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DISCRETE OPEN AND CLOSED MAPPINGS ON GENERALIZED CONTINUA AND NEWMAN'S PROPERTY

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1. Introduction. In 1930, M. H. A. Newman proved a rather remarkable theorem which has become one of the classical theorems in topology. It has many important applications. A special case of Newman's Theorem is that a periodic homeomorphism of period $n > 1$ of a sphere S onto itself must have some orbit which is not contained in a "cap" smaller than a hemisphere. The general theorem is as follows:

THEOREM ([**15**]). *Suppose that M^n is a connected (metric) n-manifold, U is a domain in M^n, and p is an integer greater than* 1. *Then there is a positive number d such that no uniformly continuous homeomorphism h of M^n onto itself of period p moves every point of U a distance $<$ d. That is, there is $x \in U$ so that the orbit of x under h has diameter $\geqq$ d.*

Ten years later (1940), P. A. Smith generalized Newman's Theorem as follows:

THEOREM ([**18**]). *Suppose that M is a locally compact Hausdorff space in which open sets are G_δ sets, the covering dimension of M is finite, N is a bounded open set in M, q is an integer greater than* 1, *p is a prime factor of q, $\mathbf{Z}_p$ is the additive group of integers* mod *p, and M is n-regular over $\mathbf{Z}_p$. Then there is a covering A of M such that no periodic homeomorphism T of period q on M can satisfy the relation $T < A$ over N. That is, $T < A$ over N if and only if for each $x \in N$,*

$$\{x, T(x)\} \subset a_x \quad \textit{where } a_x \in A.$$

The study of light open and closed mappings grew out of the study of analytic functions by Stoilow in 1928. However, it was G. T. Whyburn who developed the theory in a systematic manner in the 1930s [**21**]. Finite-to-one and discrete open and closed mappings are special cases of these light (totally disconnected point inverses or fibers) mappings.

Important contributions were made by a number of individuals in the 1950s; notable among these are Church and Hemmingsen [**4**]. However, it was Cernavskii who first observed that finite-to-one open and closed

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mappings on manifolds are enough like orbit mappings of finite group actions to possess this special Newman's property. In 1964, Cernavskii proved the following:

THEOREM ([**3**]). *Suppose that f is a finite-to-one open and closed mapping on a connected (metric) n-manifold M^n onto a Hausdorff space Y. Then*

(1) *there is a natural number k so that for each $x \in M^n$, cardinality of $f^{-1}f(x) \leqq k$ (bounded multiplicity) and*

(2) *the elements of maximal multiplicity form a dense open set in M^n.*

Furthermore,

(3) *for each open set U of M^n, there is $\epsilon > 0$ such that if f is any finite-to-one open and closed mapping of M^n onto some metric space Y and f is not a homeomorphism, then for some $x \in U$, diameter $f^{-1}f(x) \geqq \epsilon$.*

The proofs are complicated and difficult to follow. However, the type of arguments are similar to those due to Smith [**18**].

Using Alexander-Spanier cohomology, important properties of manifolds, and a topological index, Väisälä proved, in 1966, the following theorems which are corollaries of Cernavskii's Theorem.

THEOREM ([**19**]). *Suppose that each of X and Y is a connected (metric) n-manifold without boundary. If f is a discrete open and closed mapping of X onto Y, then*

(1) *f is finite-to-one and has bounded multiplicity,*

(2) *if $B_f = \{x|x \in X$ and f is not local homeomorphism at $x\}$, then interior $B_f = \emptyset$,*

(3) *for $x \in X - f^{-1}f(B_f), f^{-1}f(x)$ has maximum multiplicity k, and*

(4) $\dim B_f \leqq n - 2$.

Note that Cernavskii's Theorem does not require that Y be a manifold.

Returning to finite actions, A. Dress [**6**] gave a reasonably short proof of Newman's Theorem in 1968. There is an elegant proof of Newman's Theorem in Bredon's book [**5**, pp. 154-158] which appeared in 1972.

We were able to generalize, in 1982, a lemma of Dress [**6**] to obtain a short straightforward proof of the following theorem (cf. Cernavskii's Theorem).

THEOREM ([**11**]). *If (M, d) is a connected (closed) n-dimensional manifold, there is $\epsilon > 0$ so that if Y is a (closed) manifold and f is a finite-to-one proper open surjective mapping of M onto Y which is not a homeomorphism, then for some $y \in Y$,*

$$\operatorname{diam} f^{-1}(y) \geqq \epsilon.$$

In 1971, Duda and Haynsworth proved (1) and (2) of Cernavskii's Theorem for boundaries of certain open subsets of n-manifolds as well as a version of (3). Their work uses rather powerful tools from cohomology theory as well as the topological index used by Väisälä.

We generalize the results of Cernavskii, in particular, to locally compact, locally connected metric spaces (generalized continua). Our proofs are entirely elementary and are in the spirit of Smith and Cernavskii. We make use of basic (and easily proved) results of Whyburn from the theory of light open mappings, some useful but almost obvious results of Väisälä, and generalizations of two lemmas due to Wilder. Basic Čech homology is used along with some special coverings defined with respect to discrete open and closed mappings between generalized continua.

We define an obvious Newman's Property and show that it is, indeed, equivalent to bounded multiplicity for finite-to-one and closed mappings on certain generalized continua.

We also prove that $f(B_f)$ neither separates nor locally separates Y where f is a finite-to-one open and closed mapping of X onto Y where X is a generalized continuum with Newman's Property hereditarily. Furthermore, we establish the surprising result that if $f^{-1}f(B_f)$ separates X, then there are exactly n components $C_1, C_2, \ldots, C_n$ of this set and f is one-to-one on each. If f is one-to-one on $f^{-1}f(B_f)$, then $n = 2$ and there is an involution g of X onto X such that $g|B_f$ is the identity and f is topologically equivalent to the orbit mapping of g.

It should be clear that our work generalizes to certain non metrizable spaces using coverings and the idea of "relation $T < A$ over N" to replace "$\epsilon > 0$ and diam $f^{-1}f(x)$" as done by Smith in his proof of a Newman's Theorem [**18**].

We would like to point out that Larry Mann and collaborators Ku have used earlier results [**11**] to prove a Newman's Theorem for pseudosubmersions. See [**9**].

The hypothesis concerning openness can be weakened using work of Montgomery [**10**].

It may be possible to modify the treatment of Newman's work as given by Bredon [**5**] so as to prove some of the theorems in Sections 5 and 6. Although Bredon is concerned with actions of groups (finite, in the case of Newman's Theorem), various lemmas are quite general. Orbit mappings of finite group actions (on compact spaces) are finite-to-one open and closed mappings. The converse is false.

Finite-to-one open and closed mappings, in some instances, share other properties with actions. However, one must exercise considerable care in the use of homology since group actions are not involved. Moreover, our work is on the whole self contained and elementary.

2. Terminology, notation, and basic results. All mappings are continuous and all topological spaces are Hausdorff. A mapping f of a space X

onto a space Y is *open* (*closed*) if the image of each open (closed) set in X is open (closed) in Y. A mapping f of X onto Y is *light* if and only if for each $x \in X$, $f^{-1}f(x)$ is totally disconnected (each maximal connected subset, i.e., each component, is a point). It is *discrete* if and only if $f^{-1}f(x)$ is discrete (i.e., has no limit point) for each $x \in X$. The *branch set* B_f of f is the set of all $x \in X$ such that f is not a local homeomorphism at x. It is easy to show that B_f is closed, hence, under a closed mapping $f(B_f)$ is closed.

The *multiplicity* $N(x, f)$ of f at $x \in X$ is the cardinality of $f^{-1}f(x)$ while

$$N(f) = \text{supremum } \{N(x, f) \mid x \in X\}.$$

Let

$$K_i(f) = \{x \mid x \in X \text{ and } N(x, f) \leqq i\}.$$

Thus, it follows that $K_i(f)$ is closed.

We shall use $\bar{A}$ and int A to denote the closure of A and the interior of A, respectively. We may use Cl A to denote $\bar{A}$ as well.

A space X is a *generalized continuum* if and only if X is locally compact, connected, and locally connected. This term is due to Whyburn [**20**].

It is well known that if f is an open and closed mapping of X onto Y, then the collection

$$G_f = \{f^{-1}f(x) \mid x \in \dot{X}\}$$

is a continuous collection (continuous decomposition of X), i.e., if $\{y_i\} \to y$ in Y, then $\{f^{-1}(y_i)\}$ converges to $f^{-1}(y)$ in X.

A subset U of a space X is a *domain* if and only if U is open and connected. A mapping f of X onto Y is *proper* if and only if for each compact subset A of Y, $f^{-1}(A)$ is compact. If each of X and Y is locally compact, f is closed, and $f^{-1}(y)$ is compact for each $y \in Y$, then f is proper. In particular, if f is closed and finite-to-one, then f is proper.

A simple arc A is the homeomorphic image of the closed interval [0, 1] into a space Y.

A subset A of a space X is said to *separate X locally* at $x \in X$ if and only if for each open set U containing x, there is an open set V containing x such that $U \supset V$ and $V - A$ is not connected.

We shall make use of some lemmas and theorems which are easy to prove. For completeness, we include the proofs.

LEMMA 2.1. [**18**] *Suppose that f is a light open mapping of X onto Y such that $N(f) = k$ (a natural number). Then $N(x, f) < k$ for each $x \in B_f$.*

Proof. If $N(x_1, f) = k$ for some $x_1 \in B_f$, then let

$$f^{-1}f(x_1) = \{x_1, x_2, \dots, x_k\}.$$

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Choose k pairwise disjoint open sets U_i, $i = 1, 2, \ldots, k$, such that $x_i \in U_i$. Then the set

$$V = U_1 \cap f^{-1}\left[\bigcap_{i=1}^{k} f(U_i)\right]$$

is an open set such that $f|V$ is a homeomorphism and consequently f is a local homeomorphism at x_1 contrary to the fact that $x_1 \in B_f$.

LEMMA 2.2. [**18**] *Suppose that X is locally compact and that f is a discrete open mapping of X onto Y. Then* int $B_f = \emptyset$.

Proof. Suppose that int $B_f \neq \emptyset$. Then there is an open set U such that $B_f \supset U$ and $\bar{U}$ is compact. The restriction $g = f|U$ is an open mapping for which $N(x, g) < \infty$ for each $x \in U$. Thus,

$$U = \bigcup_{i=1}^{\infty} K_i(g).$$

By the Baire Theorem [**17**],

$$\text{int } K_i(g) = V \neq \emptyset \quad \text{for some } i.$$

Since $N(g|V) \leqq i$, Lemma 1 implies that there is a point $x \in V$ at which $g|V$, and hence f, is a local homeomorphism. However, $x \in U \subset B_f$ which is a contradiction.

LEMMA 2.3. [**19**] *Suppose that X is locally compact and locally connected. Furthermore, f is a light mapping of X onto Y. Then for each point $x \in X$ and each open set U in X, $x \in U$, there is a domain V in X, $x \in V \subset U$ such that $f|V = g$ is a closed mapping of V onto $f(V)$.*

Proof. Let $x \in X$ and $x \in U$, an open set in X. Since f is light, there is an open set W, $x \in W$, such that $U \supset W$, $\bar{W}$ is compact, and

$$\text{Bd } W \cap f^{-1}f(x) = \emptyset.$$

Choose an open set D of $f(X)$ such that $D \cap f(\text{Bd } W) = \emptyset$ and let V be the component of $f^{-1}(D)$ which contains x. Then $V \subset W$ is a domain and $g = f|V$ is a closed mapping of V onto $f(V)$.

LEMMA 2.4. [**19**] *Suppose that X is a locally connected space and that A is a closed subset of X such that* int $A = \emptyset$ *and $X - A$ is not connected. If F is the closure of the set of all points at which A separates X locally, then $X - F$ is not connected.*

Proof. Since $X - A$ is not connected, $X - A = U_1 \cup U_2$ — two disjoint nonempty open sets. Let

$$V_i = (\text{int } \bar{U}_i) - F.$$

It follows that $X - F = V_1 \cup V_2$ — two disjoint open sets such that $V_i \supset U_i$.

THEOREM 2.5. *Suppose that f is a proper light open mapping of X onto Y. If xy is a simple arc in Y and $p \in f^{-1}(x)$, then there is a simple arc pq in X such that $f|pq$ is a homeomorphism of pq onto xy. In fact, for each path P in Y (image of* $[0, 1]$ *under a continuous mapping θ) and $p \in f^{-1}\theta(0)$, there is a continuous mapping $\hat{\theta}$ of* $[0, 1]$ *into X such that*

$$\hat{\theta}(0) = p \quad \text{and} \quad (f|\hat{\theta}([0, 1])) \circ \hat{\theta} = \theta.$$

Proof. Since f is proper, $f^{-1}(xy) = A$ is compact. By a theorem of Whyburn ([**21**], 2.1, p. 186) there exists pq satisfying the conclusion of the theorem. Similarly, if $\theta[0, 1] = P$, then $f^{-1}(P) = A$ is compact and $f|A$ is light and open [**21**, 7.2, p. 147]. By a theorem of Floyd [**8**, 2, p. 574], there exists $\hat{\theta}$ satisfying the conclusion of the theorem.

LEMMA 2.6. [cf **19**, Lemma 5.3]. *Suppose that X is a generalized continuum and that f is a finite-to-one open and closed mapping of X onto Y. Furthermore, A is a closed subset of X such that $X - A$ is not connected and $N(x,f) = 1$ for each $x \in A$. Then each component C of $X - A$ is mapped by f onto a component $f(C)$ of $Y - f(A)$ and $g = f|C$ is a closed (and open, of course) mapping of C onto $f(C)$.*

Proof. Let E be a subset of C closed relative to C, that is, $E = \bar{E} \cap C$. Now, $f(C)$ is contained in a component V of $Y - f(A)$. Thus, $f(\bar{E}) \cap V$ is closed relative to V and contains $f(E)$. Hence, g is closed. Finally, $f(C) = V$ since g is open.

The following theorems are easy generalizations of theorems due to Whyburn, [**21**, pp. 147, 131, and 189, respectively].

THEOREM A. *Suppose that f is a proper open mapping of X onto Y where X is a generalized continuum. If R is a domain in Y, then $f^{-1}(R)$ has at most a finite number of components each of which maps onto R under f.*

THEOREM B. *Suppose that f is a proper light mapping of X onto Y where each of (X, d) and (Y, p) is a locally compact metric space. If $\epsilon > 0$, there is $\delta > 0$ such that for each closed and connected subset C of Y with diameter less than δ, each component of $f^{-1}(C)$ has diameter less than ϵ.*

THEOREM C. *Suppose that f is a proper light open mapping of X onto Y where X is a generalized continuum. If K is a generalized continuum in Y whose interior is dense in K, then $f^{-1}(K)$ is locally connected and each component of $f^{-1}(K)$ is a generalized continuum.*

It is easy to obtain proofs of these theorems by first observing that f is proper and Y is locally compact. Now, adapt the proofs given by Whyburn [**21**] to the situation here.

3. Finite-to-one open and closed mappings on generalized continua having Newman's Property. We shall prove, in Section 6, that certain n-dimensional generalized continua have Newman's Property hereditarily. Indeed, we give generalizations, in Section 7, to certain light open mappings on a class of generalized continua. Clearly, n-manifolds (connected and metric) as well as Peano continua are special cases of generalized continua.

We first give some easy and elementary proofs of theorems for generalized continua first proved for n-manifolds by Cernavskii [**3**], later proved by Väisälä [**19**, **20**], Duda and Haynsworth [**7**], and Church [**4**]. Our methods resemble those of Cernavskii but are far simpler using the easily proved results of Section 2 and some of the theorems proved below. We do not need to use manifold theory and the important Alexander-Spanier Cohomology Theory used in [**19**] and elsewhere. First, we state a key property which will be seen to be equivalent to the bounded multiplicity property for finite-to-one open and closed mappings on n-manifolds (and more general spaces).

Definition. A metric space (X, d) is said to have *Newman's Property* if and only if for each open set $U \subset X$, there is an $\epsilon > 0$ such that if f is an open and closed mapping defined on X (from X onto some space Y_f) and $1 < N(f) < \infty$, then for some $x \in U$, the diameter of $f^{-1}f(x) \geqq \epsilon$.

We can, of course, state Newman's Property for more general (non metrizable) spaces X using coverings.

A space X is said to have *Newman's Property hereditarily* if and only if each domain U in X has Newman's Property.

Since an n-manifold M is locally an n-manifold, it follows that M has Newman's Property hereditarily.

Definition. A mapping h of X onto X is said to be an *involution* if and only if h is a homeomorphism of period 2, but not the identity. The orbit mapping ϕ of h is a two-to-one open and closed mapping. We shall say that a finite-to-one open and closed mapping f is *simple* if and only if $N(f) = 2$.

It is well known that if U and V are disjoint open subsets of an n-manifold M such that

(1) $\text{Bd } U = \text{Bd } V$ and (2) $M \neq \bar{U} \cup \bar{V}$,

then there is no homeomorphism h of $\bar{U}$ onto $\bar{V}$ such that $h(x) = x$ for $x \in \text{Bd } U$. See [**19**, 5.2, p. 6]. Suppose that there is such a homeomorphism.

Let f be the quotient mapping consisting of the singletons on $M - (U \cup V)$ and the sets (orbits) $\{x, h(x)\}$ for $x \in U$. Thus, f is a finite-to-one open and closed mapping which is one-to-one on the open set

$M - (\bar{U} \cup \bar{V})$. This implies that M does not have Newman's Property with respect to simple mappings. Thus, this property used in [**19**, 5.2] is only a special use of Newman's Property for n-manifolds. Indeed, it is equivalent to this special case.

Notation. Suppose that f is a discrete open mapping of X onto Y. We use B to denote $f^{-1} f(B_f)$ where

$$B_f = \{x | f \text{ is not a local homeomorphism at } x\}.$$

Also,

$$A = \mathrm{Cl}\{x | x \in B \text{ and } B \text{ separates } X \text{ locally at } x\}.$$

We shall prove theorems in the remainder of this section which are of interest independently of their usefulness in establishing our main results.

THEOREM 3.1. *Suppose that X is a generalized continuum and that f is a finite-to-one open and closed mapping of X onto Y such that $X - B$ is connected. Then $N(x, f)$ is constant on $X - B$.*

Proof. Suppose that for some p and q in $X - B$, $N(p, f) \neq N(q, f)$. There is a simple arc pq from p to q in $X - B$ [cf. **21**].

Now, $f(pq)$ is a Peano continuum P in $f(X - B)$. There is a simple arc R from $f(p)$ to $f(q)$ in P. By using Theorem 2.5, it is seen that $X - B \supset f^{-1}(R)$ and $f^{-1}(R)$ is the union of a finite number of simple arcs x_iy_i such that

(1) $f(x_i) = f(p)$,

(2) $f(y_i) = f(q)$, and

(3) $f|x_iy_i$ is a homeomorphism of x_iy_i onto R.

Since $N(p, f) \neq N(q, f)$, some two of the arcs x_iy_i must have at least one point in common. Thus, some common point z is a point at which f is not a local homeomorphism. Hence $z \in B_f$. This is a contradiction. (See also a similar result in [**21**, 6.1, p. 199].)

THEOREM 3.2. *Suppose that X is a generalized continuum which has Newman's Property hereditarily and that f is a finite-to-one open and closed mapping of X onto Y. If $X - B$ is not connected, then f is one-to-one on each component C of $X - B$.*

Proof. Suppose that f is not one-to-one on some component C of $X - B$. Let

$$Q = \{x | x \in \bar{C} - C \text{ and } x \in \mathrm{Cl}(X - \bar{C})\}.$$

Then Q is closed and separates C from $X - \bar{C}$ in X. That is, there is a separation $X - Q = S \cup T$ where $S \supset C$ and $T \supset X - \bar{C}$.

Let $B_i = Q \cap K_i(f)$ for each i.

Since

$$Q = \bigcup_{i=1}^{\infty} B_i,$$

it follows from the Baire Theorem that for some k, the interior U of B_k with respect to Q is non empty. Let W be an open set in X so that $W \cap Q = U$. Now,

$$K_k(f) \supset U \quad \text{and} \quad N(x, f|U) \leqq k \text{ for each } x \in U.$$

Hence, there is $x_1 \in U$ so that

$$k \geqq N(x_1, f|W) = n \geqq N(x, f|W) \quad \text{for each } x \in U.$$

Let $x_2, x_3, \ldots, x_n$ be the other points (distinct from each other and from x_1) of $W \cap f^{-1}f(x_1)$, if any; otherwise, let $V = W$. Choose pairwise disjoint open sets W_i containing x_i for each $i = 1, 2, \ldots, n$ such that $W \supset W_i$. Let

$$V = W_1 \cap f^{-1}\left(\bigcap_{i=1}^{n} f(W_i)\right).$$

Then $N(x, f|V) = 1$ for each $x \in V \cap Q$. (We have used an idea from [**19**, proof of 5.4, p. 7].)

Next, choose a domain D containing x_1 such that $f|D$ is closed and $V \supset D$. Thus, $f|D \cap Q$ is one-to-one. Since $x_1 \in Q$ and C is a component of $X - B$,

$$D \cap (X - \bar{C}) \neq \emptyset.$$

Let $g = f|C \cup D$. Clearly, $f|C$ is closed since $f^{-1}f(B_f) = B$ is closed. Thus, g is a closed and open mapping on the domain $C \cup D$. Let m denote the quotient mapping defined with singletons on $D - C$ and the various sets $g^{-1}g(x)$ for $x \in C$. Thus, m, is an open and closed mapping of bounded multiplicity, i.e., $N(m) = N(f|C)$ which is constant by Theorem 2.1 and greater than 1 by assumption. Since m is one-to-one on the non empty open set $D - \bar{C}$, it follows that X does not have Newman's Property hereditarily. This is a contradiction. The theorem is proved.

Remark. Instead of assuming that X has Newman's Property hereditarily, assume that X has Newman's Property and that $f|\text{Bd } C$ is one-to-one where C is a component of $X - B$. Assuming that f is not one-to-one on C, let m be the quotient mapping defined by the singletons on $X - \bar{C}$ and the sets $f^{-1}f(x) \cap C$. Consequently, if $X - B$ is not connected, then m is one-to-one on the non empty set $X - \bar{C}$. This contradicts the assumption that X has Newman's Property. It follows that $f|C$ is one-to-one.

THEOREM 3.3. *Suppose that X is a generalized continuum and that f is a finite-to-one open and closed mapping of X onto Y. Furthermore, C is a component of $X - B$ (where $B = f^{-1}f(B_f)$) such that $f|C$ is one-to-one. Then C is contained in a component C' of $X - A$ (where $A = \mathrm{Cl}\,\{x|x \in B$ and B separates X locally at $x\}$) and f is one-to-one on C'.*

Proof. If $B = \emptyset = A$, then $C = X$ and f is one-to-one on X. If $B \neq \emptyset$, let

$$F = \{x|x \in \bar{C} \cap B \text{ and } x \notin \mathrm{Cl}(X - \bar{C})\}.$$

For each $x \in F$, there is a domain D_x containing x and no point of $\mathrm{Cl}(X - \bar{C})$. Clearly, $C' = C \cup F$ is connected and $F \cap A = \emptyset$. However, $A \supset B \cap (\bar{C} - F)$. It follows that C' is a component of $X - A$.

Suppose that f is not one-to-one on C'. Then there are two points p and q in C' such that $f(p) = f(q)$. There are disjoint domains D_p and D_q containing p and q, respectively, such that $f(D_p) = f(D_q)$. Consequently, there are points $x \in D_p \cap C$ and $y \in D_q \cap C$ so that $f(x) = f(y)$ which contradicts Theorem 3.2. Thus, f is one-to-one on C'.

THEOREM 3.4. *Suppose that X is a generalized continuum with Newman's Property hereditarily and that f is a finite-to-one open and closed mapping of X onto Y. Furthermore, Q is a closed subset of B such that*

(1) *C_1 and C_2 are two components of $X - Q$,*
(2) *$f(C_1) = f(C_2)$,*
(3) *$f|C_i$ is one-to-one,*
(4) *$f|C_i$ is closed (and, open, of course),*
(5) *D is a domain such that $f|D$ is closed,*
(6) *$N(x, f|D) = 1$ for each $x \in D \cap Q$, and*
(7) *$D \cap C_i \neq \emptyset$ for $i = 1, 2$.*

Then $D - (\bar{C}_1 \cup \bar{C}_2) = \emptyset$.

Proof. Suppose that $D - (\bar{C}_1 \cup \bar{C}_2) \neq \emptyset$. Let m be the quotient mapping defined by the singletons on $D - (C_1 \cup C_2)$ and the sets $f^{-1}f(x) \cap (C_1 \cup C_2)$ for each $x \in C_1$. It follows that the domain $D \cup C_1 \cup C_2$ does not have Newman's Property since m is one-to-one on $D - (\bar{C}_1 \cup \bar{C}_2)$. This is a contradiction. The theorem is proved.

THEOREM 3.5. *Suppose that X is a generalized continuum with Newman's Property hereditarily and that f is a finite-to-one open and closed mapping from X onto Y. If $X - B$ is not connected, then there is $p \in B_f$ and at least two components C_1 and C_2 of $X - B$ such that*

(1) *$p \in \bar{C}_1 \cap \bar{C}_2$,*
(2) *$f(C_1) = f(C_2)$,*
(3) *$f|C_i$ is one-to-one, and*
(4) *$f|B$ is a local homeomorphism m at p.*

Indeed, there exist components C_1' and C_2' of $X - A$ containing C_1 and C_2, respectively, such that $f|C_i'$ is one-to-one. (*Recall that* $B = f^{-1}f(B_f)$ *and that* $A = \mathrm{Cl}\,\{x|x \in B$ *and B separates X locally at* $x\}$.)

Proof. Let $B_i = B \cap K_i(f)$ for each i. As in the proof of Theorem 3.2, there is some k such that the interior U of B_k relative to B is non empty. As shown there, there is $p = x_1 \in U$ and an open set V in X containing p such that

$$U \supset V \cap B \quad \text{and} \quad N(x, f|V) = 1 \text{ for each } x \in V \cap B.$$

Since $U \supset f^{-1}f(p)$, there is no loss of generality in assuming that $p \in B_f$ and f is not a local homeomorphism at p.

There is a sequence $\{y_i\} \to f(p)$ such that for each i,

(1) $y_i \notin f(B)$ and

(2) there are points p_i and q_i in $f^{-1}(y_i)$ where $p_i \neq q_i$, $\{p_i\} \to p$, and $\{q_i\} \to p$.

For each i, there are components W_i and Z_i of $X - B$ such that $p_i \in W_i$, $q_i \in Z_i$, $f(W_i) = f(Z_i)$, and $W_i \cap Z_i = \emptyset$.

Suppose that for infinitely many i, $p \notin \bar{W}_i$. By Lemma 2.3, there is a domain D containing p so that $f|D$ is closed. Also, assume that $V \supset D$. Hence, $N(x, f|D) = 1$ for each $x \in D \cap B$. For some t, $D \cap W_t \neq \emptyset$, $D \cap Z_t \neq \emptyset$, and $p \notin \overline{W_t}$. Now, define a quotient mapping m on $D \cup W_t \cup Z_t$ with singletons on $D - (W_t \cup Z_t)$ and the sets $f^{-1}f(x) \cap (W_t \cup Z_t)$ for each $x \in W_t$. Now, $D \cup W_t \cup Z_t$ has Newman's Property. By Theorem 3.4,

$$D - (\bar{W}_t \cup \bar{Z}_t) = \emptyset.$$

This is a contradiction. Thus, there is N so that if $i > N$, $p \in \overline{W_i}$ and $p \in \overline{Z_i}$.

If there is some i and j, $i > j > N$, so that $f(W_i) \neq f(W_j)$, then choose a domain D containing p so that $f|D$ is closed and $N(x, f|D) = 1$ for each $x \in D \cap B$.

Now, $f(W_i) = f(Z_i)$. Also,

$$D - (\bar{W}_i \cup \bar{Z}_i) \neq \emptyset$$

since $p \in D$, $p \in \overline{W_j}$, and $W_j \cap (\bar{W}_i \cup \bar{Z}_i) = \emptyset$.

Again, we can define a quotient mapping on $D \cup W_i \cup Z_i$, similar to the above, and obtain a contradiction to Theorem 3.4. Thus, $f(W_i) = f(W_j)$ for each $i, j > N$ and, similarly, $f(Z_i) = f(Z_j)$ for $i, j > N$. Consequently, we have proved that there is M so that for each $i > N$, $p_j \in W_i$ and $q_j \in Z_i$ for each $j > M$.

The final part of the theorem follows from an application of Theorem 3.3.

Definition. A pair (X, Y) where each of X and Y is a space is said to have *The Invariance Of Domain Property* if and only if for any pair (U, V) such that

(1) $X \supset U$,

(2) $Y \supset V$, and

(3) there is a homeomorphism h of U onto V,

then U is open implies that V is open and conversely.

This is a well known property for pairs (X, Y) where each of X and Y is an n-manifold (without boundary).

Question. What pairs (X, Y) have the Invariance Of Domain Property?

THEOREM 3.6. *Suppose that X is a generalized continuum with Newman's Property hereditarily and that f is a finite-to-one open and closed mapping of X onto Y. Furthermore, the pair (X, Y) has the Invariance Of Domain Property. Then B_f does not locally separate X.*

Proof. Let

$$F = \text{Cl } \{x|x \in B_f \text{ and } B_f \text{ separates } X \text{ locally } X \text{ at } x\}.$$

It follows from Lemma 2.4 that if B_f separates X locally at x, then F does.

Let $B_i = F \cap K_i(f)$ for each i. Thus, by the Baire Theorem, there is some k such that the interior U of B_k relative to F is non empty. Let W be open in X such that $W \cap F = U$. Since $K_k(f) \supset U$,

$$N(x, f|W) \leqq k \quad \text{for each } x \in U.$$

Hence, there is $x_1 \in U$ so that

$$k \geqq N(x_1, f|W) = n \geqq N(x, f|W) \quad \text{for each } x \in U.$$

As shown before, there is an open set V in X so that

$$U \supset V \cap F \quad \text{and } N(x, f|V) = 1 \quad \text{for each } x \in V \cap F.$$

Since $x_1 \in F$, there is $q \in B_f \cap V$ at which B_f (and hence F) separates X locally at q. Choose a domain D containing q such that $f|D$ is closed, $V \supset D$, and $D - F$ is not connected. Since

$$\text{int } B_f = \emptyset = \text{int } B,$$

$D - B$ is not connected.

Now, D is a generalized continuum and $g = f|D$ is a finite-to-one open and closed mapping of D onto $f(D)$. Furthermore, D has Newman's Property. Now,

$$g^{-1}g(B_g) = B \cap D.$$

By Theorem 3.5, there exists $p \in B_g = B_f \cap D$, components C_1 and C_2 of

$D - B$ such that

(1) $g(C_1) = g(C_2)$,
(2) $p \in \overline{C_1} \cap \overline{C_2}$,
(3) $g|C_i$ is one-to-one, and
(4) $g|B$ a local homeomorphism at p.

There is a domain D_p containing p such that $D \supset D_p$, $g|D_p$ is closed, and

$$N(x, g|D_p) = 1 \quad \text{for each } x \in D_p \cap B.$$

There are components Z_1 and Z_2 of $D_p - B$ such that $C_i \supset Z_i$. By Theorem 3.4,

$$D_p - (\overline{Z_1} \cup \overline{Z_2}) = \emptyset.$$

Since g is one-to-one on $D_p \cap B$, it follows that $\bar{Z}_1 \cap D_p$ is homeomorphic to $g(D_p)$ which is open in Y. By the Invariance Of Domain Property, $\overline{Z_1} \cap D_p$ is open in x and in D_p. Since $\overline{Z_1} \cap D_p$ is also closed in D_p, it follows that $\overline{Z_1} \cap D_p = D_p$. This contradicts the fact that $Z_2 \subset D_p$ and $Z_1 \cap Z_2 = \emptyset$. We are forced to conclude that B_f does not separate X locally at any point.

COROLLARY 3.61. *If each X and Y is a connected n-manifold without boundary, X has Newman's Property hereditarily* (*proved in Section* 6), *and f is a finite-to-one open and closed mapping of X onto Y, then B_f neither separates X nor separates X locally at any point. Indeed,* $\dim B_f \leqq n - 2$. [*cf.* **19**, 5.4, p. 7].

Proof. If $\dim B_f \geqq n - 1$, then B_f separates X locally at some point.

Remark. We used Newman's Property to show, in the proof of Theorem 3.6, that $D_p - B$ consists of exactly the two components Z_1 and Z_2. A proof for a similar result in [**19**] where each of X and Y is an n-manifold uses the following result: Suppose that X is an n-manifold containing disjoint domains U_1 and U_2 such that $\text{Bd } U_1 = \text{Bd } U_2$ and $\overline{U_1} \cup \overline{U_2} \neq X$. Then there is no homeomorphism h of $\overline{U_1}$ onto $\overline{U_2}$ which keeps the common boundary, $\text{Bd } U_1$, fixed.

It is remarkable that this result is equivalent to Newman's Property with respect to simple mappings.

In [**3**, Theorem1], the proof is given for n-manifolds. However, it holds true for certain generalized continua as follows:

THEOREM 3.7. *Suppose that X is a generalized continuum which has Newman's Property hereditarily and f is a finite-to-one open and closed mapping of X onto Y. Then $N(f) < \infty$.*

Proof. Suppose that $N(f) = \infty$. Let

$$E_i = E_i(f) = \text{Cl}(\text{int } K_i(f) - K_{i-1}(f))$$

for each i where $K_0 = \emptyset$. For some i, $E_i \neq \emptyset$ by the Baire Theorem. We shall prove that

(1) $K_i(f) - K_{i-1}(f) = H_i(f)$ is dense in Bd E_i.

Choose $x \in$ Bd E_i such that $N(x,f) \geqq N(y,f)$ for each $y \in$ Bd E_i. Of course, $i \geqq N(x,f)$. Suppose that $i > N(x,f)$. Then there is $z \in f^{-1}f(x)$ such that $f|E_i$ is not a local homeomorphism at z. However, $f|$Bd E_i is a local homeomorphism at z by choice of x. There is a domain D_z containing z such that

$$N(p,f|D_z) = 1 \quad \text{for each } p \in D_z \cap \text{Bd } E_i,$$

$f|D_z$ is closed, and

$$D_z - \text{Bd } E_i = V_1 \cup V_2,$$

two separated sets where $E_i \supset V_1$ and $E_i \cap V_2 = \emptyset$ (Bd E_i separates int E_i from $X - E_i$).

Let m be the quotient mapping consisting of the singletons of $\bar{V}_2$ and the sets $f^{-1}f(x) \cap V_1$. Thus, m is an open and closed mapping on D_z which is one-to-one on the open set V_2. This contradicts the hypothesis that X has Newman's Property hereditarily. Thus, $N(x,f) = i$ and $H_i(f)$ is dense in Bd E_i.

Choose $x \in$ Bd E_i such that $N(x,f) = i$. There is $z \in f^{-1}f(x) \subset$ Bd E_i such that f is not a local homeomorphism at z but $f|$Bd E_i is a local homeomorphism at z. Let D_z be a domain such that $f|D_z$ is closed and

$$N(p,f|D_z) = 1 \quad \text{for each } p \in D_z \cap \text{Bd } E_i.$$

We shall prove that

(2) $\{N(q,f)\,|q \in D_z - E_i\}$ is unbounded.

If this is not the case, then there is k so that

$$N(q,f) < k \quad \text{for each } q \in D_z - E_i.$$

Define a quotient mapping m consisting of singletons on $E_i \cap D_z$ and the sets $f^{-1}f(x) \cap D_z - E_i$. Again, m is an open and closed mapping on D_z with $N(x, m) < k$. Thus, m being one-to-one on int E_i contradicts the hypothesis that X has Newman's Property hereditarily. The statement (2) is true.

Let

$$M = X - \bigcup_{i=1}^{\infty} \text{int } K_i(f).$$

Thus, if $x \in M$, then $M \supset f^{-1}f(x)$. Also, if $x \in M$ and U_x is an open set containing x, then $N(p,f|U_x)$ is unbounded. Note that

$$M \supset \bigcup_{i=1}^{\infty} \operatorname{Bd} E_i.$$

Let $M_i = M \cap K_i(f)$. By the Baire Theorem, there is k so that the interior of M_k relative to M is a non empty set U. If $x \in U$, then

$$U \supset f^{-1}f(x) \quad \text{and} \quad N(x, f) \leqq k.$$

There is an open set W in X such that

$$W \cap M = U \quad \text{and} \quad f^{-1}f(W) = W.$$

Thus, $W \cap \operatorname{Bd} E_i = \emptyset$ for $i > k$. If $W \cap \operatorname{int} E_i \neq \emptyset$ for some $i > k$, then either

(a) $\operatorname{int} E_i \supset W$ or

(b) $W \cap \operatorname{Bd} E_i \neq \emptyset$.

In case (a) $N(w, f) \leqq i$ for $w \in W$. Case (b) can not be true because

$$W \cap M = U \subset K_k(f).$$

Thus, it follows that, in any case, $\{N(p, f) | p \in W\}$ is bounded. This involves a contradiction since by statements (1) and (2), each open set containing a point of $\operatorname{Bd} E_k$ (such as W) has the property that $\{N(p, f) | p \in W\}$ is unbounded. Consequently, the theorem is true and $N(f) < \infty$.

THEOREM 3.8. *Suppose that X is a generalized continuum with Newman's Property hereditarily and that f is a finite-to-one open and closed mapping of X onto Y. If $X - B$ is not connected, then there are $n > 1$ components of $X - B$ each of which maps homeomorphically onto $Y - f(B_f)$.*

Proof. By Theorem 3.5, there is a point $p \in B_f$ and two components C_1 and C_2 of $X - B$ such that $p \in \overline{C_1} \cap \overline{C_2}$, $f(C_1) = f(C_2)$, $f|C_i$ is one-to-one, and $f|B$ is a local homeomorphism at p.

Let $C_1, C_2, \ldots, C_n$ be the components of $f^{-1}f(C_1)$. Thus $f(C_i) = f(C_1)$ and $f|C_i$ is one-to-one. Indeed, $f|\overline{C_i}$ is one-to-one. Let

$$Z = \{q | q \in \operatorname{Bd} C_i \text{ for some } i\}.$$

Now, Z is closed and $p \in Z$. Consider

$$Q = Z \cap \operatorname{Cl}(X - \bigcup_{i=1}^{n} \overline{C_i}).$$

Either

(1) $Q = \emptyset$ or

(2) $Q \neq \emptyset$.

We shall show that in case (1),

$$X = \bigcup_{i=1}^{n} \overline{C_i}.$$

If

$$X \neq \bigcup_{i=1}^{n} \overline{C_i},$$

then there is a sequence $\{q_i) \to q$ such that

$$q_i \notin \bigcup_{i=1}^{n} \overline{C_i} \quad \text{and} \quad q \in \text{Bd } C_i \quad \text{for some } i.$$

Thus, $Q \neq \emptyset$, contrary to this case.

Let $B_i = Q \cap K_i(f)$ for each i. By the Baire Theorem, there is some k so that the interior U of B_k relative to Q is non empty. By an argument similar to one used before, there is $z \in U$ and an open set V in X containing z such that

$$U \supset V \cap Q \quad \text{and} \quad N(x, f|V) = 1 \quad \text{for } x \in V \cap Q.$$

There is a domain R containing z such that $f|R$ is closed, $N(x, f|R) = 1$ for $x \in R \cap Q$, and $R \cap \overline{C_i} = \emptyset$ if $z \notin \text{Bd } C_i$. Define a quotient mapping m on R with singletons on $\overline{C_t} \cap R$ for each t so that $C_t \cap R \neq \emptyset$. (Recall that $f|\overline{C_t}$ is one-to-one) and the sets

$$f^{-1}f(x) \cap R \quad \text{for } x \in R - \bigcup_{i=1}^{n} \overline{C_i}.$$

If $x \in \text{Bd } C_i \cap R$ for some i, then either

$$x \in Q \quad \text{or} \quad x \notin \text{Cl}(X - \bigcup_{i=1}^{n} \overline{C_i}).$$

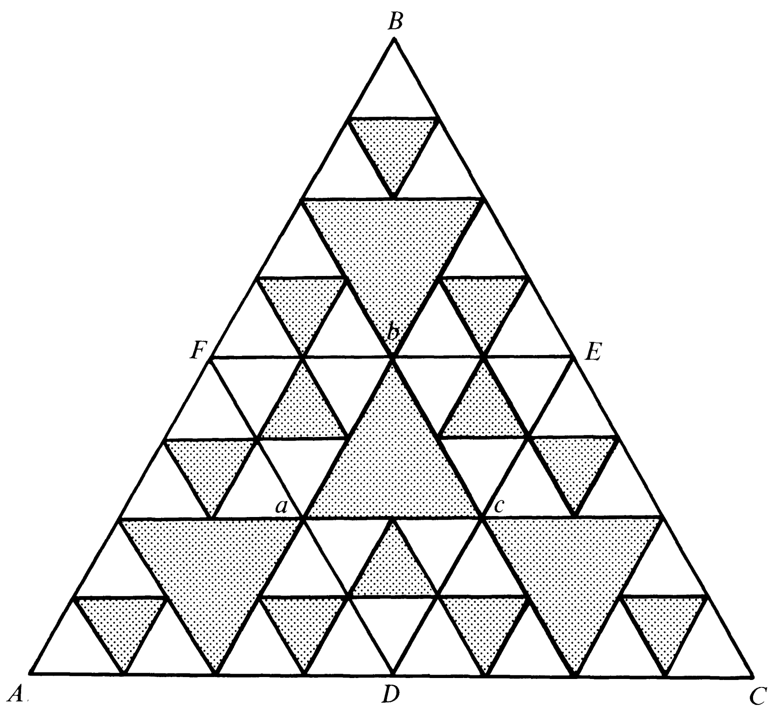
Consequently, m is an open and closed mapping on R. By Theorem 3.7, $N(m) < \infty$. Thus, X can not have Newman's Property hereditarily since m is one-to-one on the non empty open set $R \cap (\bigcup_{i=1}^{n} C_i)$, a contradiction. Thus,

$$X = \bigcup_{i=1}^{n} \overline{C_i}.$$

It follows that f maps each C_i homeomorphically onto $Y - f(B_f)$ and the theorem is proved.

Example. There is an example of a Peano continuum X and an open and closed mapping f of X onto Y such that $N(f) = 4$, $X - B$ has infinitely many components, and X has Newman's Property. However, X does not have Newman's Property hereditarily. Thus, Theorem 3.8 is false without this hereditary property.

We construct the Peano continuum X by starting with an equilateral triangle ABC containing another equilateral triangle DEF where D, E, and F are the midpoints of the sides AC, CB, and AB, respectively. In each equilateral triangle ADF, DFE, DEC, and EFB, we remove the interior of an equilateral triangle such as abc having vertices as midpoints of the sides of the larger triangle. Continue removing the interiors of equilateral triangles on the remaining triangles in this manner. We obtain a 1-dimensional Peano continuum X which has Newman's Property (see Section 6). Fold triangle ADF onto triangle DFE, BFE onto DFE, and DEC onto DFE to obtain an open and closed mapping f of X onto Y, the part of X contained in triangle DFE. Now, $N(f) = 4$ and B_f consists of the straight line intervals DE, EF, and DF. Both B_f and $B = f^{-1}f(B_f)$ separates X into infinitely many components none of which maps onto $Y - f(B_f)$.



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Example. There is a finite-to-one open and closed mapping f of a simple closed curve X (1-manifold) onto the closed interval $[0, 1] = Y$ such that $N(f) = 4$, $X - B_f$ has 4 components, and f is one-to-one on each component. This is in contrast to Theorem 3.6. In this example, the pair (X, Y) does not have the Invariance Of Domain Property. Thus, without this property, B may separate X but in a very special way.

Let X be the space consisting of

$$X_1 = \{ (x, y) \,|\, y - x = 1, -1 \leqq x \leqq 0\},$$

$$X_2 = \{ (x, y) \,|\, y - x = -1, 0 \leqq x \leqq 1\},$$

$$X_3 = \{ (x, y) \,|\, x + y = 1, 0 \leqq x \leqq 1\},$$

and

$$X_4 = \{ (x, y) \,|\, x + y = -1, -1 \leqq x \leqq 0\}$$

with the induced topology from the plane.

Let f map each of X_3 linearly onto $[0, 1]$ taking $(0, 0)$ to 1 and $(0, 1)$ to 0, X_1 linearly onto $[0, 1]$ taking $(0, 1)$ to 0 and $(-1, 0)$ to 1, X_4 linearly onto $[0, 1]$ taking $(-1, 0)$ to 1 and $(0, -1)$ to 0, and X_2 linearly onto $[0, 1]$ taking $(0, -1)$ to 0, and $(0, 1)$ to 1. Clearly, X has Newman's Property hereditarily (Section 6), but (X, Y) does not have the Invariance Of Domain Property. Now,

$$B = B_f = \{ (1, 0), (0, 1), (-1, 0), (0, -1) \}.$$

The remainder of the claim should be obvious.

One can take $X \times [0, 1]$ and $Y \times [0, 1]$ to construct a similar example in two dimensions.

COROLLARY 3.81. *Suppose that X is a generalized continuum which has Newman's Property hereditarily. If f is an open and closed mapping of X onto Y such that $N(f) = 2$ and $X - B$ is not connected, then $X - B$ consists of exactly two components C_1 and C_2. Furthermore, there is an involution h of X onto X such that $h(x) = x$ if and only if $x \in B_f$ and f is (topologically) the orbit mapping of h.*

The proof is an easy consequence of Theorem 3.8.

COROLLARY 3.82. *Suppose that X is a generalized continuum with Newman's Property and that f is a finite-to-one open and closed mapping of X onto Y. Then $f(B_f)$ does not separate Y.*

Proof. If $f(B_f)$ separates Y, then $f^{-1}f(B_f) = B$ separates X. Thus, by Theorem 3.8, there are exactly n components of $X - B$, say, $C_1, C_2, \ldots, C_n$ such that $f|C_i$ is one-to-one and

$$f(C_i) = f(C_j) = Y - f(B).$$

Thus, $Y - f(B)$ is connected. This involves a contradiction.

COROLLARY 3.83. *Suppose that X is a generalized continuum with Newman's Property hereditarily and that f is a finite-to-one open and closed mapping of X onto Y. Then $f(B_f)$ does not separate Y locally at any point.*

Proof. Let U be a domain containing y at which $f(B_f)$ separates Y locally and $U - f(B_f) = S \cup T$, two separated sets.

Let $E = \bar{S} \cap \bar{T}$. Since $N(f) < \infty$ by Theorem 3.8, there is $y \in E$ such that

$$N(y, f) = n \geqq N(x, f) \quad \text{for each } x \in f^{-1}(E) = F.$$

There exists a domain D containing y such that $f^{-1}(D)$ has exactly n components $C_1, C_2, \ldots, C_n$ such that $f(C_i) = D$ for each i. Thus, $f|(F \cap C_i)$ is one-to-one. For some k, there is $p \in B_f \cap C_k$ where $p \in F$. Now, these exist sequences $\{p_i\} \to p$, $\{q_i\} \to p$, and $\{y_i\} \to p$ such that $f(p_i) = f(q_i) \in S$ and $f(y_i) \in T$. Let W_i, Z_i, and Q_i be components of $C_k - F$ such that $p_i \in W_i$, $q_i \in Z_i$, and $y_i \in Q_i$. It may happen that $W_i = Z_i$, but clearly $f(W_i) = f(Z_i)$ in any case.

Choose a domain V containing p such that $f|V$ is closed and $C_k \supset V$. There exists t such that

$$V \cap W_t \neq \emptyset \quad \text{and} \quad V \cap Q_t \neq \emptyset.$$

Define a quotient mapping m on V with singletons on $V - (W_t \cup Z_t)$ and the sets $f^{-1}f(x) \cap (W_t \cup Z_t)$ for $x \in W_t$.

Suppose that $x \in \operatorname{Bd} W_t \cap V$. If $x \notin F$, then there is a domain N_x containing x such that $N_x \cap F = \emptyset$ and $V \supset N_x$. Thus, $W_t \supset N_x$ since W_t is a component of $C_k - F$ and $x \in \operatorname{Bd} W_t$, a contradiction. Thus, $x \in F$. Since $f|(F \cap C_k)$ is one-to-one, it follows that m is an open and closed mapping on V. Furthermore, m is one-to-one on the non empty set $V - (W_t \cup Z_t)$. This contradicts the hypothesis that the domain V has Newman's Property. Hence, the corollary is proved.

COROLLARY 3.84. *Suppose that X is a generalized continuum with Newman's Property hereditarily and that f is a finite-to-one open and closed mapping of X onto Y. If $B = B_f = f^{-1}f(B_f)$, f is one-to-one on B, and $X - B$ is not connected, then $X - B$ consists of 2 components C_1 and C_2 and there is a periodic homeomorphism h of X onto X of period 2 such that $h(x) = x$ for $x \in B$ and f is topologically the orbit mapping of h.*

Proof. By Theorem 3.8, there are exactly n components $C_1, C_2, \ldots, C_n$ of $X - B$ each of which maps homeomorphically onto $Y - f(B)$.

Let $b \in B$. Then $b \in \operatorname{Bd} C_j$ for some j. Let $1 \leqq k \leqq n$. There is a sequence $\{p_i\} \to b$ such that $p_i \in C_j$ for each i. Thus,

$$\{f^{-1}f(p_i)\} \to f^{-1}f(b) = b \quad \text{and} \quad q_i \in f^{-1}f(p_i) \cap C_k.$$

Since f is closed it follows that $\{q_i\} \to b$ and $b \in \text{Bd } C_k$.

Suppose that $n > 2$. Let m denote the quotient mapping with singletons on $\overline{C_i}$ for $i > 2$ and the sets $f^{-1}f(x) \cap (C_1 \cup C_2)$ for $x \in C_1$. Thus, m is an open and closed mapping on X which is one-to-one on the open set C_3. This contradicts the hypothesis that X has Newman's Property. Thus, $n = 2$.

Let $f^{-1}f(x) \cap C_i$ be the singleton x_i. Now, define $h(x_1) = x_2$ and $h(x_2) = x_1$. For $x \in B_f$, let $h(x) = x$. The conclusion of the corollary follows.

Example. Let X denote the unit 2-sphere

$$\{ (x, y, z) \,|x^2 + y^2 + z^2 = 1\}$$

and Y denote the 2-disk

$$\{ (x, y, 0) \,|x^2 + y^2 \leqq 1\}.$$

Let f be the vertical projection of X onto Y, i.e., $f(x, y, z) = (x, y)$. Thus, f is an open and closed mapping of the 2-manifold X onto the 2-manifold Y with boundary. Note that (X, Y) does not have the Invariance Of Domain Property. Here,

$$B_f = B = \{ (x, y, 0) \,|x^2 + y^2 = 1\}.$$

Also, f is the orbit map of an involution g on X.

4. More terminology and preliminary theorems. We shall prove that certain generalized continua possess Newman's Property. First, we state needed concepts and prove useful theorems.

A collection K of subsets of a space X is said to be *locally finite* if and only if for each $x \in X$, there is an open set U containing x which has a nonempty intersection with at most a finite number of the elements of K. A set A *meets* a set B if and only if $A \cap B \neq \emptyset$. If K is an open covering of a space X, then $N(K)$ will denote the nerve of K consisting of all elements of K (vertices) and all n-simplices

$$\sigma^n = (V_0, V_1, \ldots, V_n)$$

such that

$$V_i \in K \quad \text{and} \quad \bigcap_{i=0}^{n} V_i \neq \emptyset$$

for each natural number n. The nucleus, $N[\sigma^n]$, of σ^n is the set $\bigcap_{i=0}^{n} V_i$.

We shall be using Cech homology (cf. [**22**, chapter V]).

Suppose that K is a collection of subsets of X and $A \subset X$. Then the *star of A with respect to K* is the collection

$$\{k|k \in K \text{ and } k \cap A \neq \emptyset\}.$$

We shall make use of the well known theorems about open coverings of metric spaces (X, d). For example, for each open covering G of X, there is an open covering H of X which

(1) refines G (each element of H is contained in an element of G) and denoted by $H > G$ and

(2) H is locally finite. That is, (X, d) is *paracompact.*

If each element h of H has the property that $\bar{h} \subset g \in G$, then H *closure* refines G (denoted by $H \gg G$).

We shall use covering dimension of (X, d) which we denote by dim X. The *order* of an open covering G of X is n if and only if at most n elements of G meet and some n elements of G meet. That is, there is an n-simplex in $N(G)$ and $N(G)$ contains no k-simplex for $k > n$. It is well known that for each open covering G of a metric space (X, d) such that dim $X = n$, there is an open covering H such that

(1) H star refines G, i.e., if $h \in H$, then there is $g \in G$ such that each element of the star of h with respect to H lies in g,

(2) H is locally finite, and

(3) order $H = n + 1$ (cf. [**14**]).

A collection K of subsets of X is *closure preserving* if and only if for any subcollection H of K,

$$\bigcup_{h \in H} \bar{h} = \overline{\bigcup_{h \in H} h}.$$

Also, K is *discrete* if it is closure preserving and the closures of the elements of K are pairwise disjoint. We let K^* denote the union of the elements of K. See [**1**].

The next two theorems are useful in constructing special coverings which we use to prove that certain generalized continua possess Newman's Property. In fact, we generalize two lemmas of Wilder [**22**; 8.7 and 8.8, p. 134]. The proofs use ideas of his but our results are stronger.

THEOREM 4.1. *Suppose that Y is a generalized continuum and F is a closed subset of Y. If U is an open covering of Y, then there is a locally finite open refinement R of U which covers Y and an open set $Q \supset F$ such that if the nucleus of a simplex of $N(R)$ meets $\bar{Q}$, then it meets F. In addition, the elements of R can be taken as connected. Also, R can be chosen to star refine U. If* dim $Y = n$, *then we can choose R to also have order $n + 1$.*

Proof. There exists a locally finite open refinement W of U such that if $w \in W$, then $\bar{w}$ is compact and w is connected. Let P denote the union of all elements $w \in W$ such that $w \cap F \neq \emptyset$. Let

$$U' = \{w | w \cap \bar{P} \neq \emptyset\}.$$

Let σ^r be an r-simplex of $N(U')$. If its nucleus, $N[\sigma^r]$, meets F, then let

$$p(\sigma^r) \in F \cap N[\sigma^r];$$

otherwise, let $p(\sigma^r)$ denote any point of $N[\sigma^r]$. Let M denote the set of all these points $p(\sigma^r)$. Since W is locally finite, M has no limit point. The elements of U' are countable as $U_1, U_2, \ldots, U_i, \ldots$. Replace each U_i by a collection K_i as follows: Let

$$A_1 = (M \cap U_1) \cup [\bar{P} - \bigcup_{j=2}^{\infty} U_j].$$

Cover A_1 by a collection K_1 such that

(1) for $k \in K_1$, k is open and connected (indeed, uniformly locally connected),

(2) $U_1 \supset \bar{k}$ for each $k \in K_1$,

(3) the elements of K_1 are pairwise disjoint, and

(4) K_1 is closure preserving (use the fact that a metric space is collectionwise normal [**1**] and the collection of closed sets consisting of

$$\bar{P} \cup \bigcup_{j=2}^{\infty} U_j$$

and the points of $M \cap U_1$ is discrete).

Using induction, let

$$A_i = (M \cap U_i) \cup [\bar{P} - \bigcup_{j=1}^{i-1} K_j^* - \bigcup_{j=i+1}^{\infty} U_i].$$

Cover A_i by a discrete collection K_i of connected open sets such that for each $k \in K$, $U_i \supset \bar{k}$. Now, let R denote the collection of the elements of the various K_i and those elements of W not meeting P.

Suppose that a simplex σ^r of $N(R)$ has a nucleus $N[\sigma^r]$ that does not meet F where $\sigma^r = (b_0, b_1, \ldots, b_r)$. Then

$$x \in \overline{N(\sigma^r)} \cap F$$

would imply that

$$x \in \bigcap_{i=0}^{r} \bar{b}_i.$$

There exist $j_0, j_1, \ldots, j_r$ such that

$$\bar{b}_i \subset U_{j_i}, \quad \overline{N(\sigma^r)} \subset \bigcap_{i=0}^{r} U_{j_i} = N[\delta^r]$$

where

$$\delta^r = (U_{j_0}, U_{j_1}, \ldots U_{j_r}) \in N(U').$$

This implies that $F \cap N[\delta^r] \neq \emptyset$ and that $p(\delta^r) \in N[\sigma^r]$ which is contrary to the construction of R. Thus, if $N[\sigma^r]$ fails to meet F for $\sigma^r \in N(R)$, then

$$\overline{N[\sigma^r]} \cap F = \emptyset.$$

Let Q be any open subset of P which contains F and meets no $N[\sigma^r]$ that fails to meet F.

In case $\dim Y = n$, we first choose W to have the properties listed above and, in addition, order $W = n + 1$. The construction of R yields that order $R = n + 1$.

THEOREM 4.2. *Under the hypotheses of Theorem* 4.1, *there exists a locally finite open covering W of Y which star refines U such that if H is a subcollection of W each element of which meets F, then the intersection N of the elements of H is non empty only if N meets F. If* $\dim Y = n$, *then W exists with the additional property that order $W = n + 1$. If* $\dim Y = n$, *then W can be chosen so that in addition to the other properties, order $W = n + 1$. If* $\dim Y = n$ *and* $\dim F \leqq n - 1$, *then W can be chosen to have the additional properties that order $W = n + 1$ and order $W_F \leqq n$ where*

$$W_F = \{w|w \in W \text{ and } w \cap F \neq \emptyset\}.$$

Proof. Choose R and Q as in Theorem 4.2. As shown there, a set

$$(b_0, b_1, \ldots, b_r) = \sigma^r$$

of elements of R has a nucleus $N[\sigma^r]$ meeting F only if the same held true for $N[\delta^r]$ where

$$\delta^r = (U_{j_0}, U_{j_1}, \ldots, U_{j_r})$$

with elements in U' such that $U_{j_i} \supset \bar{b}_i$. By the way that Q was chosen, such a set $N[\sigma^r]$ must meet F if it meets Q. That is, nuclei of simplices in $N(R)$ either meet Q and hence F or lie in $Y - \bar{Q}$. Replace the collection K_i, for each i, by the collections

$$K_i' = \{k \cap Q|k \in K_i\} \quad \text{and} \quad K_i'' = \{k - F|k \in K_i\}.$$

The covering W consisting of the elements of R not in any K_i along with the elements of the various collections K_i' and K_i'' is the required covering.

In case, $\dim Y = n$ and $\dim F \leqq n - 1$, start with U in Theorem 4.2 as an open covering of Y with the property that the collection U_F of those elements of U which meet F has order $\leqq n$ relative to F (i.e., the collection

$$U(F) = \{u \cap F|u \in U_F\}$$

has order $\leqq n$) and the collection

$$U_{Y-F} = \{u|u \in U \text{ and } u \cap F = \emptyset\}$$

has order $n + 1$. Apply Theorem 4.1 and the proof above using such a covering U to obtain W. Observe that if

$$W_F = \{w | w \in W \text{ and } w \cap F \neq \emptyset\}$$

then order $W_F \leqq n$.

We shall say that W is *regular with respect to* F when W satisfies the conclusion of Theorem 4.2. In case either dim $Y = n$ or dim $Y = n$ and dim $F \leqq n - 1$, we require the appropriate additional restrictions or either W or W and W_F.

5. Special coverings. Suppose that f is a light open and closed mapping from X to Y when each of X and Y is a generalized continuum. We shall construct what we shall call special coverings of X similar to those defined by Smith [**18**]. In the case that f is finite-to-one, dim $X =$ dim $Y = n$, $F = f(B_f)$, and dim $F \leqq n - 1$, these special coverings have particularly nice properties. In fact, we shall construct the special coverings for this case. The generalization should be clear.

Suppose that U is an open covering of Y. Now, use Theorem 4.1 to obtain R refining U and satisfying all of the conditions of the theorem including conditions on the orders of R and W. Now, obtain W satisfying Theorem 4.2 such that W is regular with respect to F.

If $w \in W$, $w \notin K_i'$, and $w \notin K_i''$, then $f^{-1}(w)$ has a finite number of components each mapping onto w under f. In fact, R and W can be chosen so that if $w \in R$, the closures of the components of $f^{-1}(w)$ are pairwise disjoint. If $N(f) = k$, then for $w \in W$ where $w \cap F = \emptyset, f^{-1}(w)$ consists of exactly k components each mapping homeomorphically onto w under f.

Let $g_1, g_2 \in W$, $g_j \notin K_i'$, and $g_j \notin K_i''$ for any i and each $j = 1, 2$.

If $g_1 \cap g_2 \neq \emptyset$, then the components of $f^{-1}(g_j), j = 1, 2$, can be ordered as $g_{j1}, g_{j2}, \ldots, g_{jk}$ such that $g_{1i} \cap g_{2t} \neq \emptyset$ if and only if $i = t$.

These components of $f^{-1}(w)$ constitute a distinguished family determined by W when $W \cap F = \emptyset$. Each component will be a member of the special covering W_f. The remaining elements of W_f are as follows: If $g \in K_i'$ or $g \in K_i''$, then there exists $k \in K_i$ such that either

(1) $g = k \cap Q$ or

(2) $g = k - F$ (where Q is the open set used in the definition of K_i').

In either case, $f^{-1}(k)$ consists of a finite number of components $u_1, u_2, \ldots, u_s$ each mapping onto k under f. Now, let

$$u_i' = u_i \cap f^{-1}(Q) \quad \text{and} \quad v_i' = u_i - f^{-1}(F)$$

depending on where case (1) or (2) holds true. Now, the remaining elements of W_f are the various u_i' and v_i'. Also, the collection $\{u_1', u_2', \ldots, u_s'\}$ and $\{v_1', v_2', \ldots, v_s'\}$ are distinguished families determined by g in either case (1) or (2).

Suppose that $(v_0, v_1, \ldots, v_r) = \sigma^r$ is an r-simplex in $N(W)$. For each i, $0 \leqq i \leqq r$,

$$f^{-1}(v_i) = v_{i1} \cup v_{i2} \cup \ldots \cup v_{in_i}$$

where $\{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ is the distinguished family determined by v_i. Suppose that

$$(v_{0j_1}, v_{1j_2}, \ldots, v_{rj_r}) = \delta^r$$

is an r-simplex in $N(W_f)$. Now, there are such simplices since for each $i = 0, 1, 2, \ldots, r$; each $j = 1, 2, \ldots, n_i$; and each $k = 0, 1, \ldots, r$; there is some t so that

$$v_{ij} \cap v_{kt} \neq \emptyset$$

and conversely, for each $s = 1, 2, \ldots, n_t$, there is p so that

$$v_{ks} \cap v_{ip} \neq \emptyset.$$

That is, each member of the family determined by v_i intersects some member of the family determined by v_t and conversely. Thus, if q is in the nucleus of σ^r, $N[\sigma^r]$, then

$$f^{-1}(q) \cap v_{ij} \neq \emptyset \quad \text{for each } i = 0, 1, \ldots, r \text{ and } j = 1, 2, \ldots, n_i.$$

The orientation of δ^r is to be that of σ^r as indicated by the order given of the vertices $v_{0i_1}, v_{1i_2}, \ldots, v_{ri_r}$ of δ^r.

Since $f^{-1}(v_i)$ has n_i components, there will be at least n_i r-simplices in $N(W_f)$ which are mapped to σ^r by the simplical mapping

$$f_s:N(W_f) \rightarrow N(W)$$

induced by f. If $m = \max \{n_i | i = 0, 1, \ldots, r\}$, then there are m r-simplices in $N(W_f)$ determined by σ^r as indicated above. We say that this collection of r-simplices is the *distinguished family* of r-simplices *determined by* σ^r (more precisely, determined by $N[\sigma^r]$).

If $N[\sigma^r] \cap F = \emptyset$, then $v_i \cap F = \emptyset$ for some i (recall that the vertices of σ^r are $v_0, v_1, \ldots, v_r$ which are members of W).

A standard argument yields the following theorem.

THEOREM 5.1. *The collection of all special coverings W of Y and W_f of X are cofinal in the collection of all open coverings of Y and X, respectively.*

We now define *special projections*. Suppose that a special covering H of Y star refines a special covering G of Y. The special coverings H_f and G_f of X have the properties

(1) H_f star refines G_f and

(2) if $h \in H_f$, $h \subset g \in G_f$,

then each member of the distinguished family to which h belongs is contained in exactly one member of the distinguished family to which g belongs. Suppose that $\delta^r = (u_0, u_1, \ldots, u_r)$ is an r-simplex of $N(H_f)$. We say that a projection

$$\pi: N(H_f) \rightarrow N(G_f)$$

is *special* if and only if $\pi(\delta^r) = \sigma^m$ in $N(G_f)$, then each member δ^r_i, $i = 1, 2, \ldots, t$, of the distinguished family to which δ^r belongs is mapped by π to a member of the distinguished family to which σ^m belongs.

Next, we define the chain *operator* σ used by Cernavskii [**3**]. Observe that our constructions of the special coverings and the definition of σ is much simpler than in [**3**] for obvious reasons.

If δ^m is an m-simplex in $N(W_f)$ where W_f is a special covering, then either

(1) $N[\delta^m] \cap f^{-1}(F) \neq \emptyset$ or

(2) $N[\delta^m] \cap f^{-1}(F) = \emptyset$.

In case (1), $\sigma\delta^m = 0$. In case (2),

$$\sigma\delta^m = \sum_{i=1}^{k} \delta^m_i$$

where $\{\delta^m_i\}^k_{i=1}$ is the distinguished family to which δ^m belongs. By definition, they have the same orientation. The definition obviously extends to any chain group in the usual way.

LEMMA 5.2. *The special operator commutes with the boundary operator, i.e., for an m-simplex*, $\sigma\partial\delta^m = \partial\sigma\delta^m$.

Proof. If $N[\delta^n] \cap f^{-1}(F) \neq \emptyset$, then Theorem 4.2 and $N[\delta^n] \cap f^{-1}(F) \neq \emptyset$ imply that for each face σ^{n-1} of δ^n,

$$N[\sigma^{n-1}] \cap f^{-1}(F) \neq \emptyset.$$

Thus, $\sigma\partial\delta^n = \partial\sigma\delta^n$.

If $N[\delta^n] \cap f^{-1}(F) = \emptyset$, then consider

$$\partial\delta^n = \sum_{i=1}^{n+1} (-1)^{i+1}\delta^{n-1}_i.$$

Either

(a) $N[\delta^{n-1}_i] \cap f^{-1}(F) = \emptyset$ or

(b) $N[\delta^{n-1}_i] \cap f^{-1}(F) \neq \emptyset$.

In case (a), the distinguished family of δ^{n-1}_i contains $k(n - 1)$-simplices $\delta^{n-1}_{i1}, \delta^{n-1}_{i2}, \ldots, \delta^{n-1}_{ik}$. Each is a face of exactly one n-simplex in the distinguished family of δ^n. The labelling may be arranged so that δ_{ij}^{n-1} is a face of δ_j^n. In this case,

$$\begin{aligned}\sigma(\partial\delta^n) &= \sigma \sum_{i=1}^{n+1} (-1)^{i+1}\delta_i^{n-1} \\ &= \sum_{i=1}^{n+1} \sigma(-1)^{i+1}\delta_i^{n-1} \\ &= \sum_{i=1}^{n+1} (-1)^{i+1} \sum_{j=1}^{k} \delta_{ij}^{n-1} \\ &= \sum_{j=1}^{k} \sum_{i=1}^{n+1} (-1)^{i+1}\delta_{ij}^{n-1}.\end{aligned}$$

On the other hand,

$$\sigma\delta^n = \sum_{j=1}^{k} \delta_j^n\partial\delta_j^n = \sum_{i=1}^{n+1} (-1)^{i+1}\delta_{i(j)}^{n-1}.$$

Thus,

$$\partial(\sigma\delta^n) = \sum_{j=1}^{k} \sum_{i=1}^{n+1} (-1)^{i+1}\delta_{i(j)}^{n-1}.$$

Note that $\delta_{i(j)}^{n-1} = \delta_{ij}^{n-1}$ and $\sigma(\partial\delta^n) = \partial(\sigma\delta^n)$.

In case (b),

$$\sigma\delta_i^{n-1} = 0, \quad \sigma\delta^n = \sum_{j=1}^{k} \delta_j^n, \quad \text{and}$$

$$\partial\delta_j^n = \sum_{i=1}^{n+1} (-1)^{i+1}\delta_{ij}^{n-1}.$$

If $N[\delta_{ij}^{n-1}] \cap f^{-1}(F) \neq \emptyset$ for some j, it is true for each $j = 1, 2, \ldots, k$. Thus

$$\sigma\delta_{ij}^{n-1} = 0 \quad \text{for each } j.$$

Again, $\sigma\partial\delta^n = \partial\sigma\delta^n$ and the lemma is proved.

LEMMA 5.3. *If* dim $Y = n$, dim $F \leqq n - 1$, *and* $N(f) = k$, *then* $\sigma\sigma\delta^n = 0$ mod k *for each n-simplex in* $N(W_f)$ *where* W_f *satisfies the conclusion of Theorem* 4.2.

Proof. If $\sigma\delta^n \neq 0$, then

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$$\sigma\delta^n = \sum_{i=1}^{k} \delta_i^n$$

where $N[\delta_i^n] \cap f^{-1}(F) = \emptyset$. Now,

$$\sigma(\sigma\delta^n) = \sigma \sum_{i=1}^{k} \delta_i^n$$

where $\{\delta_i^n\}_{i=1}^k$ is the distinguished family to which δ^n belongs. Now,

$$\sigma\delta_i^n = \sum_{j=1}^{k} = \delta_j^n$$

so that

$$\sigma(\sigma\delta^n) = \sum_{i=1}^{k} \sum_{i=1}^{k} \delta_i^n = \sum_{i=1}^{k} k\delta_i^n = 0 \bmod k.$$

Observe that σ maps n-cycles to n-cycles and takes essential n-cycles to essential n-cycles. Also, δ commutes on n-chains with the special projections.

6. Finite-to-one open and closed mappings on certain generalized continua and Newman's Property. It is easy to prove that a discrete open and closed mapping on a generalized continuum is finite-to-one.

THEOREM 6.1. (Proof due to Väisälä [**19**]). *Suppose that f is a discrete open and closed mapping of X onto Y where each of X and Y is a generalized continuum. Then $f^{-1}f(x)$ is finite for each $x \in X$.*

Proof. Suppose that for some $x \in X$,

$$f^{-1}f(x) = \{x_1, x_2, \dots, x_n, \dots\},$$

an infinite set. Let d and p denote metrics for X and Y, respectively. Choose, for each i, $z_i \in X - f^{-1}f(x)$ such that

$$d(z_i, x_i) < \frac{1}{i} \quad \text{and} \quad p(f(z_i), f(x)) < \frac{1}{i}.$$

The set $Z = \{z_1, z_2, \dots, z_n, \dots\}$ is a closed set such that $f(Z)$ is not closed. This contradicts the assumption that f is closed. Consequently, f is finite-to-one.

We shall consider the class $C(k)$ $k > 1$, of all finite-to-one open and closed mappings f on a generalized n-dimensional continuum X such that

(1) f maps X onto a generalized continuum Y_f and
(2) $N(f) = k$.

We shall say that the generalized continuum X has *Newman's Property with respect to* $C(k)$ if and only if for each open set A in X, there is a positive number ϵ such that if $f \in C(k)$, then for some $x \in A$, the diameter of $f^{-1}f(x) \geqq \epsilon$. Thus, X, has *Newman's Property* if and only if X has Newman's Property with respect to $C(k)$ for each integer $k > 1$.

The main theorem of this section follows:

THEOREM 6.2. (c.f. [**2**, pp. 154-158]) *Suppose that X is an n-dimensional generalized continuum. Furthermore, for each open set A in X, the Čech homology group, $H_n(X, X - A, Z_p)$ is nontrivial for the prime p. Then X has Newman's Property with respect to $C(p)$.*

The points of maximal multiplicity of f form a dense open set in X. This restricted property is equivalent to the definition given in Section 3 when applied to connected metric n-manifolds.

Proof. Suppose that A is an open set in X. Choose an open set D such that $D \supset \bar{A}$. Let U denote a locally finite open covering of X such that if B is a locally finite open covering of X which refines U with the property that the star of A with respect to B lies in D, then any projection of $N(B)$ into $N(U)$ takes an essential n-cycle $z^n(B)$ mod $X - D$ to an essential n-cycle $z^n(U)$ mod $X - A$.

Let ϵ be the Lebesque number of the covering B. Suppose that f is an open and closed mapping of X onto a generalized continuum Y such that

(1) $N(f) = k > 1$ and

(2) if $f^{-1}f(x) \cap A \neq \emptyset$,

then $\operatorname{diam} f^{-1}f(x) < \epsilon$ and consequently lies in an element of B.

There is a special covering G_f of X which refines B such that each distinguished family of elements of G_f which covers an inverse set $f^{-1}f(x)$, $x \in X$, and meets A lies entirely in some element of B (and, hence, in D).

Consider a special projection $\pi:N(G_f) \rightarrow N(B)$ such that if some member g_i of a distinguished family $\{g_1, g_2, \dots, g_m\}$ of elements of G_f meets A, then π projects each g_i, $i = 1, 2, \dots, m$, to the same member b of B.

Suppose that σ^n is an n-simplex of $N(G_f)$. If

$$N[\sigma^n] \subset X - f^{-1}(F)$$

where $F = f(B_f)$, then the distinguished family in $N(G_f)$ which contains σ^n contains exactly ρ n-simplices.

If $z^n(G_f)$ is an essential n-cycle mod $X - D$ on A, then

$$\sigma z^n(G_f) = xz^n(G_f) \quad \text{where } x \in \mathbf{Z}_p.$$

Now,

$$\sigma\sigma z^n(G_f) = 0 = 2xz^n(G_f).$$

Either $x = 0$ or $x = 1$ in case $p = 2$. If $x = 1$, then

$$\sigma z^n(G_f) = z^n(G_f) \quad \text{and}$$

$$0 = \sigma\sigma z^n(G_f) = \sigma z^n(G_f) = z^n(G_f) = 0.$$

This is a contradiction. Hence, $x = 0$.

By construction of G_f, $\pi z^n(G_f) \neq 0$. However,

$$\sigma\pi z^n(G_f) = \pi\sigma z^n(G_f) = 0.$$

This contradicts the fact that σ takes essential n-cycles to essential n-cycles. Thus, it is false that

$$\operatorname{diam} f^{-1}f(x) < \epsilon \quad \text{for each } x \in A.$$

The theorem is proved.

The following corollaries are easy to establish.

COROLLARY 6.21. *If for each open set A of a generalized n-dimensional continuum, the Čech homology group $H_n(X, X - A, Z_p)$ is nontrivial for each prime p, then X has Newman's Property.*

COROLLARY 6.22. *Suppose that X is an n-dimensional connected metric manifold. Then X has Newman's Property.*

COROLLARY 6.23. *Suppose that each of X and Y is an n-dimensional connected metric manifold without boundary. If f is a discrete open and closed mapping of X onto Y, then* $\dim B_f \leqq n - 2$.

Proof. By Theorem 6.1, f is finite-to-one. From Theorem 6.2, X has Newman's Property. Now, by Theorem 3.8, $f(B_f)$ does not locally separate Y. Hence,

$$\dim f(B_f) \leqq n - 2.$$

If B_f locally separates X at some point $x \in X$, then there is a domain D containing x such that $f|D$ is closed, open, and $D - B_f$ is not connected. Now, $g = f|D$ is open and closed. By Theorem 3.8, $g(D) - g(B_g) = Z$ is connected. Each component C of $g^{-1}(Z)$ maps onto Z under g. Either

(1) $g^{-1}g(B_g)$ separates D and g is one-to-one on each component C of $g^{-1}(Z)$ or

(2) g is constant on the connected set $D - g^{-1}g(B_g)$.

In case (1), $B_g \supset B_f \cap D$ and since $B_g \subset B_f \cap D$,

$$B_f \cap D = B_g.$$

If C is a component of $g^{-1}(Z)$, then let

$$K = \{x | x \in B_g \text{ and } x \in \bar{C}\}.$$

Then $f|K$ is one-to-one on K. It follows that

$$\dim K = \dim f(K) \leqq n - 2.$$

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Hence, K can not separate C from $D - C$ which is a contradiction.

In case (2), we have $B_g = B_f \cap D$ and $D - B_f$ is connected. Again, we have a contradiction.

7. Generalizations. It should be clear that the dimension n of X is not so critical in the proof of Theorem 6.2 as the fact that $H_n(X, X - A, Z_p)$ is nontrivial. Thus, we can state appropriate generalizations of Theorem 6.2 which can be proved by our methods.

THEOREM 7.1. *Suppose that X is a generalized continuum. Furthermore, there is a natural number n and a prime p such that for each open set A in X, $H_n(X, X - A, Z_p)$ is nontrivial. Then X has Newman's Property hereditarily with respect to $C(p)$.*

If we consider light open and closed mappings f on a generalized continuum which are not finite-to-one, then B_f may be X [**24**]. Another generalization is stated below.

THEOREM 7.2. *Suppose that X is a generalized continuum and that L is the class of all light open and closed mappings f of X onto some generalized continuum Y_f (possibly different for different f) such that*

(1) int $B_f = \emptyset$ *and*

(2) *there is a complete sequence of special coverings $\{G_f^i\}$ of X and a prime p such that the distinguished families of r-simplices whose nuclei do not intersect $f^{-1}f(B_f)$ consist of mp r-simplices where m is a natural number.*

Furthermore, there is a natural number n such that if A is an open set in X, then $H_n(X, X - A, Z_p)$ is nontrivial. Then X has the Generalized Newman's Property hereditarily with respect to L (that is, there is $\epsilon > 0$ such that if $f \in L$, then for some $x \in U$, diam $f^{-1}f(x) \geqq \epsilon$).

Problem. Characterize those generalized continua which possess Newman's Property (Generalized Newman's Property).

The Sierpinski plane universal 1-dimensional curve S has Newman's Property. There are other obvious examples. However, what (if any) topological property (or properties) characterize them?

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