AN ESTIMATE FOR TRIGONOMETRICAL SUMS OVER SQUARE-FREE INTEGERS WITH A CONSTANT NUMBER OF PRIME FACTORS

by SEAN MC DONAGH

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1. In deriving an expression for the number of representations of a sufficiently large integer N in the form

$$N = p_1^k + p_2^k + \dots + p_{s(k)}^k,$$

where k is a positive integer, $s(k)$ a suitably large function of k and p_i is a prime number, $i = 1, 2, \dots, s(k)$, by Vinogradov's method it is necessary to obtain estimates for trigonometrical sums of the type

$$\sum_{p \leqq N^{\omega}} e(\alpha p^k) \tag{1}$$

where $\omega = 1/k$ and the real number α satisfies $0 \leqq \alpha \leqq 1$ and is " near " a rational number a/q, $(a, q) = 1$, with " large " denominator q. See Estermann (**1**), Chapter 3, for the case $k = 1$ or Hua (**2**), for the general case. The meaning of " near " and " large " is made clear below—Lemma 4—as it is necessary for us to quote Hua's estimate. In this paper, in Theorem 1, an estimate is obtained for the trigonometrical sum

$$\sum_{\pi_r \leqq N^{\omega}} e(\alpha \pi_r^k) \tag{2}$$

where α satisfies the same conditions as above and where π_r denotes a square-free number with r prime factors. This estimate enables one to derive expressions for the number of representations of a sufficiently large integer N in the form

$$N = \pi_{r_1}^k + \pi_{r_2}^k + \dots + \pi_{r_{s(k)}}^k,$$

where $s(k)$ has the same meaning as above and where π_{r_i}, $i = 1, 2, \dots, s(k)$, denotes a square-free integer with r_i prime factors.

The method used is to make sums of type (2) depend on sums of type (1) by repeated use of the equality

$$\sum_{\pi_r \leqq H} \phi(\pi_r) = \frac{1}{r}\left(\sum_{j=1}^{r} (-1)^{j-1} \sum_{\substack{p \\ p^j \pi_{r-j} \leqq H}} \sum_{\pi_{r-j}} \phi(p^j \pi_{r-j})\right), \tag{3}$$

which expresses a sum of a function ϕ over square-free integers with r prime factors in terms of sums of ϕ over integers expressible as the product of a prime power and square-free integers with fewer, $r-j$, $1 \leqq j \leqq r$, prime factors.

2. In obtaining our estimates we use the following notation. As already stated, N denotes a sufficiently large positive integer, α denotes a real number and $e(\alpha)$ stands for $e^{2\pi i\alpha}$; p, with or without a suffix, denotes a prime, π_r, $r = 0, 1, 2, \ldots$, is a square free integer with r prime factors, so that $\pi_0 = 1$ and π_1 is a prime, k is a positive integer and $\omega = 1/k$. $\sigma, \sigma_0, \sigma_1, \sigma_2$, are positive real numbers. θ is a real number satisfying $-1 \leqq \theta \leqq 1$. Constants implied by the $\ll$ and O notations depend, at most, on r and k.

3. We need the following lemmas.

Lemma 1. *Let* $\left|\alpha - \frac{q}{q}\right| \leqq \frac{1}{q^2}$, *with* $(a, q) = 1$, *and*

$$\Omega_1 = \sum_{m=f+1}^{f+q} \min\left(U, \frac{1}{2\{\alpha m\}}\right),$$

then

$$\Omega_1 \leqq 6U + q\log q.$$

Proof. Hua (2), Hilfssatz 3.5.

Lemma 2. *Let* $1 \leqq a \leqq q$, $(a, q) = 1$, *with*

$$(\log N)^{\sigma} < q \leqq N(\log N)^{-\sigma},$$

Define

$$\Omega_2 = \sum_d \sum_m e\left(\frac{a}{q}(md)^k\right),$$

where d runs through the positive integers satisfying $D < d \leqq D'$, $D' \ll D$, with $(\log N)^{\sigma_1} < D \leqq N^{\omega}(\log N)^{-\sigma_1}$, and m runs through an increasing sequence of positive integers with $m \leqq N^{\omega}d^{-1}$. Then

$$\Omega_2 \ll N^{\omega}(\log N)^{-\sigma_0}$$

if σ and σ_1 satisfy

$$\min(\sigma, \sigma_1) \geqq 2^{2k+1}\sigma_0 + 2^{3(2k-1)}.$$

Proof. Lemma 2 is Hua (2), Hilfssatz 6.3, with $P = N^{\omega}$, $f(x) = x^k$, $\sigma_3 = 0$, so that $l = 1$, $\sigma_5 = \sigma_1$ and $\sigma_6 = \sigma_1$.

In proving Hilfssatz 6.3, Hua uses Lemma 1 with $\alpha = a/q$. Using the same method of proof and the full strength of Lemma 1, Lemma 2 is seen still to hold if, in the expression for Ω_2, a/q is replaced by α, with $|\alpha - a/q| \leqq 1/q^2$. Indeed, the same method of proof yields the following lemma.

Lemma 3. *Let* $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$, $1 \leqq a \leqq q$, $(a, q) = 1$, *with*

$$(\log N)^{\sigma} < q \leqq N(\log N)^{-\sigma}.$$

Define

$$S_1 = \sum_d \xi(d) \sum_m e(\alpha(md)^k),$$

where d runs through the positive integers satisfying $D < d \leqq D'$, $D' \ll D$, with

$(\log N)^{\sigma_1} < D \leqq N^{\omega}(\log N)^{-\sigma_1}$, *m runs through an increasing sequence of positive integers with* $m \leqq N^{\omega}d^{-1}$ *and* $\xi(d) = O(1)$. *Then*

$$S_1 \ll N^{\omega}(\log N)^{-\sigma_0},$$

if σ *and* σ_1 *satisfy*

$$\min(\sigma, \sigma_1) \geqq 2^{2k+1}\sigma_0 + 2^{3(2k-1)}.$$

Proof. In proving Lemma 2, Hua uses the inequality (**2**), p. 68

$$|\Omega_2|^2 \leqq D \sum_d \left| \sum_m e\left(\frac{a}{q}(md)^k\right)\right|^2.$$

The proof depends on the derivation of an upper bound for the expression on the right hand side. By using the full force of Lemma 1, the same upper bound is seen to be an upper bound for

$$D \sum_d \left| \sum_m e(\alpha(md)^k)\right|^2,$$

if $|\alpha - a/q| \leqq 1/q^2$. Hence Lemma 3 can be proved using the same method of proof as that of Lemma 2, for since $\xi(d) = O(1)$, we have

$$|S_1|^2 \ll D \sum_d \left| \sum_m e(\alpha(md)^k)\right|^2.$$

Lemma 4. *Let* $1 \leqq a \leqq q$, $(a, q) = 1$, *with*

$$(\log N)^{\sigma} < q \leqq N(\log N)^{-\sigma}.$$

Define

$$\Omega_3 = \sum_{N^{\omega/2} \leqq p \leqq N^{\omega}} e\left(\frac{a}{q} p^k\right).$$

Then

$$\Omega_3 \ll N^{\omega}(\log N)^{-\sigma_0},$$

if

$$\sigma \leqq 2^{6k}(\sigma_0 + 1).$$

Proof. This is Hua (**2**), Satz 10, with $f(p) = p^k$, $Q = 1$ and $P = N^{\omega}$.

Again, the use of the full force of Lemma 1 in Hua's proof of Lemma 4, rather than the particular case of $\alpha = a/q$, enables one to prove that Lemma 4 still holds if, in the expression for Ω_3, a/q is replaced by α, with $\left|\alpha - \dfrac{a}{q}\right| \leqq \dfrac{1}{q^2}$. Indeed, the proof can be further modified to yield the following lemma.

Lemma 5. *Let* $1 \leqq G \leqq (\log N)^{\sigma_2}$. *Let* $\alpha = \dfrac{\alpha}{q} + \dfrac{\theta}{q^2}$, *with* $1 \leqq a \leqq q$, $(a, q) = 1$, *and*

$$(\log N)^{\sigma} < q \leqq N(\log N)^{-\sigma}.$$

Define

$$S_2 = \sum_{1 \leqq v \leqq G} \zeta(v) \sum_{N^{\omega/2} \leqq p \leqq N^{\omega}v^{-1}} e(\alpha v^k p^k),$$

where $\zeta(v)$, *for* $v = 1, 2, \ldots, G$, *satisfies* $0 \leqq \zeta(v) \ll 1$. *Then*

$$S_2 \ll N^{\omega}G(\log N)^{-\sigma_0},$$

if

$$\sigma \geqq 2^{6k}(\sigma_0 + 1).$$

Proof. The further modification of Hua's proof of Lemma 4 necessary to include the summation over v and the function $\zeta(v)$ that occur in the expression for S_2 is simple. The proof of Lemma 5 is exactly parallel to Hua's proof ((2), pp. 71-76), with the same values for the constants λ_1, λ_2, λ_3 that occur in that proof.

4. We can now prove the following theorem.

Theorem 1. *Let* $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$, $1 \leqq a \leqq q$ *and* $(a, q) = 1$ *with*

$$(\log N)^{\sigma} < q \leqq N(\log N)^{-\sigma}.$$

Then

$$S = \sum_{\pi_r \leqq N^{\omega}} e(\alpha\pi_r^k) \ll N^{\omega}(\log N)^{-\sigma_0}, \quad r = 1, 2, \ldots,$$

if

$$\sigma \geqq 2^{6k}(2^{2(k+1)}+1)\sigma_0 + 2^{6k}(2^{2(k+1)}+2^{6k-2}+1).$$

Proof. If $r = 1$, we have, by Lemma 5, with $G = 1$ and $\zeta(1) = 1$,

$$S \ll N^{\omega}(\log N)^{-\sigma_0} + N^{\omega/2} \ll N^{\omega}(\log N)^{-\sigma_0},$$

since $\sigma > 2^{6k}(\sigma_0+1)$.

Let $r > 2$, t be a positive integer $\leqq r-2$, and λ_i, $i = 1, 2, \ldots, t$, be positive integers such that $\lambda = \sum_{i=1}^{t} \lambda_i \leqq r-2$. Let $\sigma_3 = 2^{2k+1}(\sigma_0+1)+2^{3(2k-1)}$ and write $A = (\log N)^{\sigma_3}$. We use the notation Σ' throughout the remainder of this section to denote summation over t primes $p_1, \ldots, p_t$, satisfying $p_1^{\lambda_1} \ldots p_t^{\lambda_t} \leqq A$. In such sums we write $b = p_1^{\lambda_1} \ldots p_t^{\lambda_t}$ so that $b \leqq A$.

We now show that, if

$$S_3 = \Big| \sum{}' \sum_{\pi_{r-\lambda} \leqq N^{\omega} b^{-1}} e(\alpha\{b\pi_{r-\lambda}\}^k) \Big|$$

then

$$S_3 \leqq \sum_{j=1}^{r-\lambda-2} \Bigg| \sum{}' \sum_{\substack{p \\ bp^j \leqq A}} \sum_{\pi_{r-\lambda-j} \leqq N^{\omega}(bp^j)^{-1}} e(\alpha\{bp^j\pi_{r-\lambda-j}\}^k) \Bigg| + O(N^{\omega}(\log N)^{-\sigma_0}) \quad (4)$$

where, if the sum over j is empty i.e. if $\lambda = r-2$, it is defined to be zero. Repeated applications of (4) will enable us to establish Theorem 1.

Using equality (3), we write

$$S_3 = \Bigg| \sum{}' \sum_{j=1}^{r-\lambda} \frac{(-1)^{j-1}}{r-\lambda} \sum_{p} \sum_{\substack{\pi_{r-\lambda-j} \\ p^j\pi_{r-\lambda-j} \leqq N^{\omega} b^{-1}}} e(\alpha\{bp^j\pi_{r-\lambda-j}\}^k) \Bigg|$$

$$\leqq \sum_{j=1}^{r-\lambda} \Bigg| \sum{}' \sum_{p} \sum_{\substack{\pi_{r-\lambda-j} \\ p^j\pi_{r-\lambda-j} \leqq N^{\omega} b^{-1}}} e(\alpha\{bp^j\pi_{r-\lambda-j}\}^k) \Bigg|$$

$$= \sum_{j=1}^{r-\lambda} | F_j |, \text{ say.}$$

Clearly, since $r-\lambda \geqq 2$, we have

$$|F_{r-\lambda}| \leqq N^{\omega/r-\lambda}\Sigma' 1 \ll N^{\omega}(\log N)^{-\sigma_0}.$$

If $j<r-\lambda$, we write F_j in the form

$$F_j = F_j^{(1)}+F_j^{(2)}+F_j^{(3)}$$

where $F_j^{(1)}$ is the sum got from the sum which is F_j by imposing on the variables of summation the further condition $bp^j \leqq A$, $F_j^{(2)}$ is the sum got from F_j by imposing the condition $bp^j \geqq N^{\omega}A^{-1}$. $F_j^{(3)}$ is therefore got from the sum F_j by imposing the condition $A<bp^j<N^{\omega}A^{-1}$.

To estimate $F_j^{(3)}$ the range of values of bp^j is split into $\ll \log N$ ranges of the type

$$D<bp^j \leqq D', \quad D' \ll D.$$

In this way $F_j^{(3)}$ is expressed as the sum of $\ll \log N$ sums of the type considered in Lemma 3, with $\pi_{r-\lambda-j} = m$, $\sigma_1 = \sigma_3$ and $\xi(d)$ denoting the number of representations of d in the form $d = bp^j$. $\xi(d)$ is therefore $O(1)$ since it denotes the number of representations of d as the product of less than $r-\lambda$ prime factors. So, by Lemma 3, we have

$$F_j^{(3)} \ll \log N.\quad N^{\omega}(\log N)^{-(\sigma_0+1)} = N^{\omega}(\log N)^{-\sigma_0},$$

since σ and σ_3 satisfy

$$\min(\sigma, \sigma_3) \geqq 2^{2k+1}(\sigma_0+1)+2^{3(2k-1)}.$$

If $j>1$, then clearly

$$F_j^{(2)} \ll N^{\omega}\sum{}' \sum_{\substack{p \\ p^j \geqq N^{\omega}A^{-2}}} p^{-j} \ll N^{\omega}(\log N)^{-\sigma_0}.$$

To estimate $F_1^{(2)}$, we write

$$F_1^{(2)} = \sum{}' \sum_{\pi_{r-\lambda-1} \leqq A} \;\sum_{N^{\omega/2} \leqq p \leqq N^{\omega}(b\pi_{r-\lambda-1})^{-1}} e(\alpha\{bp\pi_{r-\lambda-1}\}^k)$$

$$-\sum{}' \sum_{\substack{p \geqq N^{\omega/2} \\ bp \leqq N^{\omega}A^{-1}}} \;\sum_{\pi_{r-\lambda-1} \leqq A} e(\alpha\{bp\pi_{r-\lambda-1}\}^k) = F_1^{(2)}(1)-F_1^{(2)}(2),$$

say. Lemma 5 is applicable to $F_1^{(2)}(1)$ with $G = A^2$ and $\zeta(v)$ denoting the number of representations of v in the form $v = b\pi_{r-\lambda-1}$. Then $0 \leqq \zeta(v) \ll 1$ and so, by Lemma 5, we have

$$F_1^{(2)}(1) \ll N^{\omega}A^2(\log N)^{-(\sigma_0+2\sigma_3)} \ll N^{\omega}(\log N)^{-\sigma_0}$$

since

$$\sigma \geqq 2^{6k}(\sigma_0+2\sigma_3+1) = 2^{6k}(2^{2(k+1)}+1)\sigma_0+2^{6k}(2^{2(k+1)}+2^{6k-2}+1).$$

We estimate $F_1^{(2)}(2)$ by splitting the range of values of bp into $\ll \log N$ subranges of the type

$$D<bp \leqq D', \quad D' \ll D.$$

Lemma 3 is applicable to the sum over each subrange, with $m = \pi_{r-\lambda-1}$, $\sigma_1 = \sigma_3$ and $\xi(d)$ denoting the number of representations of d in the form $d = bp$ with $p \geqq N^{\omega/2}$. Thus $\xi(d) = O(1)$ since it does not exceed the number of representations of d as the product of $\lambda+1$ primes. Hence, by Lemma 3, we have

$$F_1^{(2)}(2) \ll \log N \,.\, N^{\omega}(\log N)^{-(\sigma_0+1)} = N^{\omega}(\log N)^{-\sigma_0},$$

since σ and σ_3 satisfy

$$\min(\sigma, \sigma_3) \geqq 2^{2k+1}(\sigma_0+1)+2^{3(2k-1)}.$$

It remains to consider $F^1_{r-\lambda-1}$. Now

$$F^{(1)}_{r-\lambda-1} = \sum' \sum_{\substack{p \\ bp^{r-\lambda-1} \leqq A}} \; \sum_{\substack{\pi_1 \\ N^{\omega/2} \leqq \pi_1 \leqq N^{\omega}(bp^{r-\lambda-1})^{-1}}} e(\alpha\{bp^{r-\lambda-1}\pi_1\}^k)+O(N^{\omega/2}A).$$

Thus, applying Lemma 5, with $G = A$ and $\zeta(v)$ denoting the number of representations of v in the form $v = bp^{r-\lambda-1}$ – so that $0 \leqq \zeta(v) \ll 1$ – we have

$$F^{(1)}_{r-\lambda-1} \ll N^{\omega}\,.\,A\,.(\log N)^{-(\sigma_0+\sigma_3)}+N^{\omega/2}A \ll N^{\omega}(\log N)^{-\sigma_0},$$

since

$$\sigma \geqq 2^{6k}(\sigma_0+\sigma_3+1).$$

By combining the above estimates for $F^{(1)}_{r-\lambda-1}$, $F_j^{(2)}$ and $F_j^{(3)}$, and summing over j we get

$$S_3 \leqq \sum_{j=1}^{r-\lambda-2} |\, F_j^{(1)} \,| + O(N^{\omega}(\log N)^{-\sigma_0})$$

which is (4). In almost identical fashion, if $r \geqq 2$, it is shown that

$$|\, S \,| = \left| \sum_{\pi_r \leqq N^{\omega}} e(\alpha\pi_r^k) \right| \leqq \sum_{j=1}^{r-2} \left| \sum_{1 \leqq p^j \leqq A} \; \sum_{\pi_{r-j} \leqq N^{\omega}p^{-j}} e(\alpha\{p^j\pi_{r-j}\}^k) \right|$$
$$+O(N^{\omega}(\log N)^{-\sigma_0}), \qquad (5)$$

where an empty sum, i.e. if $r = 2$, is defined to be zero.

If $r = 2$, Theorem 1 is given by (5). If $r > 2$ relation (4) is applied repeatedly to the j-sum on the right-hand side of (5). The application of (4) $r-2$ times establishes the theorem.

5. Using Theorem 1 and Vinogradov's method one can derive expressions for the number of representations of a sufficiently large integer N in the form

$$N = \pi_{r_1}^k+\pi_{r_2}^k+\ldots+\pi_{r_{s(k)}}^k.$$

The Page-Siegel-Walfisz Theorem concerning the number of primes in an arithmetic progression is replaced by H. E. Richert's generations of it (3), Satz 1, concerning the number of square-free integers with a constant number of prime factors in an arithmetic progression. To illustrate, we now state the result, Theorem 2, proved by the author † concerning the case $k = 1$. We

† In a doctoral thesis, Trinity College, Dublin, 1965.

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define

$$W_{a,q} = \sum_{\substack{l=1 \\ (l,q)=1}}^{q} e\left(\frac{a}{q} l^k\right).$$

$\mu(q)$ and $\phi(q)$ denote the Möbius and Euler functions respectively, and B denotes Merten's constant.

Theorem 2. *Every sufficiently large odd integer N is representable in the form*

$$N = \pi_{r_1} + \pi_{r_2} + \pi_{r_3}$$

and the number of representations, $\rho(N)$, satisfies

$$\rho(N) \sim \frac{N^2}{2} \frac{(\log\log N)^{r_1+r_2+r_3-3}}{(\log N)^3} \left\{\prod_{i=1}^{3} \frac{1}{(r_i-1)!}\right\} \sum_{q=1}^{\infty} \frac{\mu(q) W_{N,q}}{\phi^3(q)}.$$

If $r_1+r_2+r_3 \geqq 4$, every sufficiently large integer N whether even or odd, is thus representable and the number, $\rho(N)$, of representations satisfies

$$\rho(N) \sim \frac{N^2}{2} \frac{(\log\log N)^{r_1+r_2+r_3-3}}{(\log N)^3} \left(\prod_{i=1}^{3} \frac{1}{(r_i-1)!}\right)$$

$$\left[\sum_{q=1}^{\infty} \frac{\mu(q) W_{N,q}}{\phi^3(q)} \left(1 + \frac{B \sum_{r_i > 1} (r_i - 1)}{\log\log N}\right) - (\log\log N)^{-1} \sum_{r_i > 1} (r_i - 1) \sum_{q=1}^{\infty} \frac{\mu(q) W_{N,q}}{\phi^3(q)} \sum_{p \mid q} 1\right].$$

The particular case $r_1 = r_2 = r_3 = 1$ of Theorem 2 is Vinogradov's Theorem (1), Chapter 3 and the case $r_1 = r_2 = 1$, $r_3 = 2$ was proved by Richert (3), Satz 2.

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University College
Galway