ON VARIOUS TYPES OF BARRELLEDNESS AND THE HEREDITARY PROPERTY OF (*DF*)-SPACES

by T. HUSAIN† and YAU-CHUEN WONG

(Received 1 April, 1975; revised 20 October, 1975)

1. Introduction. Recently, Levin and Saxon [**5**], De Wilde and Houet [**2**] defined the σ-barrelledness while Husain [**3**] defined the countable barrelledness and countable quasi-barrelledness. It is well-known that barrelled spaces are countably barrelled, and countably barrelled spaces are σ-barrelled. It is natural to ask whether there is some condition for σ-barrelled (resp. countably barrelled) spaces to be countably barrelled (resp. barrelled). Using the concept of S-absorbent sequences of sets, we are able to give such conditions in Theorem 2.5 and Corollaries 2.6 and 2.7.

Valdivia [**9**], Saxon and Levin [**8**] have shown that every vector subspace with countable codimension of a barrelled space is barrelled. Also Levin and Saxon showed in [**5**] that this hereditary property is true for σ-barrelled spaces. In §3, we show that this hereditary property is also true for countably barrelled spaces as well as for σ-barrelled (*DF*)-spaces, which is a generalization of Valdivia [**10**, Theorem 3].

The final section is devoted to some properties of S-absorbent sequences of sets which extend some results of Valdivia [**9**], De Wilde and Houet [**2**].

2. The relationship between various types of barrelledness. Let (E, T) be a Hausdorff locally convex space whose topological dual is denoted by E'. If B is a subset of E (resp. E'), then the polar of B, taken in E' (resp. E), is denoted by B^0. By a *topologizing family* (t. family, for short) for E' (resp. E) we mean a family S consisting of (convex circled) $\sigma(E, E')$-bounded subsets of E (resp. $\sigma(E', E)$-bounded subsets of E') such that $\cup\{B: B \in S\} = E$ (resp. E'). For a t. family S for E' (resp. E), the topology on E' (resp. E) of uniform convergence on S is denoted by T_S.

Let S be a t. family for E'. We denote by S^b the family of all T_S-bounded subsets of E'. Clearly S^b is again a t. family for E. The topology on E of uniform convergence on S^b is denoted by T_S^b, therefore we have $T_S^b = T_{S^b}$. Similarly we can define S^{nb} and T_S^{nb}, where $S^{nb} = S^{bb\ldots b}$ and $T_S^{nb} = T_S^{bb\ldots b}$, the superscript b being repeated n times in each case; consequently we have $T_S^{nb} = T_{S^{nb}}$ for all $n \geqq 1$.

If S is a t. family for E', let us say temporarily that S^b (resp. S^{bb}) is the *bounded-polar* (resp. *bounded-bipolar*) *family* of S, and that T_S^b (resp. T_S^{bb}) is the *bounded-polar* (resp. *bounded-bipolar*) *topology* of T_S. It is clear that $\{S^0: S \in S^b\}$ forms a neighbourhood base at 0 for the bounded-polar topology T_S^b, and that $\{B^0: B \in S^{bb}\}$ forms a neighbourhood base at 0 for the bounded-bipolar topology T_S^{bb}. If S_1 and S_2 are two t. families for E' with $S_1 \subset S_2$, then $S_2^b \subset S_1^b$.

LEMMA 2.1. *For a t. family S for E′, we have*:

(a) $S \subset S^{bb}$;

† This research was supported by an N.R.C. grant.

https://doi.org/10.1017/S001708950000286X Published online by Cambridge University Press

(b) $T_S \leqq T_S^{bb}$;
(c) $S^b = S^{3b}$ *and* $T_S^b = T_S^{3b}$.

The proof is straightforward and will be omitted. We shall see that the inclusion in (a) may be strict.

In the sequel we denote by $\beta(E, E')$ the strong topology on E, i.e., the topology of uniform convergence on all $\sigma(E', E)$-bounded subsets of E', and by $\beta^*(E, E')$ the topology of uniform convergence on all $\beta(E', E)$-bounded subsets of E'. As usual, $\tau(E, E')$ denotes the Mackey topology on E. Clearly,

$$\tau(E, E') \leqq \beta^*(E, E') \leqq \beta(E, E')$$

and

$$\tau(E', E) \leqq \beta^*(E', E) \leqq \beta(E', E).$$

It is not hard to see that each $\sigma(E, E')$-bounded subset of E is $\beta^*(E, E')$-bounded, and that dually each $\sigma(E', E)$-bounded subset of E' is $\beta^*(E', E)$-bounded.

EXAMPLES. (1) If S_f is the family of all finite subsets of E, then we have that $T_{S_f} = \sigma(E', E)$; $T_{S_f}^b = \beta(E, E')$; $T_{S_f}^{bb} = \beta^*(E', E)$. Therefore we conclude that $S_f \neq S_f^{bb}$ and $T_{S_f} \neq T_{S_f}^{bb}$, in general.

(2) If S_β is the family of all $\beta(E, E')$-bounded subsets of E, then we have $T_{S_\beta} = \beta^*(E', E)$, S_β^b is the family of all $\sigma(E', E)$-bounded subsets of E' and $T_{S_\beta}^b = \beta(E, E')$. Therefore we conclude that $S_\beta = S_\beta^{bb}$ and $T_{S_\beta} = T_{S_\beta}^{bb}$.

(3) If S_σ is the family of all $\sigma(E, E')$-bounded subsets of E, then we have $T_{S_\sigma} = \beta(E', E)$, $T_{S_\sigma}^b = \beta^*(E, E')$ and S_σ^{bb} is the family of all $\sigma(E, E')$-bounded subsets of E.

(4) Let S_c be the family of all T-compact convex circled subsets of E and let $c(E', E)$ be the topology on E' of uniform convergence on S_c. Then $\sigma(E', E) \leqq c(E', E) \leqq \tau(E', E)$; furthermore we have $T_{S_c} = c(E', E)$, $T_{S_c}^b = \beta(E, E')$ and $T_{S_c}^{bb} = \beta^*(E', E)$.

DEFINITION 2.2. Let (E, T) be a locally convex space and S a t. family for E'. Then E is said to be

(1) S-barrelled if each member in S^b is T-equicontinuous;

(2) *countably S-barrelled* if each member of S^b which is the countable union of T-equicontinuous subsets of E is T-equicontinuous;

(3) *σ-S-barrelled* if each member in S^b which is a countable set is T-equicontinuous.

If S is the family of all finite subsets of E, then E is S-barrelled (resp. countably S-barrelled, σ-S-barrelled) if and only if it is barrelled (resp. countably barrelled, σ-barrelled) under the usual terminology of **[4]** and **[6]** (resp. **[3]**, **[2]**). σ-barrelled spaces are also called ω-barrelled by Levin and Saxon **[5]**. Clearly each $\beta(E', E)$-bounded set is in S^b for any t. family for E'. Hence E is quasibarrelled (or countably quasibarrelled or σ-evaluable) if E is S-barrelled (or countably S-barrelled or σ-S-barrelled).

If S is the family of all $\sigma(E, E')$-bounded subsets of E, then E is S-barrelled (resp. countably S-barrelled, σ-S-barrelled) if and only if E is quasibarrelled (resp. countably quasibarrelled, σ-evaluable) under the usual terminology of **[4]** **[6]** (resp. **[3]**, **[2]**). Here we call σ-evaluable spaces σ-infrabarrelled.

If (E, T) is a locally convex Riesz space and if S is the family of all order-bounded subsets of E, then E is S-barrelled if and only if it is order-infrabarrelled under the usual definition of **[11]**.

As a consequence of Lemma 2.1, we have the following result.

LEMMA 2.3. *Let S be a t. family for E'. E is S-barrelled (resp. countably S-barrelled, σ-S-barrelled) if and only if E is S^{bb}-barrelled (resp. countably S^{bb}-barrelled, σ-S^{bb}-barrelled).*

In particular, (E, T) is barrelled if and only if each $\sigma(E, E')$-closed convex circled subset of E which absorbs all $\beta(E, E')$-bounded subsets of E is a T-neighbourhood of 0.

Using a standard argument, for instance, see Schaefer **[6]** and Köthe **[4**, p. 396**]**, it is easily seen that E is S-barrelled if and only if each closed convex circled subset of E which absorbs all members of S is a T-neighbourhood of 0, and that E is countably S-barrelled if and only if for any sequence (V_n) of closed convex circled T-neighbourhoods of 0, if $V = \bigcap_{n=1}^{\infty} V_n$ absorbs all members in S then V is a T-neighbourhood of 0.

In order to give a dual characterization of the σ-S-barrelledness, we require the following terminology. Let S be a t. family for E'. By an *S-absorbent sequence* (of closed sets) in E we mean a sequence $\{V_n : n \geqq 1\}$ of (closed) convex circled sets in E for which the following two conditions are satisfied:

(i) $V_n \subset V_{n+1}$ for all $n \geqq 1$;
(ii) each member in S is absorbed by some V_n.

If S is the family of all finite subsets of E, then $\{V_n : n \geqq 1\}$ is an S-absorbent sequence if and only if it is an absorbent sequence in E in the sense of **[2]**; and if S is the family of all $\sigma(E, E')$-bounded subsets of E, then $\{V_n : n \geqq 1\}$ is a σ-absorbent sequence if and only if it is a bounded-absorbent sequence in the sense of **[2]**.

PROPOSITION 2.4. *Let S be a t. family for E'. Then E is σ-S-barrelled if and only if for any S-absorbent sequence $\{V_n : n \geqq 1\}$ in E, the sequence $\{f_n : n \geqq 1\}$ is equicontinuous, where $f_n \in V_n^0$ for all $n \geqq 1$.*

Proof. Necessity. For any $S \in S$ there exists $\lambda > 0$ and $n_0 > 1$ such that $S \subset \lambda V_n$ for all $n \geqq n_0$. For each $n \geqq 1$, let $f_n \in V_n^0$. Then $|f_n(x)| \leqq \lambda$ for all $x \in S$ and $n \geqq n_0$. Since S is $\sigma(E, E')$-bounded, there exists $\mu > 0$ with $|f_n(x)| \leqq \mu$ for all $x \in S$ and $n = 1, \ldots, n_0 - 1$. Thus $\sup \{|f_n(x)| : x \in S, n \geqq 1\} \leqq \max(\lambda, \mu) < \infty$ and so $\{f_n : n \geqq 1\} \in S^b$. Hence by hypothesis $\{f_n : n \geqq 1\}$ is equicontinuous.

Sufficiency. Let $\{h_n : n \geqq 1\}$ be a T_S-bounded sequence in E'. For each $k \geqq 1$, we define

$$V_k = \{x \in E : |h_n(x)| \leqq 1 \text{ for all } n \geqq k\}.$$

Then $\{V_n : n \geqq 1\}$ is an S-absorbent sequence in E. As $h_k \in V_k^0$ for all $k \geqq 1$, we conclude from the hypothesis that $\{h_k : k \geqq 1\}$ is equicontinuous. This shows that E is σ-S-barrelled.

https://doi.org/10.1017/S001708950000286X Published online by Cambridge University Press

If S_1 and S_2 are two t. families for E' such that $S_1 \subset S_2$, then the following implications hold:

$$\begin{array}{ccccc} S_1\text{-barrelledness} & \Rightarrow & \text{countably } S_1\text{-barrelledness} & \Rightarrow & \sigma\text{-}S_1\text{-barrelledness} \\ \Downarrow & & \Downarrow & & \Downarrow \\ S_2\text{-barrelledness} & \Rightarrow & \text{countably } S_2\text{-barrelledness} & \Rightarrow & \sigma\text{-}S_2\text{-barrelledness.} \end{array}$$

Therefore it is natural to ask under what conditions on E (or E') the corresponding converse implications hold. We have the following result.

THEOREM 2.5. *Let S_1 and S_2 be two t. families for E' such that $S_1 \subset S_2$. Then (E, T) is σ-S_1-barrelled (resp. countably S_1-barrelled, S_1-barrelled) if and only if the following two conditions hold*:

(i) *E is σ-S_2-barrelled (resp. countably S_2-barrelled, S_2-barrelled)*;
(ii) *each S_1-absorbent sequence of closed sets in E is S_2-absorbent.*

Proof. Suppose that E is σ-S_1-barrelled and that $\{V_n : n \geqq 1\}$ is an S_1-absorbent sequence of closed sets in E which is not S_2-absorbent. Then there exists $B \in S_2$ such that $B \subset nV_n$ is false for all natural numbers $n \geqq 1$. For each $n \geqq 1$, let x_n, in B, be such that $x_n \notin nV_n$. As V_n is closed convex and circled, the bipolar theorem ensures that there exists $f_n \in V_n^0$ such that

$$|f_n(x_n)| > n. \tag{1}$$

As E is σ-S_1-barrelled and $\{V_n : n \geqq 1\}$ is an S_1-absorbent sequence of closed sets in E, it follows from Proposition 2.4 that $\{f_n : n \geqq 1\}$ is a T-equicontinuous sequence, and hence that $\{f_n : n \geqq 1\}$ is T_{S_2}-bounded; consequently $\{f_n : n \geqq 1\}$ must be absorbed by B^0, contrary to the inequality (1). Therefore the conditions are necessary. We show that the conditions are also sufficient.

Let $\{f_n : n \geqq 1\}$ be a T_{S_1}-bounded sequence in E'. For each $k \geqq 1$, let

$$V_k = \{x \in E : |f_n(x)| \leqq 1 \text{ for all } n \geqq k\}.$$

The T_{S_1}-boundedness of $\{f_n : n \geqq 1\}$ ensures that $\{V_n : n \geqq 1\}$ is an S_1-absorbent sequence of closed sets in E, and hence $\{V_n : n \geqq 1\}$ is S_2-absorbent by the hypotheses. On the other hand, since E is assumed to be σ-S_2-barrelled and since $f_n \in V_n^0$ for all $n \geqq 1$, it follows from Proposition 2.4 that $\{f_n : n \geqq 1\}$ is T-equicontinuous, and hence that E is σ-S_1-barrelled.

The necessity part of the proof for countably S_1-barrelled and S_1-barrelled spaces is similar and so is omitted. The sufficiency part for all cases can be handled as follows. Observe that $S_1^b \supset S_2^b$. To show that (ii) implies $S_1^b = S_2^b$, let $A \in S_1^b$ and $A \notin S_2^b$. Then there is $B \in S_2$, a sequence $\{x_n\} \subset B$ and a sequence $\{f_n\} \subset A$ such that $|f_n(x_n)| > n$ for all $n \geqq 1$. Since $V_n = \{x \in E : |f_m(x)| \leqq 1 \text{ for } m \geqq n\}$ is an S_1-absorbent sequence of closed sets in E, it follows by (ii) that it is also S_2-absorbent. Hence there exist n and λ such that $B \subset \lambda V_n$, a contradiction.

REMARK. E is σ-barrelled (resp. countably barrelled, barrelled) if and only if it is σ-infrabarrelled (resp. countably quasibarrelled, infrabarrelled) and each absorbent sequence of closed sets in E is bounded-absorbent.

COROLLARY 2.6. *Let S_1 and S_2 be two t. families for E' such that $S_1 \subset S_2$. Then:*

(a) *E is countably S_1-barrelled if and only if it is countably S_2-barrelled as well as σ-S_1-barrelled;*

(b) *E is S_1-barrelled if and only if it is σ-S_1-barrelled as well as S_2-barrelled.*

Proof. If E is countably S_1-barrelled, then it is obvious that E is countably S_2-barrelled as well as σ-S_1-barrelled. Conversely, if E is countably S_2-barrelled and if E is σ-S_1-barrelled, then by Theorem 2.5, each S_1-absorbent sequence of closed sets in E is S_2-absorbent. We conclude from Theorem 2.5 again that E is countably S_1-barrelled. This proves the assertion (a). The proof of (b) is similar.

REMARK. E is countably barrelled if and only if it is σ-barrelled and countably quasibarrelled· E is barrelled if and only if it is countably barrelled and quasibarrelled.

COROLLARY 2.7. *Let S_1 and S_2 be two t. families for E' such that $S_1 \subset S_2$. Then the following assertions hold.*

(a) *Let E be countably S_2-barrelled. Then E is countably S_1-barrelled if and only if each S_1-absorbent sequence of closed sets in E is S_2-absorbent.*

(b) *Let E be S_2-barrelled. Then E is S_1-barrelled if and only if E is σ-S_1-barrelled, and this is the case if and only if each S_1-absorbent sequence of closed sets in E is S_2-absorbent.*

Proof. (a) follows from Theorem 2.5 and Corollary 2.6 (a), while (b) follows from Corollary 2.6 (b) and the assertion (a) of this corollary.

Let E be a locally convex space. A convex circled $\sigma(E, E')$-bounded subset B of E is said to be *infracomplete* if the normed space $E(B) = \bigcup_n nB$ equipped with the norm $\|.\|_B$ defined by

$$\|x\|_B = \inf\{\lambda \geqq 0 : x \in \lambda B\} \quad (x \in E(B))$$

is complete. It is clear that every convex circled $\sigma(E, E')$-bounded and $\tau(E, E')$-sequentially complete subset of E is infracomplete. By the Banach–Mackey theorem, we see that every infracomplete subset B of E is $\beta(E, E')$-bounded (see [4, §20, 11(3)]).

Levin and Saxon [5] say that a locally convex space E has *the property* (C) (resp. *the property* (S)) if every $\sigma(E', E)$-bounded subset of E' is $\sigma(E', E)$-relatively countably compact (resp. E' is $\sigma(E', E)$-sequentially complete). As a consequence of the result mentioned above ([4, §20, 11(3)]), we obtain the following result which gives a connection between σ-barrelledness and the property (S).

PROPOSITION 2.8. *For a σ-infrabarrelled locally convex space E, the following statements are equivalent:*

(a) *E is σ-barrelled;*
(b) *E has the property (C);*
(c) *E has the property (S);*
(d) *each $\sigma(E', E)$-bounded, $\sigma(E', E)$-closed subset of E' is $\sigma(E', E)$-sequentially complete.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious. Finally, if the statement (d) holds, then by the Banach–Mackey theorem each $\sigma(E', E)$-bounded subset of E' is $\beta(E', E)$-bounded, and thus the implication (d) $\Rightarrow$ (a) follows.

Consider the vector space m of all bounded sequences with the Mackey topology $\tau(m, l_1)$. Levin and Saxon have shown in [5, Proposition 6] that $(m, \tau(m, l_1))$ is a Mackey space with the property (S) but not property (C). According to this result and Proposition 2.8, we conclude that Mackey spaces are, in general, not σ-infrabarrelled spaces.

As another consequence of the Banach–Mackey theorem, we have the following result.

PROPOSITION 2.9. *Let E be a locally convex space for which every $\sigma(E', E)$-bounded closed set is $\sigma(E', E)$-sequentially complete (equivalently, E has the property (S)). Then the following assertions hold.*

(1) *If E is infrabarrelled (in particular, bornological) then it is barrelled.*
(2) *If E is countably infrabarrelled then it is countably barrelled.*

Proof. According to the Banach–Mackey theorem each $\sigma(E', E)$-bounded subset of E' is $\beta(E', E)$-bounded, and the result follows.

Since metrizable locally convex spaces are infrabarrelled, part (1) of the preceding result is a generalization of Saxon [7, Theorem 2.7]. The following corollary is now immediate.

COROLLARY 2.10. *Let E be a locally convex space in which every $\sigma(E, E')$-bounded closed set is $\tau(E, E')$-sequentially complete (in particular, E is either $\tau(E, E')$-sequentially complete or quasi-complete). Then the following assertions hold.*

(1) *If E is σ-infrabarrelled then E is σ-barrelled and a fortiori has the property (S).*
(2) *If E is countably infrabarrelled (resp. barrelled) then it is countably barrelled (resp. barrelled).*

3. The hereditary property. Saxon, Levin [8] and Valdivia [9] have shown independently that a vector subspace with countable codimension of a barrelled space is barrelled. Also Saxon and Levin [5] have shown that a vector subspace with countable codimension of a σ-barrelled space is σ-barrelled. The same is true for countably barrelled spaces as shown by Webb [12]. We give a different and direct proof of this fact.

THEOREM 3.1. *Let M be a countable codimensional vector subspace of a countably barrelled space E. Then M is countably barrelled when furnished with the relative topology.*

Proof. In our proof we consider three cases.

(a) *M is dense in E.* In this case, the topological dual M' of M can be canonically identified with E'. Let S be a $\sigma(M', M)$-bounded subset of M' and let $\{S_n : n \geq 1\}$ be a sequence of equicontinuous subsets of M' for which $S = \bigcup_{n=1}^{\infty} S_n$. Since M is dense in E, it follows from [5, Lemma 2] that S is $\sigma(E', E)$-bounded. Further we show that each S_n is an equicontinuous subset of E'.

https://doi.org/10.1017/S001708950000286X Published online by Cambridge University Press

In fact, let S_n^0 denote the polar of S_n taken in E. Since S_n is an equicontinuous subset of M', $S_n^0 \cap M$ is a 0-neighbourhood in M; then there exists an open 0-neighbourhood U_n in E such that $U_n \cap M \subset S_n^0 \cap M \subset S_n^0$. The density of M ensures that $U_n \subset \overline{U_n \cap M} \subset S_n^0$, and hence S_n is an equicontinuous subset of E'.

Now the countable barrelledness of E implies that S is an equicontinuous subset of E' and surely an equicontinuous subset of M'. This shows that M is countably barrelled.

(b) *M is closed in E.* Let N be any algebraic complement to M in E. Since countably barrelled spaces are σ-barrelled, it follows from [7, Theorem 1.1] that N is a topological complement and has the strongest locally convex topology. Hence N is closed in E, M and E/N are topologically isomorphic. Since E is countably barrelled, by [3, Corollary 14], E/N is countably barrelled and therefore M must be countably barrelled.

(c) *General case.* Since $\overline{M}$ is a closed vector subspace of E with countable codimension, it follows from (b) that $\overline{M}$ is countably barrelled. As M is dense in $\overline{M}$, we conclude from (a) that M is countably barrelled. This completes the proof of the theorem.

COROLLARY 3.2. *Let E be a σ-barrelled (DF)-space. Then any vector subspace M of E with countable codimension is a countably barrelled (DF)-space.*

Proof. By Corollary 2.6, E is a countably barrelled (DF)-space, and hence M is a countably barrelled space by the preceding theorem. Since E has a countable fundamental system of bounded sets, and since M is a subspace, it follows that M contains a countable fundamental system of bounded subsets of M. Therefore M is a countably barrelled (DF)-space.

The preceding result was proved by Valdivia [**10**, Theorem 3] in the special case when E is barrelled.

4. Various types of absorbent sequences. Let E be a vector space. By an *increasing sequence of sets* in E we mean a sequence $\{V_n : n \geqq 1\}$ of convex circled subsets of E such that $V_n \subset V_{n+1}$ for all $n \geqq 1$. Let $\{V_n : n \geqq 1\}$ be an increasing sequence of sets in E. It is clear that $\{nV_n : n \geqq 1\}$ is an increasing sequence of sets in E, and that if E is a locally convex space then $\{\overline{V}_n : n \geqq 1\}$ is also an increasing sequence of sets in E, where $\overline{V}_n$ is the closure of V_n. An increasing sequence $\{V_n : n \geqq 1\}$ of sets in E is called an increasing sequence of (P) sets in E if each V_n has the property (P); for instance, $\{V_n : n \geqq 1\}$ is an increasing sequence of closed (resp. complete, compact, metrizable etc.) sets in E if each V_n is closed (resp. complete, compact, metrizable etc.).

It is known from §2 that the concept of S-absorbent sequences is useful for studying the relationship between various types of barrelledness. It is not hard to give an example of an increasing sequence of sets in E which is not S-absorbent. Therefore it is interesting to find some sufficient and necessary condition to ensure that increasing sequences are S-absorbent.

PROPOSITION 4.1. *Let S be a t. family for E' and suppose that $\{V_n : n \geqq 1\}$ is an increasing sequence of closed sets in E. Then it is an S-absorbent sequence if and only if for any $f_n \in V_n^0$ $(n \geqq 1)$, the sequence $\{f_n : n \geqq 1\}$ is T_S-bounded.*

Proof. Suppose that $\{V_n : n \geqq 1\}$ is S-absorbent and that $\{f_n : n \geqq 1\}$ is not T_S-bounded for some $f_n \in V_n^0$ $(n \geqq 1)$. Then there exists $B \in S$ such that $\{f_n : n \geqq 1\} \subset k^2 B^0$ is false for all natural numbers $k \geqq 1$. For each $k \geqq 1$, there exists n_k such that $f_{n_k} \notin k^2 B^0$. On the other hand, since $\{V_n : n \geqq 1\}$ is S-absorbent, there exists $\lambda > 0$ and $n_0 \geqq 1$ such that

$$V_n^0 \subset V_{n_0}^0 \subset \lambda B^0 \quad \text{for all} \quad n \geqq n_0,$$

it then follows that $f_n \in \lambda B^0$ for all $n \geqq n_0$, which contradicts the fact that $f_{n_k} \notin k^2 B^0$. Therefore the condition is necessary.

Conversely, if $\{V_n : n \geqq 1\}$ is not an S-absorbent sequence, then there exists $B \in S$ such that $B \subset nV_n$ is false for all natural numbers $n \geqq 1$. For each n, let $x_n \in B \backslash (nV_n)$ and let f_n, in V_n^0, be such that $|f_n(x_n)| > n$. Then the sequence $\{f_n : n \geqq 1\}$ is not T_S-bounded. This completes the proof.

In the sequel we always assume that E is a locally convex space and that S is a topologizing family for E'. If S_1 is another topologizing family for E' such that $S \subset S_1$, then each S_1-absorbent sequence in E must be S-absorbent. The converse is true for $S_1 = S^{bb}$ as the following result shows.

COROLLARY 4.2. *$\{V_n : n \geqq 1\}$ is an S-absorbent sequence of closed sets in E if and only if it is an S^{bb}-absorbent sequence.*

Proof. This follows from Proposition 4.1 and Lemma 2.1.

The preceding result was proved by De Wilde and Houet [**2**, Theorem 1] in the case when S is the family of all finite subsets of E.

COROLLARY 4.3. *Let S_1 and S_2 be two t. families for E' such that $S_1 \subset S_2$. Then the following statements are equivalent:*

(i) *each T_{S_1}-bounded subset of E' is T_{S_2}-bounded;*
(ii) *each S_1-absorbent sequence of closed sets in E is S_2-absorbent.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Proposition 4.1, while the implication (ii) $\Rightarrow$ (i) has been observed in Theorem 2.5.

When S_1 is the family of all finite subsets of E and S_2 is the family of all $\sigma(E, E')$-bounded subsets of E, then the implication (i) $\Rightarrow$ (ii) in the preceding result was proved by Valdivia [**9**, Theorem 6] in the case when E is barrelled, and was proved by De Wilde and Houet [**2**, Corollary 1] in the case when E is σ-barrelled.

By making use of Theorem 2.5, for a σ-barrelled space E, each $\sigma(E', E)$-bounded subset of E' is $\beta(E', E)$-bounded.

COROLLARY 4.4. *Let S_1 and S_2 be two t. families for E' such that $S_1 \subset S_2$, and let E satisfy one of the equivalent conditions* (i) *and* (ii) *of Corollary* 4.3. *If S_2 has a sequence $\{B_n : n \geqq 1\}$ such that each member of S_1 is absorbed by some B_n, then the saturated hull* ([**6**], p. 81) *of S_2 contains a countable fundamental subfamily.*

Proof. For each n, let V_n be the closed convex circled hull of $\bigcup_{j=1}^{n} B_j$. Then V_n is in the saturated hull of S_2, and $\{V_n : n \geqq 1\}$ is an S_1-absorbent sequence of closed sets in E, so by

https://doi.org/10.1017/S001708950000286X Published online by Cambridge University Press

the hypothesis, $\{V_n : n \geqq 1\}$ is S_2-absorbent. Consequently $\{nV_n : n \geqq 1\}$ is a countable fundamental subfamily of the saturated hull of S_2 because a member of the saturated hull of S_2 is either a subset, scalar multiple or an absolute convex hull of a finite number of elements of S_2.

REMARK. If E is a countably barrelled space with a sequence $\{B_n : n \geqq 1\}$ of bounded sets such that $\bigcup_{n=1}^{\infty} B_n$ is absorbing, then E is a (DF)-space.

Corollary 4.4 was proved by Valdivia [**9**, Corollary 2.6] in the case when E is barrelled.

A trivial modification of De Wilde and Houet's argument in [**2**] yields the following more general result, but for completeness we shall give the entire proof.

THEOREM 4.5. *Let E be a σ-S-barrelled space and $\{V_n : n \geqq 1\}$ an S-absorbent sequence in E. Then*

$$\overline{\bigcup_{m}^{\infty} V_m} \subset (1+\varepsilon)\bigcup_{m}^{\infty} \overline{V}_m \quad \text{for all} \quad \varepsilon > 0.$$

Proof. If $x \notin (1+\varepsilon)\bigcup_{m}^{\infty} \overline{V}_m$ for some $\varepsilon > 0$, then $x \notin (1+\varepsilon)\overline{V}_m$ for all $m \geqq 1$, and thus, for any $m \geqq 1$, there exists $f_m \in V_m^0$ such that $f_m(x) > 1+\varepsilon$. Since E is σ-S-barrelled, by Proposition 2.4, $\{f_m : m \geqq 1\}$ has a $\sigma(E', E)$-cluster point f, say, in E'; hence $f(x) \geqq 1+\varepsilon$. On the other hand, since V_n is increasing and $f_n \in V_n^0$, it follows that $f \in V_n^0$ for all $n \geqq 1$ or, equivalently $f \in \bigcap_{n \geqq 1} V_n^0 = \left(\bigcup_{n \geqq 1} V_n\right)^0$. However the inequality $f(x) \geqq 1+\varepsilon$ shows that $x \notin \overline{\bigcup_{m}^{\infty} V_m}$. This completes the proof.

REMARKS. (1) As De Wilde in [**1**, p. 212] pointed out, the condition in Theorem 4.5 that E be σ-S-barrelled can be replaced by the following condition: $\{V_n : n \geqq 1\}$ is an S-absorbent sequence in E such that for each $f_n \in V_n^0$ $(n \geqq 1)$, the sequence $\{f_n : n \geqq 1\}$ is equicontinuous.

(2) According to the preceding theorem, Corollaries 2.a–2.d in [**2**] hold for a σ-S-barrelled space.

ACKNOWLEDGEMENT. The authors are grateful to the referee for many helpful comments.

REFERENCES

1. M. De Wilde, *Summer school on topological vector spaces*, Lecture Notes in Mathematics, No. 331 (Springer-Verlag, 1973).

2. M. De Wilde and C. Houet, On increasing sequences of absolutely convex sets in locally convex spaces, *Math. Ann.* **192** (1971), 257–261.

3. T. Husain, Two new classes of locally convex spaces, *Math. Ann.* **166** (1966), 289–299.

4. G. Köthe, *Topological vector spaces I* (Springer-Verlag, 1969).

5. M. Levin and S. Saxon, A note on the inheritance of properties of locally convex spaces by subspaces of countable codimension, *Proc. Amer. Math. Soc.* **29** (1971), 97–102.

6. H. H. Schaefer, *Topological vector spaces* (Springer-Verlag, 1971).

7. S. Saxon, Nuclear and product spaces, Baire-like spaces, and the strongest locally convex topology, *Math. Ann.* **197** (1972), 87–106.

https://doi.org/10.1017/S001708950000286X Published online by Cambridge University Press

8. S. Saxon and M. Levin, Every countable-codimensional subspace of a barrelled space is barrelled, *Proc. Amer. Math. Soc.* **29** (1971), 91–96.

9. M. Valdivia, Absolutely convex sets in barrelled spaces, *Ann. Inst. Fourier Grenoble* **21** (2) (1971), 3–13.

10. M. Valdivia, On *DF*-spaces, *Math. Ann.* **191** (1971), 38–43.

11. Yau-Chuen Wong, Order-infrabarrelled Riesz spaces, *Math. Ann.* **183** (1969), 17–32.

12. J. H. Webb, Countable-codimensional subspaces of locally convex spaces, *Proc. Edinburgh Math. Soc.* **18** (1973), 167–172.

Prof. T. Husain
DEPARTMENT OF MATHEMATICS
MCMASTER UNIVERSITY
HAMILTON, ONTARIO
CANADA

Dr. Yau-Chuen Wong
DEPARTMENT OF MATHEMATICS
UNITED COLLEGE
CHINESE UNIVERSITY OF HONG KONG
HONG KONG