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DIFFERENTIAL EQUATIONS DEFINED BY THE SUM OF TWO QUASI-HOMOGENEOUS VECTOR FIELDS

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ABSTRACT. In this paper we prove, that under certain hypotheses, the planar differential equation: $\dot{x} = X_1(x,y) + X_2(x,y)$, $\dot{y} = Y_1(x,y) + Y_2(x,y)$, where (X_i, Y_i), $i = 1, 2$, are quasi-homogeneous vector fields, has at most two limit cycles. The main tools used in the proof are the generalized polar coordinates, introduced by Lyapunov to study the stability of degenerate critical points, and the analysis of the derivatives of the Poincaré return map. Our results generalize those obtained for polynomial systems with homogeneous non-linearities.

1. **Introduction and statement of main results.** Given $p, q, s \in \mathbb{N}$, we will say that *a function* $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is* (p,q)*-quasi-homogeneous of degree* s if $f(\lambda^p x, \lambda^q y) = \lambda^s f(x,y)$ for $\lambda \in \mathbb{R}$, (see [1, p. 32]). A *vector field* $X = (P,Q)\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *is called* (p,q)*-quasi-homogeneous of degree* r if P and Q are (p,q)-quasi-homogeneous functions of degrees $p+r-1$ and $q+r-1$ respectively, see [2, Chapter 7].

Observe that the above definition is the natural one for the following reasons:

(i) When $p = q = 1$, it coincides with the usual definition of homogeneous vector field of degree r.

(ii) The differential equation $\frac{dy}{dx} = \frac{Q}{P}$, associated with X, is invariant by the change of variables $\bar{x} = \lambda^p x$, $\bar{y} = \lambda^q y$.

(iii) Homogeneous vector fields can be integrated using polar coordinates whereas (p,q)-quasi-homogeneous vector fields can be integrated using the (p,q)-polar coordinates. These generalized polar coordinates were introduced by Lyapunov in his study of the stability of degenerate critical points, see [14]. In Appendix 1, we consider a small modification of these coordinates and their main properties.

The (p,q)-polar coordinates have also been applied recently to study properties of planar differential equations, see [4, 8].

In this paper we study differential equations of type:

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \big(P(x,y), Q(x,y)\big) = X(x,y) = X_n(x,y) + X_m(x,y), \tag{1}$$

where $m > n$, and X_u is a (p,q)-quasi-homogeneous vector field of degree $u-p-q+2pq$, for $u \in \{n,m\}$.

Note that when $p = q = 1$, $X = X_n + X_m$ is the sum of two homogeneous vector fields with n and m degrees of homogeneity respectively and includes quadratic differential

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equations ($p = q = n = 1$, $m = 2$) and polynomial systems with homogeneous non-linearities ($p = q = n = 1$), see [6, 7, 9, 10, 12].

In the (p,q)-polar coordinates, $(x,y) = \big(\rho^p \operatorname{Cs}(\varphi), \rho^q \operatorname{Sn}(\varphi)\big)$, defined in Appendix 1, and with a new time variable s, given by $\frac{dt}{ds} = \rho^{p+q-2pq}$, the differential equation (1) becomes

$$\dot\rho = \frac{d\rho}{ds} = \rho^{p+q+1-4pq}[x^{2q-1}P(x,y) + y^{2p-1}Q(x,y)],$$
$$\dot\varphi = \frac{d\varphi}{ds} = \rho^{-2pq}[pxQ(x,y) - qyP(x,y)].$$

Using (1) we obtain

$$\begin{aligned} \dot\rho &= \bar a_n(\varphi)\rho^n + \bar a_m(\varphi)\rho^m, \\ \dot\varphi &= b_n(\varphi)\rho^{n-1} + b_m(\varphi)\rho^{m-1}, \end{aligned} \tag{2}$$

where
$$\begin{pmatrix} \bar a_u(\varphi) \\ b_u(\varphi) \end{pmatrix} = \begin{pmatrix} \operatorname{Cs}^{2q-1}(\varphi) & \operatorname{Sn}^{2p-1}(\varphi) \\ -q\operatorname{Sn}(\varphi) & p\operatorname{Cs}(\varphi) \end{pmatrix} \begin{pmatrix} P_u\big(\operatorname{Cs}(\varphi), \operatorname{Sn}(\varphi)\big) \\ Q_u\big(\operatorname{Cs}(\varphi), \operatorname{Sn}(\varphi)\big) \end{pmatrix},$$

$u \in \{n,m\}$ and $\operatorname{Sn}(\varphi)$ and $\operatorname{Cs}(\varphi)$ are also defined in Appendix 1.

Finally taking the new coordinates r and φ and a new time variable v, given by $r = \rho^{m-n}$, $\varphi = \varphi$, $\frac{dv}{ds} = \rho^{n-1}$, the differential equation (2) writes as

$$\begin{aligned} \dot r &= \frac{dr}{dv} = a_n(\varphi)r + a_m(\varphi)r^2, \\ \dot\varphi &= \frac{d\varphi}{dv} = b_n(\varphi) + b_m(\varphi)r, \end{aligned} \tag{3}$$

where $a_u(\varphi) = \bar a_u(\varphi)\cdot(m-n)$ for $u \in \{n,m\}$.

For the values (r,φ) for which $b_n(\varphi) + b_m(\varphi)r \neq 0$, equation (3) can be transformed into a new equation as follows

$$\frac{dr}{d\varphi} = S(r,\varphi) = \frac{a_n(\varphi)r + a_m(\varphi)r^2}{b_n(\varphi) + b_m(\varphi)r}. \tag{4}$$

Most properties that we will prove for system (1) will be studied in coordinates r, φ in which this system can be written as (3) or (4). We will define the functions:

$$F(\varphi) = a_n(\varphi)b_m(\varphi) - a_m(\varphi)b_n(\varphi), \quad \text{and} \quad A(\varphi) = b_m(\varphi)F(\varphi). \tag{5}$$

Note that the function $b_m(\varphi)$ controls the infinite critical points of (1) in the (p,q) Poincaré compactification (see Appendix 2). The functions $F(\varphi)$ and $b_m(\varphi)$ control the finite critical points of (1), (see Section 2). On the other hand, $b_n(\varphi)$ gives information about the origin: if $b_n(\varphi) \neq 0$, $(0,0)$ is a critical point of center or focus type, while if $b_n(\varphi)$ vanishes, $(0,0)$ can be the α or ω-limit set for some trajectory of system (3). As the following results show, hypothesizing on A, F, or b_n we can establish the number of limit cycles in (1).

The main results are listed in the following theorems. A more detailed account of these results and related ones, such as cases $b_m(\varphi) \equiv 0$, $F(\varphi) \equiv 0$, is given at the end of Section 3.

THEOREM A. *Given system (1), assume that the function $F(\varphi)$, defined in (5), does not change sign. Thus, this system has, at most, one limit cycle and, when it exists, it is hyperbolic, and surrounds the origin.*

Furthermore, there are examples of (1), with the above hypotheses, and with one limit cycle.

THEOREM B. *Given system (1), assume that the function $A(\varphi)$, defined in (5), does not change sign. Thus, this system has, at most, two limit cycles and, when they exist, they surround the origin. Furthermore, if $b_n(\varphi)$ does not vanish, the sum of the multiplicities of the limit cycles is, at most, two.*

Moreover, there are examples of (1), with the above hypothesis, with two, one or no limit cycles.

THEOREM C. *Given system (1), assume that the function $A(\varphi)b_n(\varphi)$ does not change sign. Thus, for this system, if there are limit cycles, they surround the origin and the sum of their multiplicities is, at most, three.*

Note that Theorem C gives new information only if $A(\varphi)$ changes sign.

Theorems A, B and C generalize several results obtained for differential equations with homogeneous non-linearities to systems of type (1) (see again [7, 9, 10, 12]).

We would like to point out that most of the proofs that we present differ from the proofs that appear in the above mentioned papers. In the main, these papers use the transformation of equation (3) into an Abel differential equation (see [5, 13]) whereas our different proofs are based directly on the expression (3), although the ideas used are similar.

The organization of the paper is as follows: Section 2 contains some results on the location of the critical points and limit cycles of system (1). In Section 3, we give the proofs of Theorems A, B and C with more detailed information about the number of limit cycles. There we also consider some examples. Finally, there are three appendices. The first two of them have already been mentioned. The third one discusses how to verify the existence of n and m in such a way that a differential equation can be written in form (1).

2. **On the location of the finite critical points and the limit cycles.** In this Section we study the situation of the finite critical points and periodic orbits of system (1).

Here we will use the generalized tangent function $\mathrm{Tn}(\varphi) = \mathrm{Sn}^p(\varphi)\mathrm{Cs}^q(\varphi)$ and its inverse $\mathrm{ArcTn}(x)$, introduced in Appendix 1. Let $T = T(p,q)$ be the period of the functions $\mathrm{Sn}(\varphi)$ and $\mathrm{Cs}(\varphi)$.

Let C_{φ_0} be the half-curve of points of $\mathbb{R}^2 - \{(0,0)\}$ that has the generalized polar angle of its points equal to φ_0 in the (r,φ) coordinates considered in Section 1. Note that

$$\begin{aligned} C_{\varphi_0} \cup C_{\varphi_0+\frac{T}{2}} &= \left\{(x,y)\in\mathbb{R}^2 : \mathrm{ArcTn}\left(\frac{y^p}{x^q}\right) = \varphi_0\right\} \\ &= \left\{(x,y)\in\mathbb{R}^2 : \mathrm{ArcTn}\left(\frac{y^p}{x^q}\right) = \varphi_0 + \frac{T}{2}\right\}. \end{aligned}$$

We have the following result

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LEMMA 1. *(a) If $b_n(\varphi_1) \cdot b_m(\varphi_1) = 0$ and $b_n(\varphi_1) + b_m(\varphi_1) \neq 0$ or if $F(\varphi_1) \neq 0$ then system (3) has no critical points on C_{φ_1}.*

(b) If $b_n(\varphi_1) = b_m(\varphi_1) = 0$ then C_{φ_1} is an invariant curve for (3).

(c) If $F(\varphi_1) = 0$ and $b_n(\varphi_1) \cdot b_m(\varphi_1) < 0$, then system (3) has exactly one finite critical point on C_{φ_1}.

(d) If $F(\varphi_1) = 0$ and $b_n(\varphi_1) \cdot b_m(\varphi_1) > 0$, then system (3) has no finite critical points on C_{φ_1}.

PROOF. (a) In the first case, $b_u(\varphi_1) = 0$ for some $u \in \{n, m\}$, and then $b_n(\varphi_1) + b_m(\varphi_1)r \neq 0$ for all $r \neq 0$. To prove that if $F(\varphi_1) \neq 0$, (3) has no critical points on C_{φ_1}, note that if (r_1, φ_1) is a critical point different from the origin then $a_n(\varphi_1) + a_m(\varphi_1)r_1 = b_n(\varphi_1) + b_m(\varphi_1)r_1 = 0$, and $F(\varphi_1) = 0$.

(b) It is obvious from expression (3).

(c) If we take $r_1 = \frac{-b_n(\varphi_1)}{b_m(\varphi_1)}$, then (r_1, φ_1) is a critical point of system (3).

(d) This case follows from (c) because if (r_1, φ_1) is a critical point, then $r_1 \geq 0$. ∎

REMARK 2. (a) It is not difficult to prove that, if p and q are both odd, $f_u(\varphi + T/2) = (-1)^{u+1} f_u(\varphi)$ for f_u equals either a_u or b_u and $u \in \{n, m\}$. Therefore, in this case it is possible to relate the number of critical points on C_{φ_1} to the number of critical points on $C_{\varphi_1 + \frac{T}{2}}$ taking into account the parity of $m - n$.

(b) Where $p = q$, C_φ is the half ray through the origin with slope $\tan \varphi$.

Let K be the subset of points of $\mathbb{R}^2$ on which the angular component of the vector field (3), $\dot{\varphi}$, vanishes. In the following lemma we study the geometry of K, when it has no curves like (e) of Figure 1. We exclude this case because, as we will see in Proposition 4(i), the presence of such curves forces the non existence of periodic orbits. This lemma improves Lemma 2.2 of [7].

LEMMA 3. *Let X be the vector field associated with system (3). Then*

(a) K is the graph of the function $r = \frac{-b_n(\varphi)}{b_m(\varphi)}$.

(b) At point $a = (x_0, y_0) \in K$, $X(a)$ is tangent to the half-curve C_φ where $\varphi = \mathrm{ArcTn}(\frac{y_0^p}{x_0^q})$.

(c) If K has no curves of type (e) given in Figure 1, then K is either the finite union of curves given by sectors of type (a), (b), (c), (d) and (f) of Figure 1, or K is one of the curves which delimit the sets shown in Figure 2.

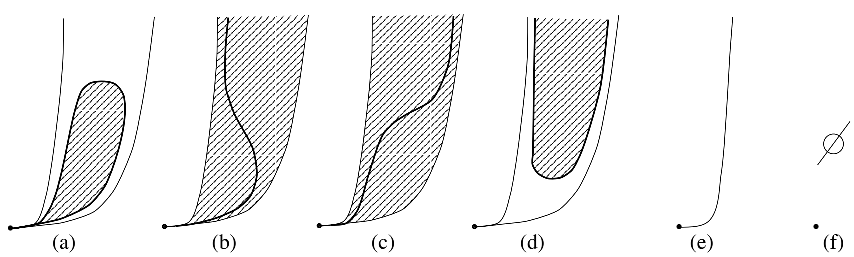

Figure 1

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FIGURE 1. The subset K can be a finite union of the curves given by these sectors. The shadowed regions in cases (b) and (c) are either positively or negatively invariant by the flow of (3). In cases (a) and (d) the same happens when one of the hypotheses assumed in Proposition 4(iii) is satisfied.

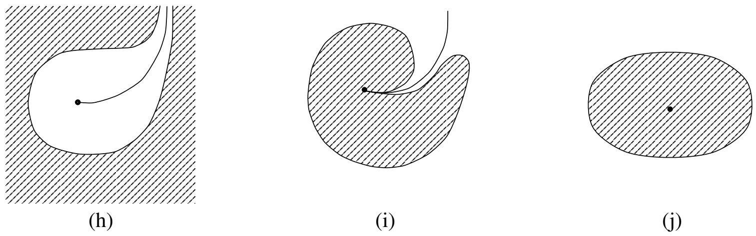

Figure 2

FIGURE 2. The subset K can be one of the curves which delimit the shadowed regions. These shadowed regions are either positively or negatively invariant by the flow of (3), when one of the hypotheses assumed in Proposition 4(iii) is satisfied.

PROOF. Parts (a) and (b) follow from direct calculations.

(c) When there are φ_1 and φ_2, not equal, and with $b_n(\varphi)b_m(\varphi) < 0$ for all φ in (φ_1, φ_2) we have (a) if $b_n(\varphi_1) = b_n(\varphi_2) = 0$; (b) if $b_n(\varphi_1) = b_m(\varphi_2) = 0$; (c) if $b_n(\varphi_2) = b_m(\varphi_1) = 0$; (d) if $b_m(\varphi_1) = b_m(\varphi_2) = 0$. When there is only one φ_1 such that $b_n(\varphi_1) = 0$, $b_m(\varphi_1) \neq 0$ and $b_n(\varphi)b_m(\varphi) > 0$, for all $\varphi \neq \varphi_1$, then we have case (f). When for all φ in some interval (φ_1, φ_2) we have $b_n(\varphi)b_m(\varphi) > 0$, then K has no points in this region and we are in case (f). When there exists φ_1 such that $b_n(\varphi_1) = b_m(\varphi_1) = 0$, then $\varphi = \varphi_1$ is invariant by the flow of system (3), and we get case (e).

When there is only one φ_1 such that $b_n(\varphi_1) \neq 0$ and $b_m(\varphi_1) = 0$ with $b_n(\varphi)b_m(\varphi) < 0$ for all $\varphi \neq \varphi_1$, we are in case (h). When there is only one φ_1 such that $b_n(\varphi_1) = 0$ and $b_m(\varphi_1) \neq 0$ with $b_n(\varphi)b_m(\varphi) < 0$ for all $\varphi \neq \varphi_1$, then the form of K is that given in (i) of Figure 2. When for all φ we have $b_n(\varphi)b_m(\varphi) < 0$, then we obtain case (j). ∎

The following proposition gives information about the periodic orbits of system (3) that surround the origin.

PROPOSITION 4. *(i) Assume that K has associated some sector of type (b), (c) or (e) of Figure 1, then equation (3) has no periodic orbits surrounding the origin.*

(ii) Assume that γ is a periodic orbit of (3) surrounding the origin, then $\gamma \cap K = \emptyset$.

(iii) Assume that one of the functions $F(\varphi)$ or $A(\varphi)$ or $A(\varphi)b_n(\varphi)$, associated with the differential equation (3), does not change sign. If γ is a periodic orbit of (3), then γ surrounds the origin. Furthermore, assume that K has associated no sectors of type (a) of Figure 1 or that the curve K is not like the curves given in (i) or (j) of Figure 2, then the origin is the only critical point surrounded by γ; otherwise γ can surround other critical points.

PROOF. (i) Let γ be a periodic orbit of system (3), then γ cannot cross those sectors, given by K, because of the sign of $b_n(\varphi) + b_m(\varphi)r$ in (3) in cases (b) and (c) (note that the shadowed regions in those sectors are either positively or negatively invariant by the flow of system (3)), or because $\varphi = \varphi_1$ is an invariant curve of system (3) in case (e).

(ii) Assume that $\gamma \cap K \neq \emptyset$. Then γ crosses K transversally because, otherwise, this contact point will be a critical point of system (3). Hence, γ must cross sectors (a) or (d) or subsets (h) or (i) or (j) in two points, R and S, because γ surrounds the origin. In essence, we will have the situation given in Figure 3, where we mark the direction of rotation of the flow of the vector field (3), by means of small arrows. We also take into account that K separates the regions where the directions of rotation are opposed. So by the uniqueness of the solutions we have a contradiction and, therefore, γ cannot surround the origin.

(iii) From the Index Theory, γ has to surround a critical point. This point belongs to the set K. Note that on K, $\dot{r}(r,\varphi) = \frac{F(\varphi)r(\varphi)}{b_m(\varphi)} = -\frac{b_n(\varphi)A(\varphi)}{b_m{}^3(\varphi)} = \frac{A(\varphi)r(\varphi)}{b_m{}^2(\varphi)}$, so $\dot{r}$ does not change sign on the connected components of K. Hence γ must surround the origin. If, in addition, K has associated no sectors of type (a) or (i) or (j) of Figures 1 and 2, the origin will be the unique critical point that γ surrounds because the shadowed regions of Figures 1 and 2 are invariant under the flow of the system (3). In the other cases γ can surround critical points different from the origin. The examples: (a) $\dot{r} = r(10-r)\cos^2\varphi$, $\dot{\varphi} = 5-(1+\sin^2\varphi)r$; (b) $\dot{r} = r(10-r)\sin^2\varphi$, $\dot{\varphi} = 5\cos^2\varphi - r$, illustrate this situation, see Figure 4. ∎

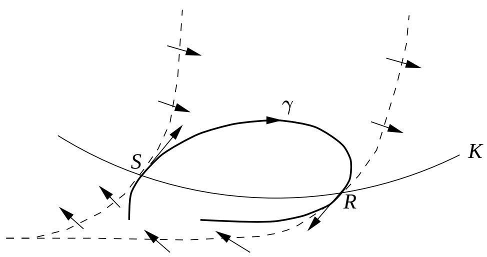

Figure 3

FIGURE 3. Standard situation that occurs when $\gamma \cap K \neq \emptyset$, and γ surrounds the origin.

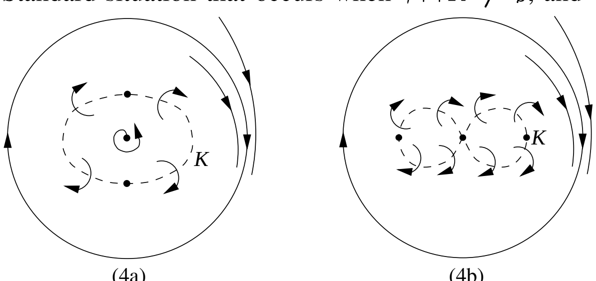

Figure 4

FIGURE 4. Limit cycles for system (3) surrounding several critical points.

COROLLARY 5. *Periodic orbits of differential equation (3) surrounding the origin can be studied as solutions of (4) satisfying* $r(\varphi_1) = r(\varphi_1 + T)$ *for any* φ_1.

PROOF. Follows from (ii) of Proposition 4. ∎

REMARK 6. (a) Note that it is possible that equation (3) has periodic orbits γ that do not surround the origin, and with $\gamma \cap K \neq \emptyset$. This can be seen simply by taking a quadratic differential equation with two limit cycles γ_1 and γ_2, one of these surrounding the origin and the other one surrounding a different critical point, see for instance [18]. Of course such a differential equation does not satisfy the hypotheses given in (iii) of Proposition 4.

(b) Observe that the result obtained in Proposition 4(ii) is true not only for periodic orbits but also for orbits turning around the origin.

Given a subset C_ψ, we define $D_\psi \subset C_\psi$ as the subset of points of C_ψ for which the Poincaré return map, h, is defined, *i.e.*, the set of points, $a \in C_\psi$ for which $h(a) := \psi(T, a)$ is defined and belongs to C_ψ, where $\psi(\varphi, a)$ is the solution of (4) such that $\psi(0, a) = a$. Note that D_ψ is always an open subset of C_ψ

PROPOSITION 7. *Assume that either the function $F(\varphi)$ or $A(\varphi)$ or $A(\varphi)b_n(\varphi)$, associated with the equation (3) does not change sign and K is not a simple closed curve (case j of Figure 2). Then there is a ψ, such that*

(i) All the periodic orbits of (3) belong to the closest connected component to the origin of D_ψ.
(ii) If $b_n(\varphi)$ does not vanish, $0 \in \overline{D_\psi}$.

PROOF. (i) If equation (3) has no periodic orbits, there is nothing to be proved. So, from Proposition 4(i), cases (b), (c) and (e) will not be considered. We can assume that there is a ψ such that C_ψ is a half curve without contact. Assume, now, that on C_ψ, D_ψ has, at least, two connected components D_1 and D_2 and equation (3) has a periodic orbit γ on D_2 (D_1 is closer to the origin than D_2). From Proposition 4(ii)–(iii) and Remark 6(b) we have $\gamma \cap K = \emptyset$, and an orbit $\tilde{\gamma}$ through a point in D_1 must, always, surround sectors like (a) of Figure 1, if K has associated some of them. Hence, if we take a point q on $C_\psi \setminus D_\psi$ between D_1 and D_2, its α-limit or ω-limit set must be non-empty. This is impossible because between $\tilde{\gamma}$ and γ there are no critical points. Thus, all periodic orbits of (3) cut D_1, and (i) follows (see Figure 5).

(ii) The proof follows from (i) taking into account that when $b_n(\varphi)$ does not vanish the origin behaves like a periodic orbit. ∎

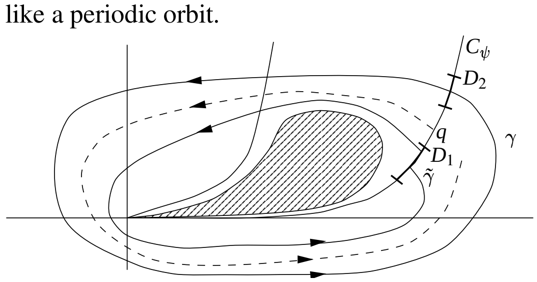

Figure 5

FIGURE 5. C_ψ with two different connected components.

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REMARK 8. When K is a simple closed curve and the other hypotheses of Proposition 7 hold, system (3) can have periodic orbits in different connected components of C_ψ. For instance, one periodic orbit turning counterclockwise and another one turning clockwise. This is the reason why this case will be studied separately in the following section. Observe that this situation can only occur when $b_n(\varphi)$ and $b_m(\varphi)$ do not vanish.

3. **Proof of Theorems A, B and C.** First, we will give some preliminary results.

PROPOSITION 9 (See [15]). *Let $h(x)$ be the return map associated with the differential equation $dr/d\varphi = S(r,\varphi)$, then*

(i) $h'(x) = \exp \int_0^T \frac{\partial S}{\partial r}\big(r(\varphi,x),\varphi\big)\, d\varphi$,

(ii) $h''(x) = h'(x)\Big[\int_0^T \frac{\partial^2 S}{\partial r^2}\big(r(\varphi,x),\varphi\big) \exp\big\{\int_0^\varphi \frac{\partial S}{\partial r}(r(s,x),s)\, ds)\big\}\, d\varphi\Big]$,

(iii) $h'''(x) = h'(x)\Big[\frac{3}{2}(\frac{h''(x)}{h'(x)})^2 + \int_0^T \frac{\partial^3 S}{\partial r^3}\big(r(\varphi,x),\varphi\big) \exp\big\{2\int_0^\varphi \frac{\partial S}{\partial r}\big(r(s,x),s\big)\, ds\big\}\, d\varphi\Big]$,

where $r(\varphi,x)$ denotes the solution of the differential equation such that $r(0,x) = x$.

Direct calculations give the following lemma,

LEMMA 10. *For equation (4) we have:*

(i) $S(r,\varphi) = \frac{a_m(\varphi)}{b_m(\varphi)} r + \frac{F(\varphi)}{b_m^2(\varphi)} - \frac{F(\varphi)b_n(\varphi)}{b_m^2(\varphi)(b_n(\varphi)+b_m(\varphi)r)}$,

(ii) $\frac{\partial S}{\partial r}(r,\varphi) = \frac{a_m(\varphi)}{b_m(\varphi)} + \frac{F(\varphi)b_n(\varphi)}{b_m(\varphi)(b_n(\varphi)+b_m(\varphi)r)^2}$,

(iii) $\frac{\partial^2 S}{\partial r^2}(r,\varphi) = \frac{-2F(\varphi)b_n(\varphi)}{(b_n(\varphi)+b_m(\varphi)r)^3}$,

(iv) $\frac{\partial^3 S}{\partial r^3}(r,\varphi) = \frac{6A(\varphi)b_n(\varphi)}{(b_n(\varphi)+b_m(\varphi)r)^4}$.

When the return map is defined we obtain the next result,

LEMMA 11. *The first derivative of the return map associated to a periodic orbit, $r(\varphi)$, of equation (4) is*

(i) $\exp\{\int_0^T \frac{a_n(\varphi)}{b_n(\varphi)}\, d\varphi\}$, *if* $r \equiv 0$,

(ii) $\exp\{-\int_0^T \frac{F(\varphi)r(\varphi)}{(b_n(\varphi)+b_m(\varphi)r(\varphi))^2}\, d\varphi\}$, *if* $r \not\equiv 0$.

PROOF. (i) follows from the expression obtained for the function $S(r,\varphi) = \frac{dr}{d\varphi}$ in Lemma 10(ii), and from Proposition 9(i).

To prove (ii), note that from equation (4),

$$0 = \int_0^T \frac{r'(\varphi)}{r(\varphi)}\, d\varphi = \int_0^T \frac{a_n(\varphi)+a_m(\varphi)r}{b_n(\varphi)+b_m(\varphi)r}\, d\varphi$$
$$= \int_0^T \frac{\big(a_n(\varphi)+a_m(\varphi)r\big)\big(b_n(\varphi)+b_m(\varphi)r\big)}{\big(b_n(\varphi)+b_m(\varphi)r\big)^2}\, d\varphi,$$

and, from this last expression, we have that

$$\int_0^T \frac{a_n(\varphi)b_n(\varphi)+a_m(\varphi)b_m(\varphi)r^2}{\big(b_n(\varphi)+b_m(\varphi)r\big)^2}\, d\varphi = \int_0^T \frac{-r\big(a_n(\varphi)b_m(\varphi)+b_n(\varphi)a_m(\varphi)\big)}{\big(b_n(\varphi)+b_m(\varphi)r\big)^2}\, d\varphi.$$

Hence, using this equality, (i) of Proposition 9 and (ii) of Lemma 10, (ii) holds. ∎

The calculations made in the following lemma are inspired by [16] and are straightforward. See also Remark 24.

LEMMA 12. *Let $r_1(\varphi) > r_2(\varphi) > r_3(\varphi)$ be three positive solutions of (4). If*

$$\mathcal{H}(\varphi) := \frac{S(r_1,\varphi) - S(r_2,\varphi)}{r_1(\varphi) - r_2(\varphi)} - \frac{S(r_1,\varphi) - S(r_3,\varphi)}{r_1(\varphi) - r_3(\varphi)} - \frac{S(r_2,\varphi)}{r_2(\varphi)} + \frac{S(r_3,\varphi)}{r_3(\varphi)}, \tag{6}$$

where $S(r_i,\varphi)$, for $i = 1, 2, 3$ is defined in (4), then we have

$$\mathcal{H}(\varphi) = \frac{A(\varphi) r_1(\varphi)\big(r_2(\varphi) - r_3(\varphi)\big)}{\big(b_n(\varphi) + b_m(\varphi) r_1\big)\big(b_n(\varphi) + b_m(\varphi) r_2\big)\big(b_n(\varphi) + b_m(\varphi) r_3\big)}. \tag{7}$$

The next lemma follows from direct computations and is based on the change of variables made in [5].

LEMMA 13. *If $b_n(\varphi)$ does not vanish, the transformation $\mathcal{T}(r,\varphi) = (\rho,\varphi)$, where*

$$\rho = \frac{r}{b_n(\varphi) + b_m(\varphi) r},$$

is a diffeomorphism between $\mathbb{R}^2 \setminus K$ and its image. Furthermore, the differential equation (4) is transformed into the following Abel differential equation:

$$\frac{d\rho}{d\varphi} = \alpha(\varphi)\rho^3 + \beta(\varphi)\rho^2 + \gamma(\varphi)\rho, \tag{8}$$

where

$$\alpha(\varphi) = \frac{b_m(\varphi)}{b_n(\varphi)}[a_n(\varphi) b_m(\varphi) - a_m(\varphi) b_n(\varphi)] = \frac{F(\varphi) b_m(\varphi)}{b_n(\varphi)} = \frac{A(\varphi)}{b_n(\varphi)},$$
$$\beta(\varphi) = \frac{1}{b_n(\varphi)}[b_n(\varphi) a_m(\varphi) - 2a_n(\varphi) b_m(\varphi)] + \frac{b_n'(\varphi) b_m(\varphi) - b_n(\varphi) b_m'(\varphi)}{b_n(\varphi)},$$
$$\gamma(\varphi) = \frac{a_n(\varphi) - b_n'(\varphi)}{b_n(\varphi)},$$

REMARK 14. Observe that, from the above lemma, the periodic orbits of (4) that do not cut K are transformed into T-periodic solutions of (8).

Following [7], equation (8) can be written in a different way as the next lemma shows.

LEMMA 15. *Equation (8) is equivalent to*

$$\frac{d\big(\rho^{-1} - b_m(\varphi)\big)}{d\varphi} = \big(\rho^{-1} - b_m(\varphi)\big)\left(\frac{F(\varphi)}{b_n(\varphi)}\rho - \frac{a_n(\varphi)}{b_n(\varphi)} + \frac{b_n'(\varphi)}{b_n(\varphi)}\right). \tag{9}$$

REMARK 16. Observe that from equation (9), when $b_n(\varphi)$ and $b_m(\varphi)$ do not vanish, $\rho(\varphi) = \frac{1}{b_m(\varphi)}$ is a T-periodic solution of (8). Note that this solution is mapped onto infinity in the (r,φ) coordinates.

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LEMMA 17. *It is not restrictive, when the function $A(\varphi)b_n(\varphi)$ does not change sign, to consider $A(\varphi)b_n(\varphi) \geq 0$ for every φ.*

PROOF. By using the following change of variables, $(r,\varphi) \longrightarrow (r, T-\varphi)$, the lemma follows. ∎

PROPOSITION 18. *Assume that the function $A(\varphi)b_n(\varphi)$ does not change sign. Then the third derivative of the Poincaré return map, h, of (4) is positive.*

PROOF. Using Lemma 17, if $A(\varphi)b_n(\varphi)$ does not change sign, one can assume that $A(\varphi)b_n(\varphi) \geq 0$. Since, for Lemma 10(iv), $\frac{\partial^3 S}{\partial r^3}(r,\varphi) = \frac{6A(\varphi)b_n(\varphi)}{(b_n(\varphi)+b_m(\varphi)r)^4} \geq 0$, it follows from Proposition 9 that $h'''(x) > 0$ for all x for which h is defined. ∎

In a similar way as in equation (4), we can define a Poincaré return map $\tilde{h}$ for equation (8) between $\varphi = 0$ and $\varphi = T$. For this map $\tilde{h}$ we have the following result which has already been proved in several other papers, see for instance [9].

PROPOSITION 19. *Assume that the function $b_n(\varphi)$ does not vanish and $A(\varphi)b_n(\varphi)$ does not change sign, then the third derivative of the Poincaré return map, $\tilde{h}$, of (8) is positive.*

PROOF. Since $\frac{\partial^3}{\partial \rho^3}\Big(\alpha(\varphi)\rho^3 + \beta(\varphi)\rho^2 + \gamma(\varphi)\rho\Big) = 6\alpha(\varphi) = 6\frac{A(\varphi)}{b_n(\varphi)}$, does not change sign, the proof follows in the same way as the proof of Proposition 18. ∎

REMARK 20. Although the conclusions of Propositions 18 and 19 are similar, we note that h and $\tilde{h}$ have different properties. For instance, while $h(x)$ is only defined for positive values of x, $\tilde{h}(x)$ has to be studied in all the real line.

First we will prove Theorems A, B and C only when $F(\varphi)$, $A(\varphi)$ or $b_n(\varphi)$ are not identically zero. The case in which one of the three functions identically vanishes is easier and is studied at the end of this section.

PROOF OF THEOREM A. From Proposition 4(iii), any periodic orbit of (3) surrounds the origin. As explained in Remark 8, we will divide the proof into two cases:

CASE a). K is not a simple closed curve.

From Proposition 7, there exists a ψ such that all periodic orbits are in the connected component D_ψ of C_ψ. Take a periodic orbit γ of (3). From Lemma 11, since F does not change sign, it is a hyperbolic stable (resp. unstable) limit cycle if $F(\varphi)$ is greater than or equal to (resp. less than or equal to) zero. Hence γ is unique.

CASE b). K is a simple closed curve.

Periodic orbits of (3) can cut different connected components of C_ψ. Of course, the proof of case a) shows that, in case b), our system has, at most, two limit cycles, one turning clockwise and another one turning counterclockwise but, as we will see, they can not coexist.

From Proposition 4, periodic orbits of (3) surround the origin, furthermore, from Remark 14 and since $b_n(\varphi)$ does not vanish, we can study the periodic orbits of (3) as T-periodic solutions of (9). Let $r(\varphi)$ be a periodic orbit of (3). It gives a T-periodic solution of (9), $\rho(\varphi)$. From Lemma 15, we have that:

$$\frac{d}{d\varphi}\ln\big(\rho^{-1}(\varphi)-b_m(\varphi)\big)=\left(\frac{F(\varphi)}{b_n(\varphi)}\rho(\varphi)-\frac{a_n(\varphi)}{b_n(\varphi)}+\frac{b_n'(\varphi)}{b_n(\varphi)}\right),$$

and since $\rho(\varphi)$ is T-periodic,

$$0=\int_0^T\frac{F(\varphi)}{b_n(\varphi)}\rho(\varphi)\,d\varphi+k, \tag{10}$$

where $k=-\int_0^T a_n(\varphi)/b_n(\varphi)\,d\varphi$. Observe that if $r_1(\varphi)$ and $r_2(\varphi)$ are two periodic orbits of (3), they induce two T-periodic solutions of (9), $\rho_1(\varphi)$ and $\rho_2(\varphi)$. We can assume that $\rho_1(\varphi)>\rho_2(\varphi)$. But since $F(\varphi)/b_n(\varphi)$ does not change sign,

$$\int_0^T\frac{F(\varphi)}{b_n(\varphi)}\rho_1(\varphi)\,d\varphi\neq\int_0^T\frac{F(\varphi)}{b_n(\varphi)}\rho_2(\varphi)\,d\varphi,$$

and this contradicts (10). Hence (3) has, at most, one periodic orbit. Using Lemma 11, it is hyperbolic. ∎

COROLLARY 21. *Given the differential equation (1), assume that $F(\varphi)\not\equiv 0$, does not change sign and that $b_n(\varphi)$ does not vanish. Set $c=\int_0^T\frac{a_n(\varphi)}{b_n(\varphi)}\,d\varphi$. Then*

a) If K is not a simple closed curve, the unique limit cycle for system (1) only exists when $\operatorname{sign}(F)\cdot c>0$.

b) If K is a simple closed curve, it divides $\mathbb{R}^2$ in two connected components, one bounded K_b and one unbounded K_u. Thus, if the limit cycle exists in system (1), it is in K_b (resp. K_u) if $\operatorname{sign}(F)\cdot c$ *is plus (resp. minus).*

PROOF. Follows easily from Lemma 11 and Theorem A. ∎

PROOF OF THEOREM B. In our hypotheses and from Proposition 4, all periodic orbits of system (1) surround the origin and do not cut K. Assume that system (1) has three limit cycles $r_1(\varphi)>r_2(\varphi)>r_3(\varphi)$. From Corollary 5, $r_i(\varphi)$, $i=1,2,3$, can be considered as positive solutions of equation (4). Since from Lemma 12, $A(\varphi)$ does not change sign, we have that $\mathcal{H}(\varphi)$ does not change sign and is a continuous function. But, on the other hand, we have that:

$$0=\log\left\{\frac{\big(r_1(\varphi)-r_2(\varphi)\big)r_3(\varphi)}{\big(r_1(\varphi)-r_3(\varphi)\big)r_2(\varphi)}\right\}\Bigg|_0^T=\int_0^T\mathcal{H}(\varphi)\,d\varphi,$$

and this contradicts the continuity of $\mathcal{H}(\varphi)$. Hence system (1) has, at most, two limit cycles. Now we have to prove that, when $b_n(\varphi)$ does not vanish, the sum of the multiplicities of the limit cycles is, at most, two. In this case, when K is not a simple closed curve, from Proposition 7 and Corollary 5, all periodic orbits of (4), included the origin,

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belong to the same connected component of D_ψ. Furthermore, from Proposition 13, the third derivative of the Poincaré return map of (4), h, is positive. Whence we conclude, from Rolle's Theorem, that $h(x) = x$ has, at most, two simple solutions besides the origin. Therefore the theorem follows. When K is a simple closed curve $b_m(\varphi)$ does not vanish. Hence, $F(\varphi) = A(\varphi)/b_m(\varphi)$ neither changes sign. Therefore from Theorem A, system (1) has, at most, one hyperbolic limit cycle and again the theorem follows. ∎

REMARK 22. As we have seen in this last proof, the conclusions of Theorem B can be improved when K is a simple closed curve. In fact, in such situation, system (1) has, at most, one limit cycle, and when it exists it is hyperbolic.

In the case where $A(\varphi) \not\equiv 0$ does not change sign and $b_n(\varphi) \neq 0$, for all φ (this is the case where the local phase portrait of the origin of system (1) is of focus or center type), we obtain a more precise distribution of limit cycles, as we can see in the next theorem. This theorem is based on [9, Theorem A].

THEOREM 23. *Assume that in system (1), $A(\varphi) \not\equiv 0$ does not change sign, $b_n(\varphi) \neq 0$, for all φ, and K is not a simple closed curve. Then Table I shows the distribution of limit cycles when $A(\varphi)b_n(\varphi) \geq 0$, according to the different values of c and d. (The case $A(\varphi)b_n(\varphi) \leq 0$ has associated the table obtained reversing the inequalities for c and d, in accordance with Lemma 17).*

	$c<0$	$c=0$			$c>0$		
		$d<0$	$d=0$	$d>0$	$d<0$	$d=0$	$d>0$
(I)	1	1	0	0	2	0	0
(II)	1	2	3	2	1	1	1

TABLE I. Maximum number of limit cycles of equation (1) when $A(\varphi)b_n(\varphi) \geq 0$. Here $c = \int_0^T \frac{a_n(\varphi)}{b_n(\varphi)}\,d\varphi$, $d = \int_0^T \frac{-2F(\varphi)}{b_n^2(\varphi)} \exp\left(\int_0^\varphi \frac{a_n(s)}{b_n(s)}\,ds\right)d\varphi$. (I) maximum number of limit cycles, taking into account their multiplicity. (II) multiplicity of the solution $r \equiv 0$.

PROOF. Using Corollary 5, to study the limit cycles of (1), it is sufficient to consider equation (4). From the hypotheses, we have that set K is not like the curve in (j) of Figure 2. Therefore, from Proposition 7, there exists some ψ such that all periodic orbits of (4) cut a connected subset of D_ψ, I and, furthermore, $0 \in \overline{I}$.

If we define $H(x) = h(x) - x$, where $h(x)$ is the Poincaré return map associated with (4) and with ψ, we have the following properties for H:

(i) $H'''(x) > 0$, for all $x \in I$ (Proposition 9(iii) and Lemma 10(iv))

(ii) $H'(0) = e^c - 1$, and $H''(0) = e^c d$ (Proposition 9 and Lemma 11(i))

Note that $x = 0$ corresponds to solution $r \equiv 0$, and the fixed points of H correspond with the periodic orbits of (4). Therefore, using (i) and (ii) and arguing as in the proof of [9,Theorem A], we obtain Table I. ∎

PROOF OF THEOREM C. From Proposition 4(iii), if (1) has some limit cycle, it surrounds the origin.

CASE a). K is not a simple closed curve.

From Proposition 7 we have that all periodic orbits cut the same connected component D_ψ of C_ψ. From Proposition 18, the third derivative of the return map, h, when it is defined, is positive. Therefore, by Rolle's Theorem, the sum of the multiplicities of the limit cycles is, at most, three.

CASE b). K is a simple closed curve.

Since in this case $b_n(\varphi)$ does not vanish, the result is that $A(\varphi)$ does not change sign and we can apply Remark 22. So, in fact, there is, at most, one limit cycle. ∎

REMARK 24. The proofs of Theorems A, B and C have been essentially based on two different methods. On the one hand, we considered the function $\mathcal{H}(\varphi)$ given in (7) and, on the other, we calculated the third derivative of the function $S(r,\varphi)$ defined in (4). Here, we are going to prove that there is a relationship between both of them.

Fixed φ, set $S(r) = S(r,\varphi)$. Remember that given $r_i \in \mathbb{R}$, $i = 1,\ldots,n$, we can define inductively the divided differences of S, as:

$$S[r_i, r_{i+1}, \ldots, r_{i+j+1}] = \frac{S[r_{i+1}, \ldots, r_{i+j+1}] - S[r_i, \ldots, r_{i+j}]}{r_{i+j+1} - r_i},$$

where $S[r_i] = S(r_i)$, see [11, Chapter 6]. It turns out that $S[\ ]$ is a symmetric function of its variables. As usual, we call it $S_{i,\ldots,i+j+1}$ for short. Then, with this notation, and using $S(0) = 0$,

$$\mathcal{H}(\varphi) = S_{1,2} - S_{1,3} - S_{2,0} + S_{3,0} = (S_{2,1,3} - S_{2,0,3})(r_2 - r_3) = S_{0,1,2,3}(r_2 - r_3)r_1.$$

At the same time, it is well known that $S_{0,1,2,3,\ldots,n} = \frac{S^{(n)}(\xi)}{n!}$, where $\xi \in \langle r_0, r_1, \ldots, r_n \rangle$. Therefore, we have that

$$\mathcal{H}(\varphi) = \frac{1}{3!} r_1(\varphi)\big(r_2(\varphi) - r_3(\varphi)\big) \frac{\partial^3 S(r,\varphi)}{\partial r^3}\bigg|_{r=\xi(\varphi, r_1(\varphi), r_2(\varphi), r_3(\varphi))}.$$

When $b_n(\varphi) \equiv 0$ or $A(\varphi) \equiv 0$, it is possible to have more precise information about the limit cycles. And we go on to deal with this below.

When $b_n(\varphi) \equiv 0$ and $b_m(\varphi) \not\equiv 0$ (in the case $b_n(\varphi) \equiv b_m(\varphi) \equiv 0$, system (3) has the solution $\varphi =$ constant, for all φ), or $A(\varphi) \equiv 0$, we can integrate system (3). Hence, in these cases, we can know exactly the trajectories of all closed solutions. Their initial conditions are given in the following lemma.

LEMMA 25. *In system (3) we assume $b_n(\varphi) \equiv 0$, $d_1 = \int_0^T \frac{a_m(\varphi)}{b_m(\varphi)}\, d\varphi$, and $d_2 = \int_0^T \frac{a_n(\varphi)}{b_m(\varphi)} \exp\big(-\int_0^\varphi \frac{a_m(s)}{b_m(s)}\, ds\big)\, d\varphi$. Thus, the following hold.*

(i) If $d_1 = d_2 = 0$, all trajectories of (3), in a neighbourhood of $r \equiv 0$, are closed.

(ii) If $|d_1| + |d_2| \neq 0$, system (3) has at most two closed solutions. Furthermore, these solutions are the ones with initial conditions

$$\left.\begin{matrix} r(0) = 0 \\ \varphi(0) = 0 \end{matrix}\right\}, \quad \textit{and} \quad \left.\begin{matrix} r(0) = \frac{d_2 e^{d_1}}{1 - e^{d_1}} \\ \varphi(0) = 0 \end{matrix}\right\}.$$

PROOF. The proof follows by direct calculations. ∎

PROPOSITION 26. *In system (3), assuming* $A(\varphi) \equiv 0$,

(i) *If* $F(\varphi) \equiv 0$ *and* $b_m(\varphi) \not\equiv 0$, *then system (3) has no limit cycles. Moreover, if* $c = \int_0^T \frac{a_n(\varphi)}{b_n(\varphi)}\, d\varphi = 0$, *then the origin is a center for system (3).*

(ii) *If* $F(\varphi) \not\equiv 0$ *and* $b_m(\varphi) \equiv 0$, *then system (3) has, at most, one limit cycle. Moreover, if* $d = \int_0^T \frac{-F(\varphi)}{b_n^2(\varphi)} \exp\big(\int_0^\varphi \frac{a_n(s)}{b_n(s)}\, ds\big)\, d\varphi$, *and c is the value given in (i), the following holds:*

  (a) *If* $d = c = 0$, *all trajectories of (3), in a neighbourhood of* $r \equiv 0$, *are closed.*

  (b) *If* $|c| + |d| \neq 0$, *system (3) has, at most, two closed solutions with initial conditions*
$$\left.\begin{array}{r} r(0) = 0 \\ \varphi(0) = 0 \end{array}\right\}, \quad \textit{and} \quad \left.\begin{array}{r} r(0) = \frac{(1-e^c)}{d} \\ \varphi(0) = 0 \end{array}\right\}.$$

(iii) *Assume* $F(\varphi) \equiv b_m(\varphi) \equiv 0$. *If* $b_n(\varphi) \equiv 0$, *then all straight lines through the origin are invariant and if* $b_n(\varphi) \not\equiv 0$ *and* $a_m(\varphi) \equiv 0$, *then the origin is a center.*

PROOF. If $F(\varphi) \equiv 0$ and assuming $b_m(\varphi) \not\equiv 0$, we have that system (3) is equivalent to $\frac{dr}{d\varphi} = \frac{a_m(\varphi)}{b_m(\varphi)} r$. With the condition $F(\varphi) \equiv 0$ and integrating this equation we obtain the solutions
$$r(\varphi) = r(0) \cdot \exp\left(\int_0^\varphi \frac{a_n(s)}{b_n(s)}\, ds\right).$$
Then (i) follows.

If $b_m(\varphi) \equiv 0$, system (3) becomes
$$\begin{aligned} \dot r &= a_n(\varphi) r + a_m(\varphi) r^2, \\ \dot\varphi &= b_n(\varphi), \end{aligned}$$
or, equivalently, $\frac{dr}{d\varphi} = \frac{a_n(\varphi)}{b_n(\varphi)} r + \frac{a_m(\varphi)}{b_n(\varphi)} r^2$, and the solutions, $r(\varphi)$, of this Riccati equation are
$$r(\varphi) = \frac{\exp\big(\int_0^\varphi \frac{a_n(s)}{b_n(s)}\, ds\big)}{-\int_0^\varphi \frac{a_m(s)}{b_n(s)} \exp\big(\int_0^s \frac{a_n(\tau)}{b_n(\tau)}\, d\tau\big)\, ds + r^{-1}(0)}.$$
From this expression, (ii) follows. The proof of (iii) is trivial. ∎

The natural generalization of the example given in [10, Proposition 6.3]
$$\begin{aligned} \dot x &= apx + \gamma y^{p/q} - (a\rho^{2pq} + \gamma x^{2q-1} y^{p/q})(px + y^{2p-1}\rho^{p+q-2pq}), \\ \dot y &= aqy - (a\rho^{2pq} + \gamma x^{2q-1} y^{p/q})(qy - x^{2q-1}\rho^{p+q-2pq}), \end{aligned}$$
where $\rho = \sqrt[2pq]{px^{2q} + qy^{2p}}$, is a system of type (1). For some values of a, γ, p, and q, this system has one or two limit cycles and it is in accordance with the hypotheses of Theorem A and B. Therefore, it shows that the results of Theorem A and B cannot be improved. We also present some different examples for which the above theorems apply. We stress that they have not homogeneous nonlinearities. Consider
$$(\dot x, \dot y) = \big(-y^{2p-1} + P_m(x, y), x^{2q-1} + Q_m(x, y)\big),$$

where P_m and Q_m are (p,q)-quasihomogeneous polynomials of degrees $m+2pq-(q+1)$ and $m+2pq-(p+1)$ respectively. For these systems $F(x,y) = -\big(x^{2q-1}P_m(x,y)+y^{2p-1}Q_m(x,y)\big)$, and $A(x,y) = F(x,y)\big(pxQ_m(x,y)-qyP_m(x,y)\big)$. For instance, for system

$$(\dot x,\dot y) = (-y+ax^3+bxy, x^3+cx^4+dx^2y),$$

with $(b+c)^2-4ad<0$, we get $F(x,y) = ax^6+(b+c)x^4y+dx^2y^2$ and taking $y=\lambda x^2$, we can prove that F does not change sign. On the other hand, consider

$$(\dot x,\dot y) = (-y+ax^5+bx^2y, x^5+cx^7+dx^4y+3bxy^2),$$

where $b = b(a,c,d) = -\frac{a(3a-d)^3+c^2(3a-d)^2+dc^2(3a-d)}{3c^3+c(3a-d)^2}$. For this system we have $F(x,y) = x\big(cx^3+(d-3a)y\big)(\alpha y^2+\beta x^3y+\gamma x^6)$, and $A(x,y) = x^6\big(cx^3+(d-3a)y\big)^2\cdot(\alpha y^2+\beta x^3y+\gamma x^6)$, where α,β and γ are real values depending on a, c and d. If we assume that $\Delta = \Delta(a,c,d) = (d+3b\lambda)^2-12b(3b\lambda^2+d\lambda+b+c)<0$, where $\lambda = \frac{c}{3a-d}$, then it can be proved that $\alpha y^2+\beta x^3y+\gamma x^6$ does not change sign. So, Theorem B can be applied to the above system under condition $\Delta<0$. We observe that this last condition is not empty because, for instance, $\Delta(a,3a-d,d) = d^2-6ad-3a^2$. In fact, when $a_n(\varphi)\equiv 0$ and $b_n(\varphi)\equiv 1$, Theorem B can be improved by using Propositions 4 and 19, and Remark 14, because in this case $\rho=0$ is a periodic orbit of multiplicity two of system (8), and then system (3) has at most one limit cycle. So the above example has at most one limit cycle.

Before ending we give, for some family of systems of type (1), a compact expression of functions F and A in complex coordinates ($z = x+iy$). Consider system

$$\begin{aligned}\dot x &= \lambda x - y + P_m(x,y),\\ \dot y &= x+\lambda y + Q_m(x,y),\end{aligned}$$

where P_m and Q_m are real homogeneous polynomials of degree m on x and y. It also writes as $\dot z = (i+\lambda)z + H_m(z,\bar z)$, where $H_m(z,\bar z)$ is a complex homogeneous polynomial of degree m on z and $\bar z$. In this setting, the functions F and A, that appear in (5), are $F = (1-m)\operatorname{Re}\big((1+\lambda i)H_m(z,\bar z)\bar z\big)$ and $A = (1-m)\operatorname{Re}\big((1+\lambda i)H_m(z,\bar z)\bar z\big)\operatorname{Im}\big(H_m(z,\bar z)\bar z\big)$, evaluated at $z = e^{i\varphi}, \bar z = e^{-i\varphi}$.

Appendix 1. Generalized Polar Coordinates. Following Lyapunov [14], we introduce the (p,q)-trigonometric functions $z(\varphi) = \mathrm{Sn}(\varphi)$ and $w(\varphi) = \mathrm{Cs}(\varphi)$, as the solutions of the Cauchy problem:

$$\begin{aligned}&\dot z = -w^{2p-1},\\ &\dot w = z^{2q-1},\\ &z(0) = \sqrt[2q]{\frac{1}{p}},\quad w(0)=0,\end{aligned}\tag{A1}$$

where p and q, are positive integers. Observe that we do not explicitly put the dependence of $\mathrm{Sn}(\varphi)$ and $\mathrm{Cs}(\varphi)$ with respect to p and q. Also note that for $p=q=1$, $\mathrm{Sn}(\varphi) = \sin(\varphi)$

and $\mathrm{Cs}(\varphi) = \cos(\varphi)$. Therefore, it is natural to say that the argument of the functions $\mathrm{Sn}(\varphi)$ and $\mathrm{Cs}(\varphi)$ is an angle.

We define $\mathrm{Tn}(\varphi)$, $\mathrm{Ctn}(\varphi)$, $\mathrm{Sec}(\varphi)$, $\mathrm{Csc}(\varphi)$, by

$$\mathrm{Tn}(\varphi) = \frac{\mathrm{Sn}^p(\varphi)}{\mathrm{Cs}^q(\varphi)}, \quad \mathrm{Ctn}(\varphi) = \frac{\mathrm{Cs}^q(\varphi)}{\mathrm{Sn}^p(\varphi)},$$
$$\mathrm{Sec}(\varphi) = \frac{1}{\mathrm{Cs}^q(\varphi)}, \quad \text{and} \quad \mathrm{Csc}(\varphi) = \frac{1}{\mathrm{Sn}^p(\varphi)}.$$

From these definitions, direct calculations give the following lemma.

LEMMA A1. *The functions defined above satisfy the following properties*

(i) $p\,\mathrm{Cs}^{2q}(\varphi) + q\,\mathrm{Sn}^{2p}(\varphi) = 1$,
(ii) $p + q\,\mathrm{Tn}^2(\varphi) = \mathrm{Sec}^2(\varphi)$,
(iii) $p\,\mathrm{Ctn}^2(\varphi) + q = \mathrm{Csc}^2(\varphi)$,
(iv) $\frac{d\,\mathrm{Sn}(\varphi)}{d\varphi} = \mathrm{Cs}^{2q-1}(\varphi)$,
(v) $\frac{d\,\mathrm{Cs}(\varphi)}{d\varphi} = -\,\mathrm{Sn}^{2p-1}(\varphi)$,
(vi) $\frac{d\,\mathrm{Tn}(\varphi)}{d\varphi} = \frac{\mathrm{Sn}^{p-1}(\varphi)}{\mathrm{Cs}^{q+1}(\varphi)}$,
(vii) $\frac{d\,\mathrm{Csc}(\varphi)}{d\varphi} = -p\frac{\mathrm{Cs}^{2q-1}(\varphi)}{\mathrm{Sn}^{p+1}(\varphi)}$,
(viii) $\frac{d\,\mathrm{Sec}(\varphi)}{d\varphi} = q\frac{\mathrm{Sn}^{2p-1}(\varphi)}{\mathrm{Cs}^{q+1}(\varphi)}$,
(ix) $\frac{d\,\mathrm{Ctn}(\varphi)}{d\varphi} = -\frac{\mathrm{Cs}^{q-1}(\varphi)}{\mathrm{Sn}^{p+1}(\varphi)}$.

LEMMA A2. $\mathrm{Sn}(\varphi)$ *and* $\mathrm{Cs}(\varphi)$ *are* T*-periodic functions (whose period is* T*) and* T *is given by*

$$T = 2p^{\frac{-1}{2q}}q^{\frac{-1}{2p}}\int_0^1 (1-t)^{\frac{(1-2p)}{2p}}t^{\frac{(1-2q)}{2q}}\,dt = 2p^{\frac{-1}{2q}}q^{\frac{-1}{2p}}\frac{\Gamma(\frac{1}{2p})\cdot\Gamma(\frac{1}{2q})}{\Gamma(\frac{1}{2p}+\frac{1}{2q})}.$$

PROOF. Since $f(z,w) = qw^{2p} + pz^{2q}$, is a first integral for system (A1), there exists $T > 0$ such that $\mathrm{Sn}(\varphi)$ and $\mathrm{Cs}(\varphi)$ are T-periodic functions.

From Lemma A1:

$$\frac{d\,\mathrm{Sn}(\varphi)}{d\varphi} = \sqrt[2q]{\left(\frac{1-q\mathrm{Sn}^{2p}(\varphi)}{p}\right)^{2q-1}},$$

so

$$\frac{\frac{d\,\mathrm{Sn}(\varphi)}{d\varphi}}{\sqrt[2q]{\left(\frac{1-q\mathrm{Sn}^{2p}(\varphi)}{p}\right)^{2q-1}}} = 1, \quad \text{or} \quad \frac{d}{d\varphi}\left(\int_0^{\mathrm{Sn}(\varphi)} \frac{\sqrt[2q]{p^{2q-1}}}{\sqrt[2q]{(1-qx^{2p})^{2q-1}}}\,dx\right) = 1,$$

hence

$$\int_0^{\mathrm{Sn}(\varphi)} \frac{\sqrt[2q]{p^{2q-1}}}{\sqrt[2q]{(1-qx^{2p})^{2q-1}}}\,dx = \varphi + k,$$

where $k=0$, because $\mathrm{Sn}(0)=0$, (from the initial conditions of the Cauchy problem (A1)). Otherwise, φ is the parameter of derivation in (A1), so the period T is given by:

$$T=4\int_0^{\mathrm{Sn}(\frac{T}{4})}\frac{\sqrt[2q]{p^{2q-1}}}{\sqrt[2q]{(1-qx^{2p})^{2q-1}}}\,dx=4\int_0^{\sqrt[2p]{\frac{1}{q}}}\frac{\sqrt[2q]{p^{2q-1}}}{\sqrt[2q]{(1-qx^{2p})^{2q-1}}}\,dx,$$

where we have used Lemma A1(i). Integrating this last expression we obtain the desired result. ∎

More properties of $\mathrm{Sn}(\varphi)$ and $\mathrm{Cs}(\varphi)$, are listed in the next lemma.

LEMMA A3. *Functions* $\mathrm{Sn}(\varphi)$ *and* $\mathrm{Cs}(\varphi)$*, satisfy the following relations:*

(i) $\mathrm{Cs}(-\varphi)=\mathrm{Cs}(\varphi)$,
(ii) $\mathrm{Sn}(-\varphi)=-\mathrm{Sn}(\varphi)$,
(iii) $\mathrm{Cs}(\frac{T}{2}-\varphi)=-\mathrm{Cs}(\varphi)$,
(iv) $\mathrm{Sn}(\frac{T}{2}-\varphi)=\mathrm{Sn}(\varphi)$,
(v) $\mathrm{Cs}(\frac{T}{2}+\varphi)=-\mathrm{Cs}(\varphi)$,
(vi) $\mathrm{Sn}(\frac{T}{2}+\varphi)=-\mathrm{Sn}(\varphi)$.

PROOF. The relations are obtained from the invariance of system (A1) under the transformations: $(z,w,t)\longrightarrow(z,-w,-t)$, $(z,w,t)\longrightarrow(-z,w,-t)$ and $(z,w,t)\longrightarrow(-z,-w,t)$. ∎

Given a point $(x,y)\neq(0,0)\in\mathbb{R}^2$, we can associate the positive real number $r=\sqrt[2pq]{px^{2q}+qy^{2p}}$, with it. Hence, $\varphi\in\mathbb{R}/[0,T]$ and r give the so-called (p,q)-polar coordinates of $\mathbb{R}^2$. In other words,

$$x=r^p\,\mathrm{Cs}(\varphi),\quad y=r^q\,\mathrm{Sn}(\varphi).$$

Using these coordinates and a new time variable, given by $\frac{dt}{ds}=r^{p+q-2pq}$, the system

$$\dot{x}=P(x,y),\quad \dot{y}=Q(x,y),$$

is transformed into

$$\begin{aligned}\dot{r}&=r^{p+q+1-4pq}[x^{2q-1}\dot{x}+y^{2p-1}\dot{y}],\\ \dot{\varphi}&=r^{-2pq}[p\dot{y}x-qy\dot{x}].\end{aligned}$$

Appendix 2. (p,q)**-Poincaré compactification.** In order to study the behaviour of the orbits in a neighbourhood of infinity we follow a generalization of the approach to the usual Poincaré compactification, [17], explained in [3].

Let $X=(P,Q)$ be a polynomial vector field of usual degree $n\geq 1$. Set $M=\{(i,j)\in\{0,1,\ldots,n\}^2\mid 0\leq i+j\leq n\}$, and

$$P(x,y)=\sum_{(i,j)\in M}a_{ij}x^iy^j,\quad Q(x,y)=\sum_{(i,j)\in M}b_{ij}x^iy^j.$$

Fixed $p, q \in \mathbb{N}$, $p \geq q$, we define the following subset of $\mathbb{Z}$, $A = \{ip + jq + 1 - p \mid (i,j) \in M\} \cup \{ip + jq + 1 - q \mid (i,j) \in M\}$. Observe that the smallest element of A is $1 - p$ and the biggest one is $np + 1 - q$. Given that $k \in \mathbb{Z}$ and $r \in \{p, q\}$, consider the subset of M, $L_k^r = \{(i,j) \in M \mid ip + jq + 1 - r = k\}$. Define the vector field:

$$X_k = \big(P_k(\varphi), Q_k(\varphi)\big) = \Big(\sum_{(i,j)\in L_k^p} a_{ij}x^iy^j, \sum_{(i,j)\in L_k^q} b_{ij}x^iy^j \Big).$$

It is clear that X_k is a (p,q)-homogeneous function of degree k. Thus $X = \sum_{k\in A} X_k$, is the decomposition of X in (p,q)-quasi-homogeneous vector fields.

The expression of $(\dot{x}, \dot{y}) = X(x,y)$ in the (p,q)-polar coordinates (see Appendix 1) is:

$$\dot{r} = r^{p+q-2pq} \sum_{k\in A} f_k(\varphi) r^{k+1},$$
$$\dot{\varphi} = r^{p+q-2pq} \sum_{k\in A} g_k(\varphi) r^k,$$

where

$$f_k(\varphi) = \mathrm{Cs}^{2q-1}(\varphi) P_k\big(\mathrm{Cs}(\varphi), \mathrm{Sn}(\varphi)\big) + \mathrm{Sn}^{2p-1} Q_k\big(\mathrm{Cs}(\varphi), \mathrm{Sn}(\varphi)\big), \quad \text{and}$$
$$g_k(\varphi) = p\,\mathrm{Cs}(\varphi) Q_k\big(\mathrm{Cs}(\varphi), \mathrm{Sn}(\varphi)\big) - q\,\mathrm{Sn}(\varphi) P_k\big(\mathrm{Cs}(\varphi), \mathrm{Sn}(\varphi)\big).$$

Putting $\rho = r^{-1}$, and replacing the old time t by a new one t_1, given by the relation $\frac{dt_1}{dt} = r^{(n+1-2q)p+1}$, we get

$$\dot{\rho} = \sum_{k\in A} f_k(\varphi) \rho^{np+2-q-k},$$
$$\dot{\varphi} = \sum_{k\in A} g_k(\varphi) \rho^{np+1-q-k}.$$

This last expression gives the (p,q)-Poincaré compactification of the vector field X. Observe that $\rho = 0$ (the equator) is invariant and the infinite critical points of X are the points with $\rho = 0$ and φ satisfying $g_{np+1-q}(\varphi) = 0$.

Finally, we would like to point out that when $p = q = 1$, this procedure gives the usual Poincaré compactification.

Appendix 3. Characterization of the polynomial differential equations given by the sum of two quasi-homogeneous vector fields. In this appendix we characterize vector fields defined by the sum of two quasi-homogeneous vector fields. This method is based on the Newton diagram, see for instance [2, Chapter 7].

Given a polynomial vector field $X = (P, Q)$, where

$$P(x,y) = \sum_{i+j=0}^{n} a_{ij}x^iy^j, \quad Q(x,y) = \sum_{i+j=0}^{n} b_{ij}x^iy^j,$$

we define its support, S_X, as the following subset of $\mathbb{R}^2$:

$$S_X = \{(i+1, j) \mid b_{ij} \neq 0\} \cup \{(i, j+1) \mid a_{ij} \neq 0\}.$$

The next lemma follows from direct computations.

LEMMA A4. *Let X be a polynomial vector field, and let p and q be natural numbers with $(p,q)=1$.*

Then, X is given by the sum of two (p,q)-quasi-homogeneous vector fields of degrees $k_1+1-(p+q)$ and $k_2+1-(p+q)$ respectively, if and only if there are two straight lines, $l_i=\{(x,y)\in\mathbb{R}^2 \mid px+qy=k_i\}$ for $i=1,2$, such that $S_X\subset l_1\cup l_2$.

Furthermore, $S_{X_i}\subset l_i$, for $i=1,2$.

Observe that, from the above lemma, in order to know if X is given by the sum of two (p,q)-quasi-homogeneous vector fields, for some p and q, it is sufficient to plot its support S_X in $\mathbb{R}^2$ and to check if it is contained in the union of two parallel straight lines.

EXAMPLE. The vector field

$$X=(y^8+x^3y^6+x^6y^4+y^{11}+x^3y^9+x^6y^7,x^8y^3+x^{11}y+x^8y^6+x^{11}y^4+x^{14}y^2+x^{17}), \tag{A2}$$

can be decomposed as the sum of two $(2,3)$-quasi-homogeneous vector fields of degrees 23 and 32. See Figure A1

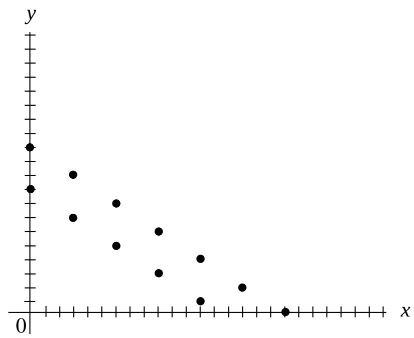

Figure A1. Support of the vector field (A2).

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