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# CONTINUATION OF COMPLEX VARIETIES ACROSS RECTIFIABLE SETS

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ABSTRACT. We continue our research on extension of complex varieties across closed subsets. While efforts are being made to deal with varieties of any dimensions, the paper primarily concerns 1-dimensional case, and the exceptional set is thus assumed to be connected with finite length. As applications of the main result, several corollaries are obtained with interesting features.

0. **Introduction.** This paper is a continuation of our previous work [Xu], in which the main goal was to find certain topological conditions on a $2k-1$ dimensional $C^1$ submanifold $E$ in a domain $\Omega$ and on a $k$-dimensional complex variety $V$ in $\Omega \setminus E$ so that $V$ can be extended analytically across $E$. To replace the smoothness of $E$, the major obstruction, following lines in [Xu], is that we no longer have a regularity theorem for the pair $(V,E)$, (*cf.* Section 3 in [Xu]). Consequently, Stokes' formula for the pair $(E, V)$, *i.e.*, the formula $d[V] = [E]$ in the sense of currents, is no longer valid here, since $E$ is only the topological boundary of $V$. To overcome this difficulty, a different method has to be used. For this purpose, we need a delicate analysis of the set $E$ which yields certain uniqueness and removable singularities results for holomorphic functions. To simplify our argument, we will concentrate the case when $E$ is a rectifiable curves. As the proof goes on, we will realize that the method adopted here may not work well for high dimensional rectifiable sets, although we believe that the conclusion is still valid there. Finally it worthwhile to note that the method given here also works in our former paper [Xu] and gives an alternative proof of the results appeared there. It can also be used to show that a 1-dimensional complex variety in a strictly pseudoconvex domain with rectifiable arc as its boundary can be parameterized as the image of some analytic mappings from the unit disc in $\mathbb{C}^1$ and Lipschitz continuous up to the unit circle. We will publish this result elsewhere.

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1. **Analysis of some planar sets.** Let $D$ be a bounded simply connected domain in $\mathbb{C}$. If the boundary of $D$ is a rectifiable curve, then it is known that a bounded function $f$ holomorphic on $D$ with vanishing nontangential boundary values on a subset $E \subset bD$ with positive length must be the zero function. The same conclusion is not true if $D$ is only

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a domain, as an example from Beurling shows that there exist domains with boundaries of finite length and nonzero bounded holomorphic functions whose nontangential limits vanish on a set with positive length. (See example after Corollary 2.5.)

Throughout the paper we denote by $\Lambda^1$ the 1-dimensional Hausdorff measure for sets in $\mathbb{C}^n$.

DEFINITION 1.1. A *continuum* in $\mathbb{C}$ is a connected compact set. A *curve* is a continuous image of a closed interval $[a,b] \subset \mathbb{R}$. A curve is called an *arc* (*simple closed curve*) if it is a homeomorphic image of a closed interval (the unit circle). A curve is called *rectifiable* if it has finite length. A closed subset $E$ is called *almost simple* if $E$ has the following decomposition:

$$E = E_0 \cup \left(\bigcup_{i=1}^{\infty} \Gamma_i\right)$$

where $E_0$ is a null set, *i.e.*, $\Lambda^1(E_0) = 0$, and each $\Gamma_i$ is a simple curve such that $\Gamma_i \cap \Gamma_j$, if not empty, has only one point, which is an endpoint for either $\Gamma_i$, $\Gamma_j$ or both.

EXAMPLE A. Define $E_0$ to be the interval $[0, i]$, $E_{2n-1}$ to be the interval $[\frac{1}{2n\pi}, \frac{1}{(2n-1)\pi}]$ and $E_{2n}$ to be the graph $\{x + i\sin\frac{1}{x} : \frac{1}{(2n+1)\pi} \leq x \leq \frac{1}{2n\pi}\}$. Then the set $E = \bigcup_{n=0}^{\infty} E_i$ is almost simple.

EXAMPLE B. Let $C$ be the Cantor ternary set in the unit interval $[0, 1]$. The midpoints of the components of $[0, 1]\setminus C$ are $1/2, 1/6, 5/6, 1/18, 5/18$, *etc.* Let $\Gamma$ be the tree in the upper half plane whose vertices are $F = \{(1/2, 1), (1/6, 1/2), (5/6, 1/2), (1/18, 1/4), (5/18, 1/4), \textit{etc.}\}$, and define $E = (C \times \{0\}) \cup \Gamma$. Then the set $E$ is almost simple.

DEFINITION 1.2. Let $D$ be a domain in $\mathbb{C}$ and let $E \subset \bar{D}$ be a closed connected set. We say that the pair $(D, E)$ has property $(Q)$ at $p \in E$ if one of the following holds.

1. If $p \in D$, then for any given neighborhood $O$ of $p$, there exists a smaller neighborhood $U$ containing $p$ so that $(D \setminus E) \cap U$ is a union of two nonempty disjoint connected and simply connected open sets $U_1$ and $U_2$ that are contained entirely in $D$, and satisfy
   (a) $\overline{U_1} \cap \overline{U_2}$ is a curve.
   (b) $p$ is an interior point of the curve $\overline{U_1} \cap \overline{U_2}$.
2. If $p \in bD$, then for any given neighborhood $O$ of $p$, there exists a smaller neighborhood $U$ containing $p$ so that $(D \setminus E) \cap U$ is either a connected and simply connected open set $U$ contained in $D$ with $\bar{U} \cap bD$ a curve that contains $p$ as an interior point, or a union of two nonempty disjoint connected, simply connected open sets contained in $D$ satisfying requirements (a) and (b) in case 1.

If $(D, E)$ has property $(Q)$ at almost all (with respect to $\Lambda^1$) of its points, then we say that $(D, E)$ has property $(Q)$.

Later on, we will see that, by Theorem 1.4 and Lemma 2.2, condition 1(a) is the same as to say that there exist two points $q_1$ $q_2$ on $\overline{U_1} \cap \overline{U_2}$ and an arc $\gamma$ with $q_1$ and $q_2$ as two end points and with $p \in \operatorname{int}(\gamma)$, such that $\overline{U_1} \cap \overline{U_2} = \gamma$. In the same manner, conditions on $\overline{U} \cap bD$ in 2 can be reformulated to a more transparent but equivalent condition.

REMARKS 1.3. **A.** Let $D_1$ and $D_2$ be two disjoint simply connected domains in $\mathbb{C}$ with $bD_1$ and $bD_2$ two simple closed curves. If $E = \overline{D_1} \cap \overline{D_2}$ is a simple curve, then the pair $(D_1 \cup \operatorname{int}(E) \cup D_2, E)$ has property $(Q)$. Note that, in a simply connected domain, not every almost simple curve with finite length has property $(Q)$. An example can be obtained by putting the set $E$ constructed in Example 2, Section 1.3.3 in [Xu] into a large disc in $\mathbb{C}$.

**B.** We list here several useful properties of continua, curves and rectifiable curves.

(a) If $\Gamma$ is a curve , then $\Gamma$ is compact, connected and locally connected. Therefore $\Gamma$ is (uniformly) locally arcwise connected ([Cu], p. 333), *i.e.*, for every $\epsilon > 0$, there exists a $\delta > 0$, such that whenever the Euclidean distance between any two points on the curve is less than $\delta$, there exists an arc $\lambda \subset \Gamma$ with diameter less than $\epsilon$ connecting these two points. Conversely the Hahn-Mazurkiewicz theorem states that any non-empty compact, connected, locally connected and metrizable space is a curve ([Cu], p. 334).

(b) If $K \subset \mathbb{C}$ is a continuum with finite 1-dimensional Hausdorff measure, then $K$ is locally connected. In this case, even more is true: $K$ is arcwise connected ([Fa], p. 34).

**C.** Besicovitch proved, in his fundamental paper [Be] on the structure of certain planar sets, that every rectifiable curve is an almost simple rectifiable curve. Independently, Ważewski ([Wa]) proved that same is true for every continuum with finite length by showing that every continuum with finite length is actually a rectifiable curve.

THEOREM 1.4. *If $E$ is a rectifiable curve in a domain $D$ such that $E$ contains only finitely many simple closed curves, then the pair $(D, E)$ has property $(Q)$.*

PROOF. By the Besicovitch decomposition for rectifiable curves, we may write $E = \Gamma_0 \cup (\bigcup_{i=1}^{\infty} \Gamma_i)$ with each $\Gamma_i$ a simple curve in $D$. Define:

$$F_i = \bigcup_{j \neq i,} (\Gamma_j \cap \Gamma_i), \quad F = \Gamma_0 \cup \Big( \bigcup_{i=1}^{\infty} F_i \Big).$$

Then $\Lambda^1(F) = \Lambda^1(F_i) = 0$. It is enough to show that $(D, E)$ has property $(Q)$ at every point $p \in E \setminus F$.

Suppose that $p \in \Gamma_1 \setminus F$ and $O$ is a neighborhood of $p$. Since $\Gamma_1$ is a simple curve inside the domain $D$ and since $E$ contains only finitely many simple closed curves, we can find a smaller neighborhood $O'$ of $p$ so that

(1) $O' \setminus \Gamma_1 = S_1 \cup S_2$ with each $S_i$ nonempty connected and simply connected.

(2) There is no simple closed curve in $E$ that is contained entirely inside $O'$.

(3) $E \cap O'$ is connected. This follows since $E$ is uniformly locally arcwise connected.

Denote by $\gamma_1 = bO' \cap \overline{S_1}, \gamma_2 = bO' \cap \overline{S_2}$ and define

$$J = \{i > 1 : \Gamma_i \cap \Gamma_1 \neq \phi, \Gamma_i \cap \Gamma_1 \in \Gamma_1 \cap O'\}$$
$$J_i = \{j \in J : \Gamma_j \cap S_i \neq \phi\}$$
$$I_i = \{j \in J_i : \Gamma_j \cap \gamma_i \neq \phi\} \subset J_i$$

for $i = 1, 2$. Fix an $i$, say $i = 1$. For $i_1 \neq i_2$ in $J_1$, $\Gamma_{i_1} \cap \Gamma_{i_2} = \phi$ by (2). Consider two cases.

(A) If $I_1$ is the empty set, then $S_1 \setminus (\bigcup_{j\in J_1} \Gamma_j)$ is connected. As $b\big(S_1 \setminus (\bigcup_{j\in J_1} \Gamma_j)\big) = bS_1 \cup (\bigcup_{j\in J_1} \Gamma_j)$ is connected, $S_1 \setminus (\bigcup_{j\in J_1} \Gamma_j)$ is simply connected.

(B) If the set $I_1$ is nonempty, we denote by $\{p_i\}$, $\{q_i\}$ the set of points $\{\Gamma_i \cap \Gamma_1\}_{i\in J_1}$ and $\{\Gamma_i \cap \gamma_1\}_{i\in I_1}$ respectively. If $I_1$ is a finite set, then there exist $i_0, j_0$ in $I_1$ such that the domain bounded by

$$(\overline{p_{i_0}q_{i_0}})_{\Gamma_{i_0}} \cup (\overline{q_{i_0}q_{j_0}})_{\gamma_1} \cup (\overline{q_{j_0}p_{j_0}})_{\Gamma_{j_0}} \cup (\overline{p_{j_0}p_{i_0}})_{\Gamma_1}$$

is simply connected, here $(\overline{pq})_\gamma$ denotes the arc segment from $p$ to $q$ along the curve $\gamma$. Thus we reduce to the case $(A)$. If the set $I_1$ is an infinite set, then our point $p$ can not be a limit point of $\{q_i\}$ since the length of $E$ is finite. Thus the indices $i_0$ and $j_0$ still exist and we are done.

In all cases, we can find a subdomain $U_1$ that is contained entirely in $S_1$ such that $bU_1$ meets set $E$ only along set $\Gamma_1$, *i.e.*, $U_1 \setminus E$ is connected. Since $b(U_1 \setminus E)$ is connected, this leads to the conclusion that $U_1 \setminus E$ is simply connected. Following the same lines, we obtain $U_2$ in $S_2$. By taking $U = U_1 \cup U_2 \cup \Gamma_1$, we finish our proof.

The condition that $E$ contain finitely many simple closed curves can be replaced by the condition that the Čech cohomology group $\check{H}^1(E, \mathbb{Z})$ has finite rank.

COROLLARY 1.5. *Let $E$ be a rectifiable curve in a domain $D$ so that $D\setminus E$ is connected. Then $(D,E)$ has property $(Q)$.*

Next we consider the case when the set $E$ is in the boundary of the domain.

THEOREM 1.6. *If $D$ is a simply connected domain with $bD$ a rectifiable curve and if $E \subset bD$ is a rectifiable curve, then $(D,E)$ has property $(Q)$.*

PROOF. We need only to show that for almost all points $p \in E \subset bD$, if $O$ is a neighborhood of $p$, then a smaller neighborhood $B \subset O$ of $p$ can be found such that $B \cap D = B \cap (D \setminus E)$ is connected and simply connected. As the set $E$ is contained in the boundary of $D$, it is enough to construct an arc $\lambda$ that is contained in $O$ with two end points $q_1, q_2$ on $bD$ distinct from $p$. Then the domain $B$ enclosed by $\lambda \cup (\overline{q_1q_2})$ has the desired property, where $\overline{q_1q_2}$ is the curve contained in $bD$ from $q_1$ to $q_2$, via $p$. The existence of such an arc $\lambda$ can be verified by the same way as we did in the proof of Theorem 1.4, since we assume $bD$ to be rectifiable. Thus we finish our proof.

What happens to domains without assumption of simply connectivity? In general Theorem 1.6 is no longer valid. For example, we let $D$ be the domain in $\mathbb{C}^1$ obtained by deleting a sequence of small discs from the unit disc such that the unit circle is the set of limit points of these discs. Then certainly $(D, S^1)$ does not have property $(Q)$. Keeping this example in mind, we consider a domain $D \subset \mathbb{C}$ with $bD$ of finite length. We first decompose the boundary as

$$bD = \bigcup_i \delta_i,$$

where $\{\delta_i\}$ are the path-connected components of $bD$. Each $\delta_i$ is either a single point or a rectifiable curve. Assume that the set $\{\delta_i : \delta_i \text{ is a single point}\}$ is at most countable. We call a point $p \in bD$ a *limit point* of the family $\{\delta_i\}$ if every neighborhood of $p$ intersects infinitely many distinct $\delta_i$'s. Let

$$(bD)^\sharp = \{q \in bD : q \text{ is a limit point of the family } \{\delta_i\}\}.$$

If $E \subset bD$ is a rectifiable curve, then $(D, E)$ has property $(Q)$ at every point of $E \setminus (bD)^\sharp$. For if $p \in E \setminus (bD)^\sharp$, then there exists a neighborhood $U$, such that $U \cap bD$ is pathwise connected. Therefore, combining with Theorem 1.6, we have

COROLLARY 1.7. *Let $E$ be a rectifiable curve contained in the closure of a bounded domain $D$ with $bD$ of finite length. If $\Lambda^1\big(E \cap (bD)^\sharp\big) = 0$, then $(D, E)$ has property $(Q)$.*

2. **Holomorphic functions on domains with rectifiable boundaries.** Throughout this section, we let $D$ be a bounded domain in $\mathbb{C}$, and denote by $H^\infty(D)$ the set of all bounded holomorphic functions on $D$. If the boundary of $D$ is a rectifiable simple closed curve, then follows from a result by Smirnov [Gl] that for every $f \in H^\infty(D)$, the nontangential limit of $f$, which is denoted by n. t. $\lim_{z \to \zeta} f(z)$, exists for almost all of $\zeta \in bD$. Furthermore, if we define $f(\zeta) = \text{n. t.} \lim_{z \to \zeta} f(z)$ if the right-hand side exists, and $f(\zeta) = 0$ otherwise, then the following Cauchy formula holds for all $z \in D$

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

By using this integral formula, we can easily obtain

THEOREM 2.1. *Let $S_1$ and $S_2$ be two disjoint simply connected domains in $\mathbb{C}$ so that $E = bS_1 \cap bS_2$ is a rectifiable arc, and let $D = S_1 \cup S_2 \cup \operatorname{int}(E)$. If $f$ is holomorphic and bounded on $S_1 \cup S_2$ and if for almost all $\zeta \in E$*

$$\text{n. t.} \lim_{z \in S_1, z \to \zeta} f(z) = \text{n. t.} \lim_{z \in S_2, z \to \zeta} f(z),$$

*then the function $f$ continues holomorphically into all of $D$.*

This is a well-known result, but we will give a proof here, since our later results (*cf.* Theorem 2.3, *etc.*) are motivated from this proof.

PROOF. Let $\zeta$ be an arbitrary point on $\operatorname{int}(E)$. Choose an arbitrary neighborhood $U$ of $\zeta$ contained in $D$, so that $bU$ is a closed simple rectifiable curve. Let $\nu_i = S_i \cap bU$, then $\nu_i \cup (E \cap U)$ is a simple rectifiable curve, which bounds a simply connected domain $D_i$ in $S_i$, and meets $bS_i$ only along the closed subset $E \cap U$. Let $f_i = f|_{D_i}$, $i = 1, 2$, then $f_i \in H^\infty(D_i)$. By the above result of Smirnov, we have

$$f_i(z) = \frac{1}{2\pi i} \int_{bD_i} \frac{f_i(z')}{z' - z}\, dz'$$

for all $z \in D_i$. Now we define

$$F(z) = \frac{1}{2\pi i} \int_{\nu_1 \cup \nu_2} \frac{f(z')}{z' - z}\, dz'$$

Then $F(z)$ is holomorphic on the domain $D_1 \cup D_2 \cup (E \cap U)$. On the other hand, the opposite orientations on the rectifiable set $E \cap U$ induced from two domains $D_1$ and $D_2$ yield

$$F(z) = f_1(z) + f_2(z) \quad .$$

If $z \in D_1$, then Cauchy integral theorem leads to $f_2(z) = 0$. Therefore $F(z) = f_1(z)$, *i.e.*, $F|_{D_1} = f|D_1$. Similarly, $F|_{D_2} = f|_{D_2}$. By uniqueness of continuation, $F|_{S_i} = f|_{S_i}$ and our proof completes.

Note that in above theorem, the assumption that the curve $E$ to be simple is redundant. In fact, we can prove following

LEMMA 2.2. *Let $D_1$ and $D_2$ be two bounded disjoint connected and simply connected domains in $\mathbb{C}$ with rectifiable boundaries. If $E = \overline{D_1} \cap \overline{D_2}$ is a rectifiable curve, then $E$ is either a simple curve or a simple closed curve.*

Almost surely topologists know stronger versions of this lemma, but we could not quote any reference. For completeness of the paper, we include a proof below.

PROOF. First we prove that $E$ contains no proper subset that is a simple closed curve. Suppose not, then there is a simple closed curve $\lambda \subset E$, $E \neq \lambda$. Let $\mathbb{C} \setminus \lambda = U_0 \cup U_\infty$ with $\infty \in U_\infty$. Then either $D_i \subset U_0$, or $D_i \subset U_\infty$ for $i = 1, 2$. Since $E \subset bD_1 \cap bD_2$ is a rectifiable curve, Theorem 1.6 implies that both $(D_1, E)$ and $(D_2, E)$ have property $(Q)$. Thus following the proof of Theorem 1.4, we can find a rectifiable arc $\gamma \subset \lambda$ and a connected, simply connected open set $B_1 \subset D_1$ such that $bB_1 \cap E = \gamma$. If $D_1 \subset U_0$, then $B_1 \subset U_0$. Again, since $(D_2, \gamma)$ has property $(Q)$, a rectifiable subarc $\delta \subset \gamma$ and a connected, simply connected open set $B_2 \subset D_2$ can be found such that $bB_2 \cap E = \delta$. As $bB_1 \cap bB_2 = \delta$, $B_1 \cap B_2 = \phi$, and near every point $\xi \in \operatorname{int}(\delta)$ $\delta$ divides $\mathbb{C}$ into two parts, $B_2$ has to lie entirely inside $U_\infty$. Thus $D_2 \subset U_\infty$, and $E = \overline{D_1} \cap \overline{D_2} \subset U_0 \cap U_\infty = \lambda$ is a simple closed curve. Same proof applies if $D_1 \subset U_\infty$.

Secondly, If $E$ is not a simple closed curve (and hence contains no proper subset that is a simple closed curve), we need to show that $E$ is a simple curve. Suppose not, then there exists a point $p \in E$ so that $E$ is not simple at $p$. Denote by $E = \phi([0, 1])$ with $\phi$ a continuous map and $p = \phi(t_0)$ with $t_0 < 1$. Then $p \in bD_1$ has more than one prime ends. Let $\chi: U \longrightarrow D_1$ be the Riemann mapping from the unit disc $U$ in $\mathbb{C}$. Then $p$ will correspond to at least two distinct points, say $q_1$ and $q_2$, on the unit circle. If we connect $q_1$ and $q_2$ by a rectifiable arc $\lambda$ that lies entirely inside $U$ (except its endpoints $q_1$ and $q_2$) and define a domain $U_\lambda$ enclosed by $\lambda$ and by the short arc $\overline{q_1q_2}$ on the unit circle, then $\gamma_\lambda = \chi(\lambda)$ is a simple closed curve and $D_\lambda = f(U_\lambda)$ is a connected open subset in $D_1$. Let $\mathbb{C} \setminus \gamma_\lambda = S_0^\lambda \cup S_\infty^\lambda$ with $\infty \in S_\infty^\lambda$. Then for every fixed $\lambda$, we can not have both $E \cap S_0^\lambda \neq \phi$ and $E \cap S_\infty^\lambda \neq \phi$, for otherwise either $D_1 \subset S_0^\lambda$ or $D_1 \subset S_\infty^\lambda$ and both cases will leads to either $E \subset S_0^\lambda$ or $E \subset S_\infty^\lambda$. Therefore we assume that for every such an arc $\lambda$, either $E \subset S_0^\lambda$ or $E \subset S_\infty^\lambda$. Moreover, for any two such arcs $\lambda_1$ and $\lambda_2$ with $\lambda_1 \cap \lambda_2 = \{q_1, q_2\}$, since $\chi$ preserves orientation and maps interior points in $U$ to interiors points in bounded domain $D_1$, $E \subset S_0^{\lambda_1}(S_\infty^{\lambda_1})$ will imply $E \subset S_0^{\lambda_2}(S_\infty^{\lambda_2})$, and $U_{\lambda_1} \subset U_{\lambda_2}$ will imply $S_0^{\lambda_1} \subset S_0^{\lambda_2}$,

$S_\infty^{\lambda_2} \subset S_\infty^{\lambda_1}$. Thus, as the curves $\{\lambda\}$ sweep out the whole unit disc $U$, $\{\chi(\lambda)\}$ will fill the whole domain $D_1$. If $E \subset S_\infty^{\lambda_1}$ for one such $\lambda_1$, $E \cap (\bigcup_\lambda S_0^\lambda) = \{p\}$, *i.e.*, $E \cap D_1 = \{p\}$, an absurd. Similar contradiction will be produced if we assume $E \subset S_0^{\lambda_1}$ for one $\lambda_1$. Therefore such a point $p$ does not exist at all and we finish our proof.

To generalize the result in Theorem 2.1, we consider a simply connected domain $D$ with $bD$ a rectifiable curve. We need to introduce orientations on the set $bD$ and define nontangential limits for bounded holomorphic functions defined on $D$.

Let $D$ be a simply connected domain in $\mathbb{C}$ with $bD$ a rectifiable curve. Denote by $\chi$ a Riemann mapping from the unit disc $U$ in $\mathbb{C}$ to $D$. If $E \subset bD$ is an arc and if every point on $E$ has two prime ends, then $\chi^{-1}(E)$ consists of two arcs, say $I_1$ and $I_2$, on the unit circle $S^1$. If we give $S^1$ the induced orientation from the unit disc, then both $\chi(I_1)$ and $\chi(I_2)$ have induced orientations from $I_1$ and $I_2$ respectively. Thus we can define the positive and negative sides of $E$. Similarly, if $f \in H^\infty(D)$, then $f \circ \chi \in H^\infty(U)$. Therefore the nontangential limit of $f \circ \chi$ exists almost everywhere on $S^1$. Since $bD$ is a rectifiable curve, Theorem 1.6 implies that $(D, bD)$ has property $(Q)$. Moreover $\chi' \in H^1(U)$, and thus it has nontangential boundary values at almost every point of $bU$. If $F$ denotes the set of points on $S^1$ where radial limit does not exist, then $\Lambda^1\big(\chi(F)\big) = 0$. Thus we can find a set $E_0 \subset bD$ with zero length such that at every point $\zeta \in bD \setminus E_0$, $(D, bD)$ has property $(Q)$ and $bD$ has a tangent line. For $\zeta \in bD \setminus \big(E_0 \cup \chi(U_0)\big)$, if $\zeta$ has only one prime end, then there is a unique $\xi \in S^1$ corresponding to $\zeta$. Otherwise, two points $\xi_1$ and $\xi_2$ on $S^1$ exist as preimages of $\zeta$ under $\chi$. In the former case, if $\lambda$ is an arbitrary curve in $U$ ending at $\xi$ and approaching $bD$ transversely, then $\lambda^* = \lambda \circ \chi$ is a curve in $D$ that meets $bD$ nontangentially at point $\zeta$, and we define

$$\text{n.t.} \lim_{z \in D, z \to \zeta} f(z) = \lim_{z \in \lambda^*} f(z).$$

For the latter case, if $\lambda_1$ and $\lambda_2$ are two curves in $U$ ending at $q_1$ and $q_2$ respectively and approaching $bU$ transversely, then $\lambda_1^* = \lambda_1 \circ \chi$ and $\lambda_2^* = \lambda_2 \circ \chi$ are two distinct curves that meet $bD$ at $\zeta$ transversely. Moreover $\lambda_1^*$ and $\lambda_2^*$ lie on different sides of $D$ at $\zeta$, say $\lambda_1^* \subset D^+$ and $\lambda_2^* \subset D^-$ with $D^+$ $(D^-)$ the positive (negative) side of $D$ at $\zeta$. Thus we can define nontangential limits from both sides in a similar way. The existence and uniqueness follow from corresponding results about $H^1(U)$. Moreover, the nontangential limit is also independent of the choice of the mapping $\chi$. This can be easily verified.

THEOREM 2.3. *Let $D$ be a bounded simply connected domain in $\mathbb{C}$ with rectifiable boundary. Denote by $bD = \Gamma \cup E$ the decomposition of the boundary into its exterior-boundary $\Gamma = b(\mathbb{C} \setminus \bar{D})$ and remaining part $E \subset \operatorname{int}(\bar{D})$. If $E$ is a set of finite length and if $E = E_0 \cup (\bigcup_{i=1}^\infty \Gamma_i)$ with each $\Gamma_i$ an arc, then every $f \in H^\infty(D)$ has non-tangential limit almost everywhere on $\Gamma \cup (\bigcup_{i=1}^\infty \Gamma_i)$. Moreover, if we let*

$$\text{for } \zeta \in \Gamma, \quad f_0(\zeta) = \begin{cases} \text{n.t.} \lim_{z \in D, z \to \zeta} f(z) & \text{if n.t. lim exists at } \zeta \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for } \zeta \in \Gamma_i, \quad f_i^\pm(\zeta) = \begin{cases} \text{n.t.} \lim_{z \in D^\pm, z \to \zeta} f(z) & \text{if n.t. lim exists at } \zeta \\ 0 & \text{otherwise} \end{cases}$$

*where* $D_i^\pm$ *are two simply connected open sets such that* $\zeta \in \overline{D_i^+} \cap \overline{D_i^-} \cap \Gamma_i$, $\Lambda^1(\overline{D_i^+} \cap \overline{D_I^-} \cap \Gamma_i) > 0$, *and* $D^+$ *(*$D^-$*) lies entirely inside the positive (negative) side of* $E$, *then we have the following general Cauchy integral formula*

$$(*) \qquad f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f_0(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\sum_{i=1}^{\infty}\int_{\Gamma_i}\frac{f_i^+(\zeta) - f_i^-(\zeta)}{\zeta - z}\,d\zeta$$

*for* $z \in D$. *In particular, if* $f_i^+(\zeta) = f_i^-(\zeta)$ *holds almost everywhere on* $\Gamma_i$ *for all* $i \geq 1$, *then the function* $f(z)$ *is holomorphic on the domain* int $(\bar{D})$.

It is worthwhile to note that the existence of above domains $D^+$ and $D^-$ follows from Theorem 1.4, *i.e.*, the pair $(\bar{D}, E)$ has property $(Q)$, since $E \subset \bar{D}$ contains no simple closed curve. For if $\gamma \subset E$ is a simple closed curve, then $\gamma$ separates $\mathbb{C}$ into two parts, say $\Omega_\infty$ and $\Omega_0$ with $\infty \in \Omega_\infty$. Since $D$ is connected, either $\Omega_\infty \cap D = \phi$ or $\Omega_0 \cap D = \phi$. If $\Omega_\infty \cap D = \phi$, then $D \subset \Omega_0$ and $\gamma \subset \Gamma$. If $\Omega \cap D = \phi$, then $\Omega \subset \mathbb{C} \setminus D$ and hence $\gamma = b\Omega_0 \subset b(\mathbb{C} \setminus D) = \Gamma$. Both cases lead to a contradiction.

PROOF OF THEOREM 2.3. Let $\chi$ be a one-to-one holomorphic mapping from the unit disc in $\mathbb{C}$ onto $D$ by the Riemann mapping theorem. Since we assume that $D$ is bounded and $\Lambda^1(bD) < \infty$, a result from Pommerenke [Po] states that $\chi$ can be extended continuously to $\bar{U}$. Therefore, with boundary correspondence, $\chi$ maps $S^1$ onto $bD$. As $bD$ is rectifiable, $\chi' \in H^1(U)$, and hence non-zero nontangential limits of $\chi'$ exist almost everywhere on $S^1$. Moreover, since $\Gamma$ is the exterior boundary of the domain $D$, every point on $\Gamma \setminus E$ has only one prime end, while every point on $\Gamma_i$ has two prime ends by property $(Q)$. Thus $\chi^{-1}(\zeta)$ is well-defined for $\zeta \in \Gamma$ and $\chi^{-1}(\zeta)$ contains two points when $\zeta \in S^1$. We denote these two points by $\xi^+$ and $\xi^-$. There exist small neighborhoods $N(\xi^+)$, $N(\xi^-)$ of $\xi^+$ and $\xi^-$ respectively such that so $\chi$ maps $N(\xi^\pm) \cap U$ into $D^\pm$. Thus

$$f_i^\pm(\zeta) = \text{n. t.} \lim_{w \to \xi^\pm} f \circ \chi(w).$$

Since $f \circ \chi \in H^\infty(U)$, we have for $w \in U$,

$$f(z) = f \circ \chi(w) = \frac{1}{2\pi i}\int_{S^1}\frac{f \circ \chi(\xi)}{\xi - w}\,d\xi.$$

On the other hand, for each fixed $i$, we have

$$\int_{\Gamma_i}\frac{f_i^+(\zeta) - f_i^-(\zeta)}{\zeta - z}\,d\zeta = \int_{\Gamma_i}\frac{f \circ \chi(\xi^+) - f \circ \chi(\xi^-)}{\zeta - z}\,d\zeta = \int_{\chi^{-1}(\Gamma_i)}\frac{f \circ \chi(\xi)}{\chi(\xi) - \chi(w)}\chi'(\xi)\,d\xi.$$

Since $\chi$ maps $S^1$ onto $bD$, the right hand side of (*) is the same as

$$\frac{1}{2\pi i}\int_{S^1}\frac{f \circ \chi(\xi)}{\xi - w}\,d\xi + \frac{1}{2\pi i}\int_{S^1} f \circ \chi(\xi)\left(\frac{\chi'(\xi)}{\chi(\xi) - \chi(w)} - \frac{1}{\xi - w}\right)d\xi.$$

For each fixed $z \in D$, the integrand in the second term is holomorphic in $\xi \in U$ and therefore the integration over $S^1$ must vanish. The proof is complete.

COROLLARY 2.4. *Let $D$ be a domain in $\mathbb{C}$. If $E$ is a rectifiable curve in $D$ such that $(D, E)$ has property $(Q)$, then for every $f \in H^\infty(D \setminus E)$ with the property that for almost all points on $E$, the two nontangential limits from both positive and negative sides of $E$ agree, there exists an $F \in H^\infty(D)$ such that $F|_{D\setminus E} = f$.*

Given a domain $D$ in $\mathbb{C}$. A closed subset $E \subset bD$ is called a *uniqueness* set if every $f \in H^\infty(D)$ whose nontangential limits vanish almost everywhere on $E$ (with respect to its length) is identically equal to zero. For example, if $D$ is simply connected with rectifiable boundary, then every subset in $bD$ with positive length is a uniqueness set. For a general bounded domain $D \neq \mathbb{C}$, if we retain the notions from Corollary 1.7, then, by locally applying the Cauchy integral formula, we can easily obtain

COROLLARY 2.5. *Let $D$ be a bounded domain in $\mathbb{C}$, and let $E$ be a rectifiable curve in $\bar{D}$.*

*(1) If $E \subset bD$ and if $(D, E)$ has property $(Q)$, then $E$ is a uniqueness set for $H^\infty(D)$.*
*(2) If $E \subset D$ and if $D \setminus E$ is connected, then $E$ is a uniqueness et for $H^\infty(D \setminus E)$.*

The following concrete example, due to Beurling, gives an illustration that how subtle the subject becomes when we deal with domains with rectifiable boundaries instead of simply connected domains with simple curves as their boundaries.

EXAMPLE (BEURLING). Define function $\phi$ for $|z| < 1$ by

$$\varphi(z) = \prod_{i=2}^{\infty}\left(1 - z^n \exp\left(\frac{n}{\log n}\right)\right)$$

Then $\varphi(z) \in O(|z| < 1)$. Let $\{\alpha_k\}$ be the zeros of $\varphi$ in the unit disc with $|\alpha_1| \leq |\alpha_2| \leq \cdots$ and let

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{\varphi'(\alpha_k)(z - \alpha_k)}$$

$$\Delta_k = \left\{z \in C^1 : |z - \alpha_k| < \frac{1}{k^2}\right\}$$

$$D = \{|z| < 1\} \setminus \left\{\bigcup_{k=1}^{\infty} \Delta_k\right\}$$

$$f_n(z) = \sum_{k=1}^{n} \frac{1}{\varphi'(\alpha_k)(z - \alpha_k)}$$

Then we have following facts:

(1) $\Lambda^1(bD) < \infty$,
(2) $\Delta_k \cap \Delta_l = \phi, k \neq l$ and $\Delta_k \subset \{|z| < 1\}, k = 1, \ldots$.
(3) $f_n$ is a rational function with poles inside $\bigcup_{k=1}^{n} \Delta_k$. Therefore $f_n \in O(\mathcal{D})$.
(4) $f$ is bounded over $D$, since $|\varphi'(\alpha_k)| \geq e^{\sqrt{k}}$ and $|z - \alpha_k| > \frac{1}{k^2}$.
(5) $f_n \longrightarrow f$ uniformly on $D$. In fact $\{f_n\}$ converges uniformly on $\mathbb{C} \setminus \bigcup_{k=1}^{\infty} \Delta_k$. Hence $f \in O(\bar{\mathcal{D}})$.
(6) $f \equiv 0$ outside $\{|z| < 1\}$ and $f \neq 0$ in $D$.

For the proof of these claims, we refer the book by Stout ([St], p. 346). Notice that in this example the rectifiable boundary contains infinitely many simple closed curves and $(bD)^\sharp =$ the unit circle in $\mathbb{C}$, so that Corollary 2.4 does not apply here.

3. **Extendibility of analytic curves across rectifiable curves.** In the papers [Ru] and [We], Rudin and Wermer gave examples showing that disjoint analytic discs can abut along arcs of large Hausdorff dimension, and yet neither coincide nor be analytic continuations of each other. This raises the question as whether the two could be analytically continued if the arc has finite length. The main result in this section shows, as a special case, that this is true.

DEFINITION 3.1. Let $E$ be a closed subset in a domain $\Omega \subset \mathbb{C}^n$ and let $V$ be a 1-dimensional complex variety in $\Omega \setminus E$. We say that the pair $(V, E)$ has property $(Q)$ at point $p \in E$ if the following is satisfied: For every open set $B \subset \Omega$ that contains $p$, there exists a smaller open set, say $U, p \in U \subset B$ such that

$$U \cap V = \bigcup_{i=1}^{l(p)} V_i,$$

where $l(p) \geq 2$ is a finite positive integer, $\{V_i\}_{i=1}^{l(p)}$ are nonempty irreducible 1-dimensional complex varieties in $U \setminus E$ such that for $i \neq j$, $V_i \cap V_j \cap U$ is a zero dimensional variety (possibly empty) and $\{(\overline{V_i} \cap \overline{V_j}) \setminus (V_i \cap V_j)\} \cap U$ is a rectifiable curve that contains $p$. The pair $(V, E)$ has property $(Q)$ if it has it at almost all (with respect to $\Lambda^1$) points of $E$.

THEOREM 3.2. *Let $E$ be a rectifiable curve in a domain $\Omega \subset \mathbb{C}^n$ and let $V$ be a 1-dimensional complex variety in $\Omega \setminus E$ so that $(\bar{V} \setminus V) \cap \Omega = E$. If the pair $(V, E)$ has property $(Q)$, then $E$ is removable in the sense that $\bar{V} \cap \Omega$ is also a 1-dimensional complex variety in $\Omega$.*

Note that the rectifiable set $E$ is only assumed to be connected. It may have non-zero $\check{H}^1(E, \mathbb{Z})$. Therefore this theorem is not contained in our previous work [Xu1]. Also we require in Definition 3.1 that only finitely many irreducible subvarieties meet at $p$ because of following example.

EXAMPLE 3.3. Let $E$ be the real line in $\mathbb{C}^1$ and let $W_1$ be the union of countable many disjoint open discs in the upper half plane so that the closure of $W_1$ contains the set $E$. If we let $W_2 = \{x + iy : y < 0, x - iy \in W_1\}$ and let $V = W_1 \cup W_2$, then of course $V \cup E$ is not a variety in $\mathbb{C}^1$. It can be easily seen that near every point on the real line, $V$ has infinitely many irreducible branches.

PROOF OF THE THEOREM. The proof given here is lengthy. Therefore we divide it into several steps.

STEP 1. Some Preliminary Preparation.

We first prove the result in the case $n = 2$. Define two subsets of the closed set $E$ as follows:

$$E_0 = \{p \in E, (V, E) \text{ has property } (Q) \text{ at } p\}$$
$$E^\sharp = \{p \in E, \bar{V} \text{ is analytic at the point } p.\}$$

It is enough to show that $\bar{V}$ is a complex variety near each point of $E_0$, *i.e.*, $E_0 \subset E^\sharp$, since $\Lambda^1(E \setminus E_0) = 0$, $E \setminus E^\sharp$ is closed, and our conclusion follows from a result of Shiffman [Sh], which states that a 1-dimensional complex variety can be extended across a closed subset with zero length.

Let $p$ be a point in $E_0$, say $p = 0$. Since $\bar{V} \cap \Omega = V \cup E$ and since $\Lambda^1(E) < \infty$, we have that $\Lambda^3(\bar{V}) = 0$. For almost all complex lines $L$ through $p = 0$, the set $\bar{V} \cap L$ has zero length. Fix such an $L$. It follows that for almost all discs $D_L \subset L$ with center 0, $bD_L \cap (\bar{V} \cap L) = \phi$. If we let $\Sigma = L^\perp$ and $\mathbb{C}^2 = \Sigma \times L$, then $\bar{V} \cap (\{0\} \times bD_L) = \phi$. Since $\bar{V}$ is closed, we can find a small disc $D_\Sigma$ in $\Sigma$ with center 0 such that

$$\bar{V} \cap (D_\Sigma \times bD_L) = \phi.$$

The projection $\pi_\Sigma : \bar{V} \cap (D_\Sigma \times D_L) \longrightarrow D_\Sigma$ is thus a proper mapping. Choose coordinate system in $\mathbb{C}^2$ such that $\Sigma = \{(z_1, 0)\}$ and $L = \{(0, z_2)\}$ and let $\pi_1 = \pi_\Sigma : \mathbb{C}^2 \longrightarrow \mathbb{C}^1_{(z_1)}$ be defined by $\pi_1(z_1, z_2) = z_1$, $B = B_1 \times B_2$ with $B_1 = D_\Sigma$, $B_2 = D_L$. Then we have the following properties:

(1) $\pi_1 : \bar{V} \cap B \longrightarrow \pi_1(\bar{V} \cap B)$ is a proper mapping.

(2) $\pi_1|_{V \cap B}$ is locally a biholomorphic mappings away from a discrete subset $E'$ of $V \cap B$.

(3) For the set
$$B^* = \big(B_1 \setminus \pi_1(E \cap B)\big) \times B_2,$$
we have
$$\pi_1 : (V \setminus E') \cap B^* \longrightarrow \pi_1\big((V \setminus E') \cap B^*\big)$$
is a finite covering map. For if $K$ is a compact subset of $\pi_1(V \cap B^*)$, then $\pi_1^{-1}(K)$ is a compact subset of $\bar{V} \cap B$. Since $\pi_1^{-1}(K) \cap \pi_1^{-1}\pi_1(E \cap B) = \phi$, the set $\pi_1^{-1}(K)$ is indeed compact in $V \cap B^*$. The mapping is thus a proper mapping from $V \cap B^*$ onto its image, and therefore is a finite covering map away from the set $E'$.

We call the coordinate system from the pair $\Sigma \times L$ a *normal coordinate system* and the $B$'s the associated *normal neighborhoods*. Since the set of lines $L$ satisfying above properties is dense in $G(2, 1)$; indeed, it is of full measure in $G(2, 1)$, we will, in the following proof, change our normal coordinate system (and hence the associated normal neighborhoods) from time to time to simplify our argument. First for each fixed normal coordinate system, we adopt the following notations:

$$E^1 = \pi_1(E \cap B),$$
$$\pi_1(V \cap B) \setminus E^1 = \bigcup_{j=1}^{\lambda_1} U_j^1$$

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where $\{U_j^1\}$ are the connected components of the set on the left, and

$$K^1 = \pi_1(E'),$$
$$V_j^1 = V \cap B^* \cap \pi_1^{-1}(U_j^1) \setminus \pi_1^{-1}(K^1),$$
$$\Gamma_{jk}^1 = \overline{U_j^1} \cap \overline{U_k^1},$$
$$\mu_j^1 = \text{sheet number of analytic covering map } \pi_1|_{V_j^1}$$

Note that the number $\lambda_1$ may equal to $\infty$, since the set on the left in above second equality may have infinitely many components. Since $\bar{V} \cap (B_1 \times bB_2) = \phi$,

$$b(\bar{V} \cap B) = (E \cap \bar{B}) \cup \big(V \cap (bB_1 \times B_2)\big),$$

and

$$\pi_1\big(b(\bar{V} \cap B)\big) \subset E^1 \cup bB_1.$$

Thus $bU_j^1 \subset bB_1 \cup E^1$. Moreover, we claim that the set $E^1 \cup bB_1$ is connected. First we can shrink the neighborhood $B_1$ (the projection $\pi_1$ remains proper) so that $E \cap (bB_1 \times B_2) \neq \phi$. If $\gamma$ is a connected component of the set $E \cap B$, then the set $\gamma$ meets the set $bB_1 \times B_2$, since we assume that the curve $E$ is connected. Therefore the connected set $\pi_1(\gamma)$ has to meet $bB_1$ and the set $E^1 \cup bB_1$ is thus connected. Therefore the connected set $U_j^1$ has connected boundary and hence is simply connected. We have the following facts, the proof of which can be found in Chirka [Ch], p. 239:

(1) For each $a \in E^1$, $\pi_1^{-1}(a) \cap V \cap B$ is of dimension zero.

(2) For each $a \in \Gamma_{jk}^1$, the number of points (counting multiplicity) of $\pi_1^{-1}(a) \cap \bar{V} \cap \bar{B}$ will not exceed $min(\mu_j^1, \mu_k^1)$.

(3) The set $V \cap U \cap \pi_1^{-1}(U_j^1)$ is the zero set of some monic polynomial $F_j^1(z_1, z_2)$ with degree $\mu_j^1$ in $z_2$ and with coefficients analytic functions of $z_1$. Moreover, for $z_1 \in U_j^1 \setminus K^1$, the polynomial $F_j^1(z_1, z_2)$ has only simple roots in $z_2$.

STEP 2. Some Lemmas.

Next we are going to prove several lemmas that allow us to use the results in Section 2.

LEMMA 3.4. *For each j and k, if $\Lambda^1(\Gamma_{jk}^1) > 0$, then $\mu_j^1 = \mu_k^1$.*

PROOF. Let $\Delta_j^1(z_1)$ be the discriminant of the polynomial $F_j^1(z_1, z_2)$. Then $\Delta_j^1(z_1) \neq 0$ for $z_1 \in U_j^1 \setminus K^1$. Moreover, as $\bar{V} \cap B$ is bounded in $\mathbb{C}^2$, $\Delta_j^1(z_1)$ is bounded and holomorphic on $U_j^1 \setminus K^1$. Since $\Lambda^1(K^1) = 0$ and since $K^1$ is closed, $\Delta_j^1(z_1)$ is a bounded holomorphic function on whole $U_j^1$. Therefore, $\Delta_j^1(z_1)$ has nontangential limit almost everywhere along $bU_j^1$. Moreover, the uniqueness theorem for $H^\infty$ functions on a simply connected domain with rectifiable boundary shows that the subset

$$\Sigma_j^1 = \{z_1 \in bU_j^1 : \text{ n.t. limit of } \Delta_j^1 \text{ exists at } z_1 \text{ and equals zero}\}$$

has zero length. In particular,

$$\Lambda^1\big(\pi_1(E \setminus E_0)\big) = 0,$$
$$\Lambda^1(\Sigma_j^1 \cap \Gamma_{jk}^1) = 0,$$
$$\Lambda^1(\Sigma_k^1 \cap \Gamma_{jk}^1) = 0,$$

and

$$\Lambda^1\Big(\Gamma^1_{jk}\setminus\big(\Sigma^1_j\cup\Sigma^1_k\cup\pi_1(E\setminus E_0)\big)\Big)>0.$$

For any $a\in\Gamma^1_{jk}\setminus\big(\Sigma^1_j\cup\Sigma^1_k\cup\pi_1(E\setminus E_0)\big)$, there are two sequences $\{a_m\}\subset U^1_j\setminus K^1$ and $\{b_m\}\subset U^1_k\setminus K^1$ so that

(1) $\Delta^1_j(a_m)\neq 0$ and $\Delta^1_k(b_m)\neq 0$.

(2) $\Delta^1_j(a_m)\longrightarrow\Delta^1_j(a)\neq 0$ and $\Delta^1_k(b_m)\longrightarrow\Delta^1_k(a)\neq 0$.

Moreover, since all the roots of $F^1_j(z_1,z_2)$ and $F^1_k(z_1,z_2)$ lie inside $\bar V\cap B$ and all are holomorphic functions, by passing to the subsequences of $\{a_m\}$ and $\{b_m\}$, we can assume that $F^1_j(a_m,z_2)$ and $F^1_k(b_m,z_2)$ converge to $F^1_j(a,z_2)$ and $F^1_k(a,z_2)$ respectively. The discriminant of $F^1_j(a,z_2)$ ($F^1_k(a,z_2)$) equals $\Delta^1_j(a)$ ($\Delta^1_k(a)$ resp.) and $F^1_j(a,z_2)$ ($F^1_k(a,z_2)$) is a monic polynomial of degree $\mu^1_j$ ($\mu^1_k$ resp.). Since the zero set of $F^1_j(a,z_2)$ ($F^1_k(a,z_2)$) is $\pi_1^{-1}(a)\cap\bar V\cap\overline{V^1_j}$ ($\pi_1^{-1}(a)\cap\bar V\cap\overline{V^1_k}$ resp.), and since all roots are simple, we can denote the zero sets as:

$$\pi_1^{-1}(a)\cap\bar V\cap\overline{V^1_j}=\{\omega^j_1(a),\dots,\omega^j_{\mu^1_j}(a)\},$$
$$\pi_1^{-1}(a)\cap\bar V\cap\overline{V^1_k}=\{\omega^k_1(a),\dots,\omega^k_{\mu^1_k}(a)\}$$

with $\omega^j_\alpha(a)\neq\omega^j_\beta(a)$ and $\omega^k_\alpha(a)\neq\omega^k_\beta(a)$ if $\alpha\neq\beta$. Each $\omega^j_\alpha(a)$ ($\omega^k_\beta(a)$) is the limit of $\omega^j_\alpha(a_m)$ ($\omega^k_\beta(b_m)$ resp.). For each fixed $m$, if $\alpha\neq\beta$, then $\omega^j_\alpha(a_m)\neq\omega^j_\beta(a_m)$ and $\omega^k_\alpha(b_m)\neq\omega^k_\beta(b_m)$. There are two cases we have to consider separately.

(1) If $\omega^j_\alpha(a)$ does not belong to the set $B\cap E$, say $\omega^j_\alpha(a)\in W_1$, then for large $m$, $\omega^j_\alpha(a_m)\in W_1\setminus E'$. Since the mapping $\pi_1$ is biholomorphic near the point $\omega^j_\alpha(a)$ and the sequence $\{b_m\}$ converges to the point $a$, $\pi_1^{-1}(b_m)$ has a point which is also close to $\omega^j_\alpha(a)$, say $\omega^k_\beta(b_m)\in\pi_1^{-1}(b_m)$. Thus we obtain

$$\omega^k_\beta(a)=\lim\omega^k_\beta(b_m)=\lim\omega^j_\alpha(a_m)=\omega^j_\alpha(a).$$

We have established a mapping from $\pi_1^{-1}(a)\cap W_1\cap\overline{V^1_j}$ to $\pi_1^{-1}(a)\cap W_1\cap\overline{V^1_k}$. Moreover, this mapping is one-to-one, since we assume that $\Delta^1_j(a)\neq 0$ and $\Delta^1_k(a)\neq 0$. This leads to the relation

$$|\pi_1^{-1}(a)\cap W_1\cap\overline{V^1_j}|\leq|\pi_1^{-1}(a)\cap W_1\cap\overline{V^1_k}|,$$

here we use $|*|$ to denote the cardinality of the set. Similarly we can prove the reversed inequality, and this implies that

$$|\pi_1^{-1}(a)\cap V\cap\overline{V^1_j}|=|\pi_1^{-1}(a)\cap V\cap\overline{V^1_k}|.$$

(2) If $\omega^j_\alpha(a)\in E_0\cap B$, we can find a small neighborhood $U\subset B$ so that $\omega^j_\alpha(a)\in U$ and $V\cap U=W'_1\cup W'_2$, and $W'_1$, $W'_2$ have the property in Definition 3.1. There are two cases. Either there are infinitely many $m$'s, say $\{m_{j_1}\}$ and $\{m_{j_2}\}$, so that $\{a_{m_{j_1}}\}$ and $\{b_{m_{j_2}}\}$ are both contained in the same $\pi_1(W'_1)\setminus K^1$ (or $\pi(W'_2)\setminus K^1$), $a_{m_{j_1}}\longrightarrow a$, $b_{m_{j_2}}\longrightarrow a$, or we have that for large index $m$, $a_m\in\pi_1(W'_1)\setminus K^1$ and $b_m\in\pi_1(W'_2)\setminus K^1$. That $\pi_1$ is a local

biholomorphism on both $W_1' \setminus E'$ and $W_2' \setminus E'$ implies that there exist two indices $\alpha$ and $\beta$ with $1 \leq \alpha \leq \mu_j^1$, $1 \leq \beta \leq \mu_k^1$, such that, in the former case, $\omega_\alpha^j(a_{m_{j_1}}) \in W_1' \setminus E'$, $\omega_\beta^k(b_{m_{j_2}}) \in W_1' \setminus E'$, and hence

$$\omega_\alpha^j(a) = \lim \omega_\alpha^j(a_{m_{j_1}}) = \lim \omega_\beta^k(b_{m_{j_2}}) = \omega_\beta^k(a),$$

in the latter case, $\omega_\alpha^j(a_m) \in W_1' \setminus E'$, $\omega_\beta^k(b_m) \in W_2' \setminus E'$, and hence

$$\omega_\alpha^j(a) = \lim \omega_\alpha^j(a_m) = \lim \omega_\beta^k(b_m) = \omega_\beta^k(a).$$

Similar to (1) above, both lead to

$$|\pi_1^{-1}(a) \cap E \cap \overline{V_j^1}| = |\pi_1^{-1}(a) \cap E \cap \overline{V_k^1}|.$$

Combining above two cases, we reach to the conclusion that $\mu_j^1 = \mu_k^1$, and the lemma is proved.

The connected components $\{U_j^1\}$ that appeared in the above lemma depend on the choice of coordinate system and on the projection $\pi_1$. Since the number of branches of $V \cap B$ increases if we decrease the neighborhood $B$, by the definition of property $(Q)$, the number of branches in $B \cap V$ is a finite number $m_B \geq 2$ for each fixed normal neighborhood $B$. From now on we fix a normal coordinate system and a normal neighborhood $B$ of the point $p = 0$. Let $W_1$ and $W_2$ be two branches of $V \cap B$ so that $(\overline{W_1} \cap \overline{W_2}) \setminus (W_1 \cap W_2)$ is a rectifiable curve containing $p = 0$. We denote this curve by $\Gamma$, and by $\pi$ the projection $\pi_1$. Since $\pi(W_1)$ and $\pi(W_2)$ are two connected subsets of $B_1$, the above argument shows that they have connected boundaries. Hence they are simply connected. Moreover, as the mapping $\pi|_{\overline{W_i} \cap B}$ is proper, the set

$$b\pi(W_1) \cap b\pi(W_2) \supset \pi\big(b(W_1 \cap B^*)\big) \cup \pi\big(b(W_2 \cap B^*)\big) \supset \pi\big(b(W_1 \cup W_2)\big),$$

and the latter set contains $\pi(\Gamma)$, which has positive length. Therefore by the proof of Lemma 3.4, the sheet number for the mapping $\pi^{-1}$ on $\pi(W_1) \setminus \pi(\overline{W_1} \cap \overline{W_2})$ is the same as that on the set $\pi(W_2) \setminus \pi(\overline{W_1} \cap \overline{W_2})$. We denote this positive integer by $\mu$. On the one hand, since the set $\Gamma$ is in the boundaries of $W_1$ and $W_2$, the number of points in $\pi^{-1}(\zeta) \cap B$ will not exceed the number $\mu$ for almost all $\zeta \in \pi(\Gamma)$. On the other hand, by applying the uniqueness theorem for holomorphic functions to the discriminant of proper analytic mapping $\pi|_{W_i}$, we know that for almost all $\zeta \in \pi(\Gamma)$, $\pi^{-1}(\zeta)$ contains exactly $\mu$ points in $\overline{W_1 \cup W_2} \cap \bar{B}$. Moreover, by our assumption about property $(Q)$, we can assume that for each of these $\mu$ points in the fiber, property $(Q)$ holds. Next we will prove a lemma that enables us to use results in Section 2.

LEMMA 3.5. *The simply connected domains $\pi(W_1)$ and $\pi(W_2)$ are disjoint, and their boundaries inside $B_1$ meet only along the rectifiable curve $\pi(\Gamma)$.*

PROOF. Suppose $\pi(W_1) \cap \pi(W_2)$ is a nonempty open subset of $B_1$. Choose a nonempty open connected component $\omega$ of it. We first show that if $\zeta \in b\omega$, then either $\zeta \in b\pi(W_1) \cap$

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$\pi(W_2)$ or $\zeta \in b\pi(W_2) \cap \pi(W_1)$. This is so, because

$$\begin{aligned} b\omega \subset b\big(\pi(W_1) \cap \pi(W_2)\big) &= \overline{\pi(W_1) \cap \pi(W_2)} \setminus \big(\pi(W_1) \cap \pi(W_2)\big) \\ &\subset \big(\overline{\pi(W_1)} \cap \overline{\pi(W_2)}\big) \setminus \big(\pi(W_1) \cap \pi(W_2)\big) \\ &= \big(b\pi(W_1) \cap \pi(W_2)\big) \cup \big(b\pi(W_2) \cap \pi(W_1)\big). \end{aligned}$$

Therefore we may assume that

$$\Lambda^1\big(b\omega \cap b\pi(W_1) \cap \pi(W_2)\big) > 0.$$

Let $\mu_i(\zeta)$ be the number of points in $\pi^{-1}(\zeta) \cap \overline{W_i} \cap B$. If $\zeta \in b\pi(W_1) \cap \pi(W_2)$ and if $P$ is an arbitrary point in the set $\pi^{-1}(\zeta) \cap \overline{W_1 \cup W_2} \cap B$, then $P$ cannot be in $W_1$, since the mapping $\pi|_{W_1}$ is an open mapping. Therefore $P$ lies either in $bW_1 \cap bW_2 \cap B$ or in $bW_1 \setminus \overline{W_2}$, or in $W_2$. Suppose $P \in bW_1 \setminus \overline{W_2}$. Then property $(Q)$ implies that for a small neighborhood $U$ of $P$, there exist two branches $V_1$ and $V_2$ of $(W_1 \cup W_2) \cap U$ the closure of which meet along a curve containing $P$. Since $P \notin \overline{W_2}$, both $V_1$ and $V_2$ are branches of $W_1 \cap U$. Since $\pi(V_1) \subset \pi(W_1)$, $\pi(V_2) \subset \pi(W_1)$, and $\zeta \in \pi(\overline{V_1} \cap \overline{V_2}) \subset \pi(\Gamma)$, the discriminant of $\pi^{-1}|_{W_1}$ vanishes at the point $P$. Therefore for almost all points $\zeta \in b\pi(W_1) \cap \pi(W_2)$,

$$\pi^{-1}(\zeta) \subset (bW_1 \cap bW_2 \cap B) \cup W_2.$$

Now as $\zeta \in \pi(W_2)$, there exists at least one such $P$ that belongs to $W_2$, which is certainly not in $\overline{W_1}$. Therefore together with the points in $\pi^{-1}(\zeta) \cap bW_1 \cap bW_2 \cap B$, we have that $\mu_1(\zeta) < \mu_2(\zeta)$. But, as we showed before, for almost all $\zeta \in b\pi(W_1)$, $\mu_1(\zeta) = \mu$ and for $\zeta \in \pi(W_2)$ $\mu_2(\zeta) = \mu$. We obtain a contradiction that $\mu < \mu$. Thus the two simply connected domains $\pi(W_1)$ and $\pi(W_2)$ are disjoint.

Moreover, above proof also gives that for $\zeta \in b\pi(W_1) \cap b\pi(W_2)$, $\pi^{-1}(\zeta) \cap B \subset bW_1 \cap bW_2 \cap B = \Gamma$. Therefore, by the properness of the mapping $\pi$ on the set $\overline{W_1 \cup W_2}$,

$$\begin{gathered} b\pi(W_1) \cap b\pi(W_2) \cap B_1 \subset \pi(\Gamma) = \pi(bW_1 \cap bW_2 \cap B) \\ \pi(bW_1 \cap B) \cap \pi(bW_2 \cap B) \subset \big(b\pi(W_1) \cap b\pi(W_2)\big) \cap B_1. \end{gathered}$$

*i.e.*,

$$b\pi(W_1) \cap b\pi(W_2) \cap B_1 = \pi(\Gamma).$$

This finishes the proof of the lemma.

Now we have two simply connected domains $F_1 = \pi(W_1)$ and $F_2 = \pi(W_2)$ with rectifiable boundaries so that $F_1 \cap F_2 = \phi$ and $\overline{F_1} \cap \overline{F_2}$ is a rectifiable curve $\gamma = \pi(\Gamma)$. Moreover, by the proof of Lemma 3.5, for almost all $\zeta \in \gamma$, $\pi^{-1}(\zeta) \subset \overline{W_1} \cap \overline{W_2} \setminus W_1 \cap W_2 = \Gamma$. By using Lemma 2.2, the curve $\gamma$ is either an arc or a simple closed curve. In both cases, the pairs $(F_i, \gamma)$ have property $(Q)$ by Theorem 1.6 for $i = 1, 2$. That is to say that for almost all $q \in \gamma$, there exist a rectifiable arc $\gamma_i$ in $F_i$ with $\gamma_i \cap \gamma = \{q_1, q_2\}$, a rectifiable subarc $\gamma_0$ of $\gamma$ with endpoints $q_1\ q_2$, and $q \in \operatorname{int}(\gamma_0)$, such that the domain $D_i \subset F_i$ bounded by the curve $\gamma_i \cup \gamma$ is simply connected, $D_1 \cap D_2 = \phi$, and $\overline{D_1} \cap \overline{D_2} = \gamma$.

LEMMA 3.6. *$bD_i$ contains no simple closed rectifiable curve other than $\gamma_i \cup \gamma_0$.*

PROOF. Let us assume $i = 1$. All we need to show is that $bD_1 \setminus (\gamma_0 \cup \gamma_1)$ contains no simple closed curve. Suppose not, so that it contains a simple closed rectifiable curve $\delta$. Then $\delta \subset \gamma \subset \pi(E)$ and $\delta$ bounds a Jordan domain $D_0$ in $\mathbb{C}$ with $D_0 \cap F_1 = \phi$. If $\zeta \in \delta$, then $\pi^{-1}(\zeta)$ contains no point that belongs to $W_1$, since $\zeta$ is a boundary point of $\pi(W_1)$ and $\pi$ is an open mapping on $W_1$. Therefore a point in $\pi^{-1}(\zeta)$ is either in $bW_1 \cap bW_2$ or in $bW_1 \setminus \overline{W_2}$. The later case can hold only for points lying in a set of zero length, as we showed in Lemma 3.5. Therefore for almost all points $\zeta \in \delta$, $\pi^{-1}(\zeta) \subset bW_1 \cap bW_2$, which implies that $\delta \subset \pi(bW_1) \cap \pi(bW_2) = \pi(\Gamma)$. Since $\pi(W_1) \cap \pi(W_2) = \phi$, $\pi(W_2) \subset D_0$ and $\delta = \pi(\Gamma)$ that is impossible by our choice of the set $\delta$. This completes the proof.

LEMMA 3.7. *Let $D$ be either $D_1$ or $D_2$. Then* int $(\bar{D})$ *is a Jordan domain, i.e., a simply connected domain with rectifiable simple closed curve as its boundary.*

PROOF. For convenience, we denote by $D$ the simply connected domain $D_1$. Let $\mathbb{C} \setminus \bar{D} = \bigcup_{i=0}^{\infty} \Omega_i$ with each $\Omega_i$ connected and with $\Omega_0$ the unbounded component. Then

1. $b\Omega_i$ is connected, since the domain $D$ is simply connected. Therefore each domain $\Omega_i$ is simply connected.

2. $b\Omega_i \subset bD$. For if $p \in b\Omega_i$, then $p \notin D$, and therefore

$$p \in \big(bD \cup (\mathbb{C} \setminus \bar{D})\big) \cap b\Omega_i \subset bD.$$

3. $b\Omega_0$ is a rectifiable simple closed curve. This follows from our Lemma 2.2 since we know that both $D$ and $\Omega_0$ are simply connected with rectifiable boundaries and that $b\Omega_0 = \bar{D} \cap \overline{\Omega_0}$.

Let

$$bD \setminus b\Omega_0 = \bigcup_{i=1}^{\infty} \nu_i$$

where each $\nu_i$ is a rectifiable curve with one end lying on $b\Omega_0$, and $\nu_i \cap \nu_j = \phi$ for $i \neq j$. By Lemma 3.6, each $\nu_i$ is contained in the Jordan domain $D_0$ bounded by $b\Omega_0$, and each $\nu_i$ contains no closed simple rectifiable curve. To proceed the proof, we show first that $\bigcup_{i=1}^{\infty} \nu_i \subset \operatorname{int}(\bar{D})$.

Fix an $i$, say $i = 1$. Since $\nu_1$ is a rectifiable curve, Theorem 1.4 and 1.6 imply that for almost all points $p \in \nu_1$, following property holds:

There exist $p_1, p_2$ in $\nu_1$ (depending on $p$), a rectifiable arc $\overline{p_1p_2} \subset \nu_1$ with $p_1, p_2$ as endpoints and with $p$ as an interior point, and two simply connected domains $U_1$ $U_2$ contained in $D$ such that $U_1 \cap U_2 = \phi$, $\overline{U_1} \cap \overline{U_2} = \overline{p_1p_2}$.

Thus $\overline{U_1} \cup \overline{U_2} \subset \bar{D}$, and $p \in \operatorname{int}(\overline{U_1} \cup \overline{U_2})$, which is a nonempty open set contained entirely in $\operatorname{int}(\bar{D})$. Therefore $p \in \operatorname{int}(\bar{D})$. Thus if we let

$$\nu_i' = \{p \in \nu_i : (D, \nu_i) \text{ has property } (Q) \text{ at point } p\},$$

then $\nu_i' \subset \operatorname{int}(\bar{D})$. Let $\nu = \bigcup_{i=1}^{\infty}(\nu_i \setminus \nu_i')$. Then $\Lambda^1(\nu) = 0$. Hence for each $p \in \nu$, there exists a disc $B_\delta(p)$ with center $p$, with radius $\delta$, such that $bB_\delta(p) \cap \nu = \phi$ and $B_\delta(p) \subset D_0$.

Since $B_\delta(p) \cap (\bigcup_{i=1}^{\infty} \nu_i') \subset \operatorname{int}(\bar{D})$, $B_\delta(p) \setminus \nu \subset \operatorname{int}(\bar{D})$. Now as $\Lambda^1(\nu) = 0$, every point in $\nu$ can be approached by sequence of points that are in $B_\delta(p)$. Therefore

$$\nu \subset \overline{B_\delta(p)} \subset \overline{\operatorname{int}(\bar{D})} = \bar{D},$$

*i.e.*, $B_\delta(p) \subset \bar{D}$. So $p \in \operatorname{int}(\bar{D})$, which shows that $\nu \subset \operatorname{int}(\bar{D})$, and hence $\bigcup_{i=1}^{\infty} \nu_i \subset \operatorname{int}(\bar{D})$.

By this claim, we can easily finish the proof of Lemma 3.7 by observing following inclusions,

$$D_0 = D \cup (\bigcup_{i=1}^{\infty} \nu_i) \subset \operatorname{int}(\bar{D}) \subset D_0.$$

STEP 3. Final Proof.

We define for $\zeta \in D_i \setminus \pi(K_1)$ holomorphic functions $\{\alpha_j^i(\zeta)\}_{j=1}^{\mu}$ by

$$\pi^{-1}(\zeta) \cap W_i = \{\alpha_1^i(\zeta), \ldots, \alpha_\mu^i(\zeta)\},$$

where $K_1$ is the set of singular locus for the proper mapping $\pi$ on $V \cap B$. Then each $\alpha_j^i$ is a bounded holomorphic function in its domain. Since the set $K_1$ has zero length, $\alpha_j^i$ can be extended as a bounded holomorphic function to $D_i$. Therefore, we can first extend these bounded holomorphic functions to the corresponding Jordan domain $\operatorname{int}(\overline{D_i})$ by invoking Theorem 2.3. Denote the extended function by $\alpha_j^i$. Moreover, the nontangential limits of $\alpha_j^i$ exist almost everywhere on the set $\gamma_0$,and, by rearranging the order among these functions, we obtain

$$\text{n.t.} \lim_{z \in D_1, z \to \zeta} \alpha_j^1(z) = \text{n.t.} \lim_{z \in D_2, z \to \zeta} \alpha_j^2(z).$$

Thus we can extend the function $\alpha_j^1$ analytically across rectifiable arc $\gamma_0$ to $D_2$, by Theorem 2.1, and obtain an analytic function $G_j$ in $\operatorname{int}(\overline{D_1} \cup \overline{D_2})$ that contains $\operatorname{int}(\gamma_0)$. Therefore $(\overline{W_1} \cup \overline{W_2}) \cap B$ is analytic near the points on $\pi^{-1}(\gamma_0)$. Since for almost all points on $\gamma$ property $(Q)$ holds, that implies that $(\overline{W_1} \cup \overline{W_2}) \cap B$ is a variety. Our theorem has been proved for $n = 2$.

Now we turn to the proof of the theorem for general $n > 2$. The main principle is the same as we already did for $n = 2$, while minor changes need to be made. Take a point $p \in E_0$. Then, as $\Lambda^3(\bar{V}) = 0$, we can find a coordinate system $\{z_1, \ldots, z_n\}$ with $p$ and for each fixed $i$ a small neighborhood $B_i$ of $p$ such that the projection map $\pi_i : \mathbb{C}^n \longrightarrow \mathbb{C}_{z_i}^1$ is proper on $\bar{V} \cap B_i$. Fix an $i$, say $i = 1$ and let $z' = (z_2, \ldots, z_n)$. We retain the notations from above proof and use so-called *canonical defining functions*, used in the book by Chirka ([Ch], p. 47) to replace the polynomials $F_j^1$ by a system of polynomials $\{\Phi_I^j(z_1, z')\}_{|I| = \mu_j^1}$ that are polynomials in $z_1$ with degree $\mu_j^1$ and have holomorphic coefficients in $z'$, *i.e.*, on each $\pi_1^{-1}(U_j^1) \cap V \cap B \setminus E'$,

$$\Phi_I^j(z_1, z') = \sum_{|J| \leq \mu_j^1} \phi_{IJ}(z_1)(z')^J, \quad |I| = \mu_j^1.$$

Thus away from a discrete set $E'$, which is the set of branch points for the projection $\pi_1|_{V\cap B}$, the common zeros

$$\alpha^j(z_1) = \{\alpha^j_1(z_1), \ldots, \alpha^j_{\mu^1_j}(z_1)\}$$

of above system are the points of the fiber $\pi_1^{-1}(z_1) \cap V \cap B$. Each vector-valued holomorphic function $\alpha^j_i(z_1)$ is bounded and hence extends over $U^1_j$, which we still denote by $\alpha^j_i(z_1)$. Therefore,

$$(*) \qquad \pi_1^{-1}(z_1) \cap \bar{V} \cap B \supseteq \bigcap_{|I|=\mu^1_j} \{\Phi^j_I(z_1, z') = 0\}$$

for $z_1 \in \overline{U^1_j} \cap \pi_1(E \cap B)$. For $m = n-1$, define a $\binom{m+\mu^1_j-1}{m-1} \times m$ matrix

$$M^j(z_1, z') = \left(\frac{\partial \Phi_I(z_1, z')}{\partial z'}\right).$$

Then $(z_1, z') \in E'$ if and only if rank $M^j < m$. We claim that for almost all (with respect to $\Lambda^1$) $z_1 \in \pi_1(E \cap B) \cap \overline{U^1_j}$, if $1 \le i_1 < i_2 \le \mu^1_j$, then $\alpha^j_{i_1}(z_1) \ne \alpha^j_{i_2}(z_1)$, *i.e.*, $\pi_1^{-1}(z_1) \cap \bar{V} \cap B$ consists of $\mu^1_j$ distinct points and equality in $(*)$ above holds. The reason for our claim is that the discriminant of every $m \times m$ minor in $M^j(z_1, z')$ is a bounded holomorphic function on $\pi_1^{-1}(U^1_j)$, and there is at least one $m \times m$ minor, say the first $m$ rows, that has non-zero determinant $\Delta^j(z_1, z')$ on $\pi_1^{-1}(U^1_j) \setminus E'$, *i.e.*, for every $z_1 \in U^1_j \setminus \pi_1(E')$, $\Delta^j\big(z_1, \alpha^j_i(z_1)\big) \ne 0$ for each $1 \le i \le \mu^1_j$. Our claim follows from the uniqueness theorem in the case $n = 2$. Thus, a similar argument in Lemma 3.4 and in its follow-up shows that there exist two indices $\nu$ and $\beta$ with $\alpha^j_\nu(z_1) = \alpha^k_\beta(z_1)$ for almost all $z_1 \in \Gamma^1_{jk}$, provided that $\Lambda^1(\Gamma^1_{jk}) > 0$. By applying removable singularity theorem in Section 2 to each component, we reach the conclusion that $V^1_j \cup V^1_k \cup \operatorname{int}(\Gamma^1_{jk})$ is an analytic variety if $\Lambda^1(\Gamma^1_{jk}) > 0$. Finally, for the set $\Gamma^1_{jk}$ with zero length, we use the same argument in the last part of the proof when $n = 2$ to show that $\bar{V} \cap \Omega$ is also analytic. Therefore our proof is complete.

Another way to reduce the problem from $n > 2$ to $n = 2$ is to use almost single-sheeted projections from Chirka's book [Ch], p. 38.

## 4. Some corollaries.

COROLLARY 4.1. *Let $E$ be a rectifiable curve in a domain $D$. If $V_1$ and $V_2$ are two 1-dimensional irreducible varieties in $D \setminus E$ such that $bV_1$ and $bV_2$ are two rectifiable curves, and such that $E \subset \overline{V_1} \cap \overline{V_2}$, then either $V_1 = V_2$, or $V_1 \cup E \cup V_2$ is a 1-dimensional variety in $D$.*

The proof of this corollary is a simple corollary of Theorem 3.2, after we prove that property $(Q)$ holds in this case. For this purpose, we need a lemma that tells us in case that the boundary of a 1-dimensional variety is a rectifiable curve, there can only be, for almost all points on the boundary, finitely many local branches; indeed the number of branches near almost all points on the boundary cannot exceed 2. The result was proved in various places. The most recent one was included in M. Lawrence's thesis [La].

LEMMA. *Let $V$ be a 1-dimensional variety in a domain $D$ such that the boundary $bV$ is connected and of finite length. Then for almost all points $p \in bV$ and for any neighborhood $U$ of $p$ in $\mathbb{C}^n$, there exists a neighborhood $B \subset U$ of $p$ such that the number of branches in $V \cap B$ that adjacent to $p$ is at most 2.*

The proof of this lemma is a standard argument by invoking analytic projections to certain normal coordinate planes. By using a result about amply adjacent neighborhoods for the image of the point $p$ under projections, the result follows from the uniqueness theorem for holomorphic functions. For details, see the proof of the lemma in Section 2.2 in [La].

PROOF OF COROLLARY 4.1. Suppose that $V_1 \neq V_2$. Then irreducibility of both $V_1$ and $V_2$ implies that $V_1 \cap V_2$ is of at most zero dimension. The lemma above gives that both $V_1$ and $V_2$ have at most 2 local branches adjacent to almost all points on $E$. Thus $E \subset \overline{V_1} \cap \overline{V_2}$ implies that property $(Q)$ holds at almost all points of $E$, and our result follows from Theorem 3.2.

A special case of Corollary 4.1 when $E$ is a closed analytic arc was done by Globevnik and Stout in [GS] by applying a version of the "reflection principle" for varieties. Another version of our main result can be stated as following corollary, here we use $\mathbb{D}$ as the unit disc in $\mathbb{C}$.

COROLLARY 4.2. *Let $f$, $g$ be two proper holomorphic maps from $\mathbb{D}$ to a domain $\Omega$ in $\mathbb{C}^N$ that extent continuously to $\overline{\mathbb{D}}$. If there exist two arcs $I_f$ and $I_g$ contained in $b\mathbb{D}$ such that $\Gamma_1 = f(I_f)$, $\Gamma_2 = g(I_g)$ are two subsets in $b\Omega$ with $\Gamma_1 \cap \Gamma_2$ a rectifiable curve, then either $f(\mathbb{D}) = g(\mathbb{D})$, or $f(\mathbb{D}) \cup \operatorname{int}(\Gamma_1 \cap \Gamma_2) \cup g(\mathbb{D})$ forms a 1-dimensional complex variety.*

In particular, if $\Omega$ is a strictly pseudoconvex domain in $\mathbb{C}^N$ and if $f$ and $g$ are as above such that $f(b\mathbb{D}) \cap g(b\mathbb{D})$ contains a rectifiable curve, then $f(\mathbb{D}) = g(\mathbb{D})$.

It is an open problem whether the second part of above corollary still holds if $f(b\mathbb{D}) \cap g(b\mathbb{D})$ is a totally disconnected closed set with positive 1-dimensional Hausdorff measure, and $N > 2$. When $N = 2$ and $\Gamma_1, \Gamma_2$ are simple rectifiable curves, the problem was solved by Globevnik and Stout in [GS] under the assumption that $\Omega = \mathbb{B}_2$. Recently a similar version of this corollary was given by Alexander in [Al3].

PROOF OF COROLLARY 4.2. By properness of $f$ and $g$, $V_1 = f(\mathbb{D})$ and $V_2 = g(\mathbb{D})$ are two 1-dimensional irreducible complex varieties. Suppose that $f(\mathbb{D}) \neq g(\mathbb{D})$. Take an arbitrary point $p \in \operatorname{int}(\Gamma_1 \cap \Gamma_2)$, then for any neighborhood $U$ of $p$, there exists a smaller one $B$ of $p$ such that $f(\mathbb{D}) \cap B \neq g(\mathbb{D}) \cap B$ and $B \cap (\Gamma_1 \cap \Gamma_2) \subset \operatorname{int}(\Gamma_1 \cap \Gamma_2)$. For otherwise, $U \cap f(\mathbb{D}) = U \cap g(\mathbb{D})$ implies that $f(\mathbb{D}) = g(\mathbb{D})$ by the uniqueness theorem for irreducible varieties. Let $V = V_1 \cup V_2$. Since $f$ and $g$ are continuous on $\overline{\mathbb{D}}$, $\overline{f(\mathbb{D})} = f(\mathbb{D}) \cup f(b\mathbb{D})$ and $\overline{g(\mathbb{D})} = g(\mathbb{D}) \cup g(b\mathbb{D})$. Thus $V \cap B$ is a 1-dimensional variety in $B \setminus (\Gamma_1 \cap \Gamma_2)$. Let $W_1 = V_1 \cap B$ and $W_2 = V_2 \cap B$. Then $W_1 \cap W_2$ is a 0-dimensional variety (possibly empty) and $\big((\overline{W_1} \cap \overline{W_2}) \setminus (W_1 \cap W_2)\big) \cap B \subset f(b\mathbb{D}) \cap g(b\mathbb{D}) \cap B = \Gamma_1 \cap \Gamma_2 \cap B$. Our Corollary 4.1 implies that $V_1 \cup \operatorname{int}(\Gamma_1 \cap \Gamma_2) \cup V_2$ is a 1-dimensional variety.

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When $\Omega$ is a strictly pseudoconvex domain in $\mathbb{C}^N$ and $\Gamma \subset f(b\mathbb{D}) \cap g(b\mathbb{D}) \subset b\Omega$ is a rectifiable curve, then either $f(\mathbb{D}) = g(\mathbb{D})$, or $V = f(\mathbb{D}) \cup \operatorname{int}(\Gamma) \cup g(\mathbb{D})$ is a 1-dimensional complex variety. But the latter case can not be happen, for otherwise the variety $V$ will meet the boundary $b\Omega$ of strictly pseudoconvex domain $\Omega$ in a set consisting of interior points of $V$, which contradicts the maximum principle for the strictly subharmonic functions. This leads to the conclusion that $f(\mathbb{D}) = g(\mathbb{D})$.

It can be easily seen that following is a variation of Corollary 4.1 and 4.2.

COROLLARY 4.3. *Let $V_1$ and $V_2$ be two 1-dimensional irreducible varieties contained in a strictly pseudoconvex domain D. If $\overline{V_1} \cap \overline{V_2}$ is a rectifiable curve that lies in the boundary of D, then $V_1 = V_2$.*

## REFERENCES

**[Al1]** H. Alexander, *Polynomial hulls and linear measure*, Lecture Notes in Math. **1276**(1987), 1–11.
**[Al2]** ———, *Areas of projections of analytic sets*, Invent. Math. **16**(1972), 335–341.
**[Al3]** ———, *The ends of varieties*, preprint.
**[Be]** A. Besicovitch, *On the fundamental geometric properties of linear measurable plane sets of points, II*, Math. Ann. **115**(1938), 296–329.
**[Ch]** E. Chirka, *Complex analytic sets*, Kluwer Acad. Publ., 1989.
**[CK]** E. Chirka, and G. Henkin, *Boundary properties of holomorphic functions of several complex variables*, J. Soviet Math. **5**(1976), 612–687.
**[Cu]** H. Cullen, *Introduction to general topology*, D. C. Heath and Company, 1967.
**[Fa]** K. Falconer, *The geometry of fractal sets*, Cambridge Univ. Press, 1985.
**[Fo]** F. Forstneric, *Regularity of varieties in strictly pseudoconvex domains*, Publ. Mat. **32**(1988), 145–150.
**[GS]** J. Globevnik, and E. Stout, *The ends of discs*, Bull. Soc. Math. France **114**(1986), 175–195.
**[Gl]** G. Goluzin, *Geometric theory of functions of a complex variable*, Amer. Math. Soc., 1969.
**[La]** M. Lawrence, *Polynomial hulls and geometric function theory of several complex variables*, Ph.D Thesis. Univ. of Washington, 1991.
**[Po]** Ch. Pommerenke, *On analytic functions with cluster sets of finite linear measure*, Michigan Math. J. **34**(1987), 93–97.
**[Ro]** W. Rothstein, *Zur theorie der analytischen Mengen*, Math. Ann. **174**(1967), 8–32.
**[Ru]** W. Rudin, *Subalgebra of spaces of continuous functions*, Proc. Amer. Math. Soc. **7**(1956), 825–830.
**[Sh]** B. Shiffman, *On the continuation of analytic sets*, Math. Ann. **185**(1970), 1–12.
**[Sh2]** ———, *On the removal of singularities of analytic sets*, Michigan Math. J. **15**(1970), 111–120.
**[St]** E. Stout, *The theory of uniform algebras*, Bogden and Quigley, Inc., Publ., 1971.
**[Wa]** T. Ważewski, *Rectifiable continua in connection with absolutely continuous functions and mappings*, (Polish). Ann. Polon. Math. **3**(1927), 9–49.
**[We]** J. Wermer, *Polynomial approximation on an arc in $\mathbb{C}^3$*, Ann. Math. **62**(1955), 269–270.
**[Xu]** Y. Xu, *Extension of complex varieties across $C^1$ manifolds*, Michigan Math. J. **40**(1993), 399–410.

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