJ)$ is $Z/2$ generated

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by μ_2, and $\mu_2\eta = 0$ instead of μ_3. We assume then from this point forward that BSJ is a completely arbitrary space that satisfies the conditions stated in this paragraph.

THEOREM. *Diagram* (A) *does not exist.*

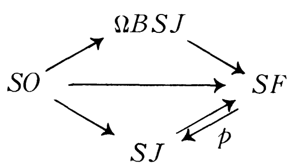
STANDING HYPOTHESIS: Diagram (A) exists.

The idea of the proof is to use our standing hypothesis to construct the diagram of Lemma 5 and apply the methods of [**4**] to obtain a contradiction. The diagram of Lemma 5 differs from the corresponding diagram of [**4**] only in that $S(RP^{12})$ appears in place of $S(RP)$. Since the contradiction that forms the cornerstone of our proof occurs in cohomology degree 13, it turns out that $S(RP^{12})$ is sufficient.

We shall need several lemmas to establish our theorem, and we shall use our standing hypothesis for each of these lemmas. Since all of our obstructions are 2-primary, please understand that all exact sequences, homotopy equivalences, and isomorphisms are 2-primary.

LEMMA 1. *ΩBSJ and SJ have the same homotopy type.*

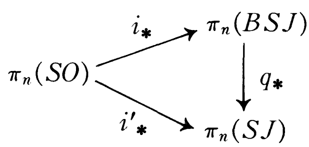
Proof. We have the homotopy commutative diagram where p is the product



splitting map of Sullivan. The resulting map

$$BSJ \xrightarrow{q} SJ$$

induces isomorphisms in homotopy except possibly in dimensions $8k + 1$ and $8k + 2$ because of the diagram



in which i_* and i'_* are surjective except in dimensions $8k + 1$ and $8k + 2$.

In SF, $\mu_{8k+2} = 4\nu_{8k+3}$, where μ_{8k+2} is the Adams class of [**1**], η is the Hopf map, and ν_{8k+3} is the generator of $\operatorname{Im}(J)$. Then $q_*\mu_{8k+2} = \mu_{8k+2}$. For if $q_*\mu_{8k+2} = 0$, then $q_*4\nu_{8k+2} = 0$ and q_* is not an isomorphism in dimension $8k + 3$, contrary to what we have already proved. Since q_* is therefore an isomorphism in dimension $8k+2$, it must also be bijective in dimension $8k+1$. Otherwise $q_*\mu_{8k+1} = a\rho_{8k-1}\eta\eta$ where $a = 0$ or 1. Then

$$\begin{aligned} q_*\mu_{8k+2} &= q_*\mu_{8k+1}\eta \\ &= \rho_{8k-1}\eta\eta\eta \\ &= 0 \end{aligned}$$

because $\rho_{8k-1}\eta\eta\eta = 0$ in SO. This contradiction proves that q_* is an isomorphism in dimension $8k+1$. Thus q is an homotopy equivalence.

We now recall some facts about the homology and cohomology of SO, BSO, SF, and BSF. These facts can be deduced from [**5**].

$$H^*(BSF) = P[w_i|2 \leqq i] \otimes E[e_I|I \in S],$$
$$H^*(SF) = P[p_{2i-1}|1 \leqq i] \otimes P[w_I|I \in S],$$
$$\sigma w_{2i} = p_{2i-1}, \text{ and } \sigma e_I = w_I.$$

The tensor splittings are splittings of $A(2)$ Hopf algebras. The p_{2i-1} are primitive and

$$\Delta w_n = \sum\{w_i \otimes w_j|i+j=n\}.$$

The set S is described by Milgram, but we need not discuss it here. The maps

$$BSO \to BSF \quad \text{and} \quad SO \to SF$$

send the w_i of BSF to the w_i of BSO and the p_{2i-1} of SF to the primitive indecomposables of SO; these latter primitive indecomposables are also denoted by p_{2i-1}.

LEMMA 2. *There are subsets* $\{w_i|2 \leqq i \leqq 15\}$ *and* $\{e_i|3 \leqq i \leqq 15\}$ *of* $H^*(BSJ)$ *and an* $A(2)$ *algebra epimorphism*

$$P[w_i|2 \leqq i \leqq 15] \otimes E[e_i|3 \leqq i \leqq 15] \to H^*(BSJ) \to 0 \text{ in degrees } \leqq 15.$$

We have $Sq^i w_j$ *as given by the Wu relations* [**10**], *and* $Sq^i e_j$ *is as in* $H^*(BBSO)$ *mod decomposables in* $H^*(BSJ)$ *that involve the* e_i (*see* [**4**]).

Proof. Since the composition

$$H^*(SO) \overset{i^*}{\leftarrow} H^*(SJ) \overset{j^*}{\leftarrow} H^*(SF)$$

is surjective by the remarks preceding this proof, it follows that i^* is surjective. Since i^* is surjective, the Serre spectral sequence of the fibration

$$SO \to SJ \to BSO$$

splits and

$$H^*(SJ) = P[p_{2i-1}|i \geqq 1] \otimes P[w_i|2 \leqq i].$$

Since $j : SJ \to SF$ is an H-map by our standing hypothesis, the first factor is an $A(2)$ Hopf subalgebra of $H^*(SJ)$. It is automatic that the second factor is an $A(2)$ subalgebra, but its coalgebra structure is not obvious.

The map $j : SJ \to SF$ is a homotopy equivalence in dimensions $\leqq 5$. Thus the map

$$j^* : H^*(SF) \to H^*(SJ)$$

is a Hopf algebra isomorphism in degrees $\leqq 5$. Rechoose the w_i of $H^*(SJ)$ so that they are the images of the w_I of $H^*(SF)$ in degrees $i = 1$ and 2. Rename these w_I in SF, calling them w_2 and w_3. They are both primitive.

In $H_*(SF)$, the square of the dual of w_2 is nonzero because the dual of w_2 lies in the polynomial algebra part of $H_*(SF)$. Therefore the dual of this nonzero square is a class w_4 such that

$$\Delta w_4 = w_4 \otimes 1 + w_2 \otimes w_2 + 1 \otimes w_4.$$

Since the square of the dual of w_2 is primitive, w_4 is indecomposable. Replace the w_I of degree 4 by w_4, and let

$$w_5 = Sq^1 w_4$$
$$w_6 = Sq^2 w_4 + w_2 w_4$$
$$w_7 = Sq^3 w_4 + w_2 w_5 + w_3 w_4.$$

We have now defined indecomposable Stiefel-Whitney classes for SF and SJ of degrees $\leqq 7$ such that

$$\Delta w_n = \sum \{w_i \otimes w_j | i + j = n\}.$$

They are indecomposable because $H^*(SJ)$ and $H^*(SO \times BSO)$ are isomorphic as $A(2)$ algebras. Indecomposability in $H^*(SJ)$ implies indecomposability in $H^*(SF)$.

The primitive classes of degree $\leqq 7$ in $H^*(SJ)$ are therefore

$$p_1, p_2^2, p_3, p_1^4, p_5, p_3^2, p_7$$
$$w_2, w_3, w_2^2, w_5 + w_2 w_3, w_3^2, w_7 + w_5 w_2 + w_4 w_3 + w_2^2 w_3.$$

Straightforward calculations show that they are primitive, and they are the only primitives because

$$j_* : H_*(SJ) \to H_*(SF)$$

is injective. By the structure theorem for Hopf algebras, $H_*(SJ)$ and $H_*(SO \times BSO)$ are isomorphic as algebras. Thus $H_*(SJ)$ has two decomposables of each degree, so that $H^*(SJ)$ has two primitives of each degree.

We have now displayed primitives of $H^*(SF)$ that map onto the primitives of $H^*(SJ)$ in degrees $\leqq 7$. We can extend this result to show that

$$j^* : PH^*(SF) \to PH^*(SJ)$$

is surjective in degrees $\leqq 14$ as follows: Let $8 \leqq n \leqq 14$. If n is even, then the primitives of degree n are the squares of the primitives of degree $n/2$. Thus they are in $\mathrm{Im}(j^*)$. If n is odd, then

$$Sq^{n-7}(w_7 + w_5w_2 + w_4w_3 + w_2^2w_3)$$

is an indecomposable primitive of degree n. Thus j^* is surjective in degree n.

We can now observe that

$$j_* : QH_*(SJ) \to QH_*(SF)$$

is injective in degrees $\leqq 14$. Let $x_i \in H_i(SF)$ denote the dual of the primitive in the p_{2n-1}. Let y_I and y_i denote the duals of the primitives in the w_I and w_i. Then

$$j_*x_i = x_i$$

$$j_*y_i = y_i \quad \text{if } 2 \leqq i \leqq 14.$$

Thus the map

$$j^* : \mathrm{Ext}_{H_*(SF)}(Z/2, Z/2) \to \mathrm{Ext}_{H_*(SJ)}(Z/2, Z/2)$$

sends σx_i to σx_i and σy_i to σy_i, where

$$\mathrm{Ext}_{H_*(SF)}(Z/2, Z/2) = P[\sigma x_i | 1 \leqq i] \otimes E[\sigma y_i | 2 \leqq i \leqq 14] \otimes E[\sigma y_I | I \notin S']$$

and $S' = \{I | w_i \text{ replaces } w_I\}$.

Let $E(G)$ denote the Eilenberg-Moore spectral sequence such that

$$E_2 = \mathrm{Ext}_{H_*(G)}(Z/2, Z/2) \quad \text{and} \quad E_\infty = H^*(BG).$$

Since $E(SF)$ collapses by [**5**], the σx_i and σy_i are permanent cycles in $E(SJ)$. Therefore they are the only indecomposables of $H^*(BSJ)$ in degrees $\leqq 15$. The epimorphism

$$H^*(BSO) \leftarrow H^*(BSJ)$$

implies that there are no relations among the σx_i. That the $(\sigma y_i)^2$ are 0 in $H^*(BSF)$ implies that they are 0 in $H^*(BSJ)$. Thus in degrees $\leqq 15$, $H^*(BSJ)$ has the asserted algebra structure. The $A(2)$ structure follows from the diagram below, in which the left vertical map is now clearly an isomorphism of $A(2)$ modules, for it is surjective and, by counting, injective.

$$\begin{array}{ccc} PH^*(SJ) & \xleftarrow{\text{epic}} & PH^*(SF) \\ \| \uparrow \sigma & & \| \uparrow \sigma \\ QH^*(BSJ) & \longleftarrow & QH^*(BSF) \end{array}$$

LEMMA 3. *The map*

$$j : BSJ \to BSF$$

induces an isomorphism

$$[CP^{2n}/CP^{2n-2}, BSJ] \to J(CP^{2n}/CP^{2n-2})$$

for $n \geqq 1$.

Proof. The cofibration sequence

$$S^{4n-1} \xrightarrow{\eta} S^{4n-2} \to CP^{2n}/CP^{2n-2} \to S^{4n} \xrightarrow{\eta} S^{4n-1}$$

gives rise to the map of exact sequences

$$\begin{array}{ccccccccc}
\tilde{K}O(S^{4n-1}) & \leftarrow & \tilde{K}O(S^{4n-2}) & \leftarrow & \tilde{K}O(CP^{2n}/CP^{2n-2}) & \leftarrow & \tilde{K}O(S^{4n}) & \leftarrow & \tilde{K}O(S^{4n-1}) \\
\downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* \\
[S^{4n-1}, BSJ] & \leftarrow & [S^{4n-2}, BSJ] & \leftarrow & [CP^{2n}/CP^{2n-2}, BSJ] & \leftarrow & [S^{4n}, BSJ] & \leftarrow & [S^{4n-1}, BSJ].
\end{array}$$

If n is even, diagram (C) becomes

$$\begin{array}{ccccccc}
0 & \leftarrow & \tilde{K}O(CP^{2n}/CP^{2n-2}) & \leftarrow & \tilde{K}O(S^{4n}) & \leftarrow & 0 \\
 & & \downarrow i_* & & \downarrow i_* & & \\
0 & \leftarrow & [CP^{2n}/CP^{2n-2}, BSJ] & \leftarrow & [S^{4n}, BSJ] & \leftarrow & 0.
\end{array}$$

The right hand vertical map is surjective. Therefore the left hand vertical map is surjective. Thus each element of $[CP^{2n}/CP^{2n-2}, BSJ]$ maps into $J(CP^{2n}/CP^{2n-2})$. Notice too that $[CP^{2n}/CP^{2n-2}, BSJ]$ is $Z/2^k$, generated by $i_*(\nu^{2n-1})$ where ν is the realification of the stable canonical line bundle over CP^{2n}. By [**2**], $J(CP^{2n}/CP^{2n-2})$ is also $Z/2^k$ generated by ν^{2n-1}. This proves the lemma in case n is even.

If n is odd, diagram (C) becomes

$$\begin{array}{ccccccccc}
0 & \leftarrow & Z/2 & \leftarrow & \tilde{K}O(CP^{2n}/CP^{2n-2}) & \overset{\alpha}{\leftarrow} & Z & \leftarrow & 0 \\
\downarrow & & \downarrow i_* & & \downarrow i_* & & \downarrow \text{epic} & & \downarrow \\
Z/2 & \leftarrow & Z/2 \oplus Z/2 & \leftarrow & [CP^{2n}/CP^{2n-2}, BSJ] & \overset{\eta^*}{\leftarrow} & Z/8 & \leftarrow & Z/2 .
\end{array}$$

By [**2**], the group $\tilde{K}O(CP^{2n}/CP^{2n-2})$ is free abelian on the generator ν^{2n-1}. This generator maps nontrivially to $Z/2$. The α image of the generator of Z is

$$\nu^{2n} = 2\nu^{2n-1}.$$

Since $i_* \neq 0$ on $Z/2$, we know that $i_*\nu^{2n-1} \neq 0$. Therefore $i_*\nu^{2n-1}$ and $i_*\nu^{2n}$

generate $[CP^{2n}/CP^{2n-2}, BSJ]$. Indeed, this argument shows that $[CP^{2n}/CP^{2n-2}, BSJ]$ is cyclic of order 8 or 16, generated by $i_*\nu^{2n-1}$. The correct order is 8, for the homotopy table of BSJ shows that $\mu_{8k+3}\eta = 4\nu_{8k+4}$, so that the map $Z/2 \to Z/8$ is nontrivial. Since the middle vertical map is again surjective, $[CP^{2n}/CP^{2n-2}, BSJ]$ maps into $J(CP^{2n}/CP^{2n-2})$. Since the generator of the former group maps again to the generator of the latter group, the map is again bijective.

LEMMA 4. *The map $BSJ \to BSF$ induces an isomorphism of exact sequences*

$$\begin{array}{ccccccc} 0 \leftarrow [CP^{2n-2}, BSJ] & \leftarrow & [CP^{2n}, BSJ] & \overset{q^*}{\leftarrow} & [CP^{2n}/CP^{2n-2}, BSJ] & \leftarrow 0 \\ \downarrow & & \downarrow & & \downarrow & \\ 0 \leftarrow J(CP^{2n-2}) & \leftarrow & J(CP^{2n}) & \leftarrow & J(CP^{2n}/CP^{2n-2}) & \leftarrow 0 \end{array}$$

if $n \geqq 1$.

Proof. The lower sequence is exact by [**2**]. If $n = 1$, then CP^{2n-2} is a one point space. Therefore the left hand groups are both 0 and q is the identity. The result then follows from Lemma 3. Continuing inductively, since $[CP^{2n}, BSJ]$ is generated by $i_*\nu^k$ for $1 \leqq k \leqq 2n$ and $i_*\nu^{2n-1} (= q^*\, i_*\nu^{2n-1})$, the middle vertical map is well defined. Since the left and right hand vertical maps are isomorphisms by inductive hypothesis and by Lemma 3 respectively, it follows that the middle vertical map is an isomorphism.

Let $\lambda : RP^{12} \to CP^6$ be the restriction of the nontrivial map $RP \to CP$. Let C_λ denote the mapping cone.

LEMMA 5. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} C_\lambda & \xrightarrow{f} & BSO \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle i} \\ S(RP^{12}) & \xrightarrow[g]{} & BSJ \end{array}$$

satisfying the following conditions:

(1) *In Z cohomology, $f^*P_3 = -4z_{12}$.*

(2) *In $Z/2$ cohomology, $f^*w_{12} = f^*w_{13} = 0$.*

(3) *In $Z/2$ cohomology, $g^*e_{12} = g^*e_{13} = 0$.*

(Here z_{12} denotes the generator of $H^{12}(C_\lambda; Z) = Z$.)

Proof. The composition

$$RP^{12} \xrightarrow{\lambda} CP^6 \xrightarrow{\eta^3 - \eta} BU \xrightarrow{r} BSO \xrightarrow{J} BSF$$

is trivial and

$$J : KSO(RP^{12}) \to [RP^{12}, BSF]$$

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is injective. Thus the composition $r \circ (\eta^3 - \eta) \circ \lambda$ is trivial and we obtain the commutative diagram

$$\begin{array}{ccc} CP^6 & & \\ \downarrow & \searrow^{r(\eta^3 - \eta)} & \\ C_\lambda & \xrightarrow[f]{} & BSO \end{array}$$

Since all Stiefel-Whitney classes of the nontrivial map $SRP \to BSO$ are non-zero, f can be chosen so that $f^*w_{12} = 0$. We choose f in this way and notice that $f^*w_{13} = 0$ since $w_{13} = Sq^1 w_{12}$.

The composition along the top of the diagram

$$\begin{array}{ccccccc} CP^6 & \xrightarrow{\eta^3 - \eta} & BU & \xrightarrow{r} & BSO & \xrightarrow{J} & BSF \\ & & & & {\scriptstyle i}\searrow & & \nearrow{\scriptstyle j} \\ & & & & & BSJ & \end{array}$$

is trivial. Thus $i \circ r \circ (\eta^3 - \eta) = 0$ because

$$j_* : [CP^6, BSJ] \to [CP^6, BSF]$$

is injective by Lemma 3. This fact enables us to fill g into our diagram. Since $i^*e_{12} = 0$, we have $h^*g^*e_{12} = 0$. Since h^* is an isomorphism in even degrees of $Z/2$ cohomology, $g^*e_{12} = 0$. By Lemma 2, we have $g^*e_{13} = 0$ because $e_{13} = Sq^1e_{12}$ modulo decomposables.

We calculate $f^*(P_3)$ as follows: Let P and C denote the total Pontryagin and Chern classes respectively. Then

$$\begin{aligned} P(r(\eta^3 - \eta)) &= C(cr(\eta^3 - \eta)) \\ &= C(\eta^3 + \bar{\eta}^3 - \eta - \bar{\eta}) \\ &= \frac{C(\eta^3)C(\bar{\eta}^3)}{C(\eta)C(\bar{\eta})} \\ &= \frac{(1 + 3x)(1 - 3x)}{(1 + x)(1 - x)} \\ &= \frac{1 - 9x^2}{1 - x^2} \\ &= 1 - 8x^2 - 8x^4 - 8x^6 \end{aligned}$$

where x is the generator of $H^2(CP^6; Z)$. Since the map

$$H^{12}(C_\lambda; Z) \to H^{12}(CP^6; Z)$$

sends z_{12} to $2x^6$, we see that $f^*P_3 = -4z_{12}$.

The proof of our theorem now proceeds exactly as in [**4**].

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