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ON THE DENSITY TYPE TOPOLOGIES IN HIGHER DIMENSIONS

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Abstract

The topologies of the density type in Euclidean space of dimension higher than one are introduced. Definitions are based on a notion of density point connected with a set of sequences of real numbers. Our purpose is to study properties of these topologies and connections between them.

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Following an observation that the notion of a density point (see [6]) of a measurable subset of the real line can be described by using a fixed sequence $\{n\}_{n\in\mathbb{N}}$, Filipczak and Hejduk [1] introduced the notion of a density point of a measurable subset of the real line with respect to a fixed unbounded and nondecreasing sequence of positive reals. They proved that this notion coincides with that of a classical density point if and only if the sequence in question tends to infinity not too fast.

We wish to investigate a similar notion, but on the plane and in Euclidean space of dimension higher than two, where, even in the classical case, the situation is more complicated (see [5, 6]). We shall use differentiation bases consisting of intervals of a special type.

We begin by recalling some basic definitions. Let $\mathcal{L}_2$ stand for the family of all Lebesgue measurable sets on the plane and let λ_2 stand for two-dimensional Lebesgue measure.

We say that a point $(x_0, y_0) \in \mathbb{R}^2$ is an ordinary density point of the set $A \in \mathcal{L}_2$ if and only if

$$\lim_{h\to 0^+} \frac{\lambda_2(A \cap ([x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]))}{4h^2} = 1.$$

We say that a point $(x_0, y_0) \in \mathbb{R}^2$ is a strong density point of the set $A \in \mathcal{L}_2$ if and only if

$$\lim_{h\to 0^+, k\to 0^+} \frac{\lambda_2(A \cap ([x_0 - h, x_0 + h] \times [y_0 - k, y_0 + k]))}{4hk} = 1.$$

Obviously if (x_0, y_0) is a strong density point of A then it is also an ordinary density point of A, but the converse need not be true.

As usual, let $\Phi_0(A)$ denote the set of all ordinary density points of a set $A \in \mathcal{L}_2$ and let $\Phi_s(A)$ denote the set of all strong density points of $A \in \mathcal{L}_2$.

For brevity, let $R((x, y), a, b)$ stand for the rectangle $(x-a, x+a) \times (y-b, y+b)$, where $x, y \in \mathbb{R}$, $a, b \in \mathbb{R}_+$, and $S((x, y), a) := R((x, y), a, a)$.

Let $\mathcal{S}$ be the family of all unbounded and nondecreasing sequences of positive reals. Sequences $\{s_n\}_{n\in\mathbb{N}} \in \mathcal{S}$ are denoted by $\langle s \rangle$. We divide $\mathcal{S}$ into two sets:

$$\mathcal{S}_0 := \left\{ \langle s \rangle \in \mathcal{S} : \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0 \right\}$$

and

$$\mathcal{S}_+ := \mathcal{S} \setminus \mathcal{S}_0 = \left\{ \langle s \rangle \in \mathcal{S} : \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} > 0 \right\}.$$

Definition 1. Let $\langle s \rangle, \langle t \rangle \in \mathcal{S}$. For $A \in \mathcal{L}_2$ we define an operator

$$\Phi_{\langle s \rangle \langle t \rangle}(A) = \left\{ (x, y) \in \mathbb{R}^2 : \lim_{n\to+\infty} \frac{\lambda_2(A \cap R((x, y), 1/s_n, 1/t_n))}{4/s_n t_n} = 1 \right\}.$$

It is well known (see [6]) that the fact that (x, y) belongs to $\Phi_0(A)$ is equivalent to the fact that

$$\lim_{n\to\infty} \frac{\lambda_2(A \cap R((x, y), h_n, k_n))}{4h_n k_n} = 1$$

for each pair of sequences of positive numbers $\{h_n\}_{n\in\mathbb{N}}$, $\{k_n\}_{n\in\mathbb{N}}$ tending to 0 and for which there exists a number $\alpha \in (0, 1)$ (called the parameter of regularity) such that $\alpha < h_n k_n^{-1} < \alpha^{-1}$ for each $n \in \mathbb{N}$.

With the latter we introduce a relation between sequences from $\mathcal{S}$.

Let $\langle s \rangle, \langle t \rangle \in \mathcal{S}$. We say that $\langle s \rangle$ is regular to $\langle t \rangle$ (written $\langle s \rangle \operatorname{reg} \langle t \rangle$) if there exists a number $\alpha \in (0, 1)$ such that $\alpha < s_n t_n^{-1} < \alpha^{-1}$ for each $n \in \mathbb{N}$.

Here are some elementary properties of this relation.

Property 2. The relation reg is an equivalence relation in $\mathcal{S}$.

Property 3. If $\langle s \rangle \in \mathcal{S}_+$ and $\langle t \rangle \in \mathcal{S}_0$ then $\langle s \rangle$ cannot be regular to $\langle t \rangle$.

Property 4. Let $\langle s \rangle, \langle t \rangle \in \mathcal{S}$, $\langle s \rangle \operatorname{reg} \langle t \rangle$ and put $k_n = \max(s_n, t_n)$ for $n \in \mathbb{N}$. Then $\langle k \rangle \operatorname{reg} \langle s \rangle$.

Just from the definitions and the condition equivalent to the definition of an ordinary density we get the following proposition.

Proposition 5. *For any $A \in \mathcal{L}_2$,*

$$\bigcap_{\substack{\langle s \rangle, \langle t \rangle \in \mathcal{S} \\ \langle s \rangle \operatorname{reg} \langle t \rangle}} \Phi_{\langle s \rangle \langle t \rangle}(A) = \Phi_0(A).$$

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COROLLARY 6. *For any* $A \in \mathcal{L}_2$ *and for any* $\langle s\rangle, \langle t\rangle \in \mathcal{S}$ *such that* $\langle s\rangle$ reg $\langle t\rangle$,

$$\Phi_0(A) \subset \Phi_{\langle s\rangle\langle t\rangle}(A).$$

Let $\langle n\rangle$ denote the increasing sequence of all natural numbers.

PROPOSITION 7. *For any* $A \in \mathcal{L}_2$,

$$\Phi_{\langle n\rangle\langle n\rangle}(A) \subset \Phi_0(A).$$

PROOF. Let $A \in \mathcal{L}_2$ and $(x, y) \in \Phi_{\langle n\rangle\langle n\rangle}(A)$, that is,

$$\lim_{n\to+\infty} \frac{\lambda_2(A \cap S((x, y), 1/n))}{4/n^2} = 1. \tag{1}$$

Let $\{h_n\}_{n\in\mathbb{N}}$ be a nonincreasing sequence tending to 0. Set

$$\underline{h}_n = \max\left\{\frac{1}{k} : k \in \mathbb{N} \wedge \frac{1}{k} \le h_n\right\} \quad \text{and} \quad \overline{h}_n = \min\left\{\frac{1}{k} : k \in \mathbb{N} \wedge \frac{1}{k} \ge h_n\right\}$$

for $n \in \mathbb{N}$. Then the quotients $\underline{h}_n/h_n$ and $\overline{h}_n/h_n$ tend to 1. Since

$$\frac{\underline{h}_n^2}{h_n^2}\,\frac{\lambda_2(A \cap S((x, y), \underline{h}_n))}{4\underline{h}_n^2} \le \frac{\lambda_2(A \cap S((x, y), h_n))}{4h_n^2} \le \frac{\overline{h}_n^2}{h_n^2}\,\frac{\lambda_2(A \cap S((x, y)\overline{h}_n))}{4\overline{h}_n^2},$$

Equation (1) gives

$$\lim_{h\to0^+} \frac{\lambda_2(A \cap S((x, y), h))}{4h^2} = 1. \qquad \square$$

COROLLARY 8. *For any* $A \in \mathcal{L}_2$,

$$\Phi_0(A) = \Phi_{\langle n\rangle\langle n\rangle}(A).$$

PROPOSITION 9. *For every* $\langle s\rangle \in \mathcal{S}_+$, $\langle u\rangle \in \mathcal{S}$ *and for every* $A \in \mathcal{L}_2$,

$$\Phi_{\langle s\rangle\langle s\rangle}(A) \subset \Phi_{\langle u\rangle\langle u\rangle}(A).$$

PROOF. Let $\langle s\rangle \in \mathcal{S}_+$, $\langle u\rangle \in \mathcal{S}$, $A \in \mathcal{L}_2$ and $(x, y) \in \Phi_{\langle s\rangle\langle s\rangle}(A)$. Denoting the complement of A by B, we assert that $\lim_{n\to+\infty} \lambda_2(B \cap S((x, y), 1/s_n))/(4/s_n^2) = 0$ and $\liminf_{n\to+\infty} s_n/s_{n+1} = g > 0$.

Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_0$, we get

$$\frac{\lambda_2(B \cap S((x, y), 1/s_n))}{4/s_n^2} < \epsilon \cdot \frac{g^2}{4} \quad \text{and} \quad \frac{s_n}{s_{n+1}} > \frac{g^2}{4}.$$

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There exists $k_0 \in \mathbb{N}$ such that $s_{n_o} \le u_{k_0}$. Fix $k \in \mathbb{N}$, $k > k_0$. There exists $n \in \mathbb{N}$, $n \ge n_0$, such that $s_n \le u_k \le s_{n+1}$. Thus

$$\begin{aligned}\frac{\lambda_2(B \cap S((x,y),1/u_k))}{4/u_k^2} &\le \frac{\lambda_2(B \cap S((x,y),1/s_n))}{4/(s_{n+1})^2} \\ &= \frac{\lambda_2(B \cap S((x,y),1/s_n))}{4/s_n^2} \cdot \left(\frac{s_{n+1}}{s_n}\right)^2 < \epsilon \cdot \frac{g^2}{4} \cdot \frac{4}{g^2} = \epsilon,\end{aligned}$$

so $(x, y) \in \Phi_{\langle u\rangle\langle u\rangle}(A)$. □

Corollary 10. *For every $\langle s\rangle \in \mathcal{S}_+$, $\langle u\rangle \in \mathcal{S}_0$ and for every $A \in \mathcal{L}_2$,*

$$\Phi_{\langle s\rangle\langle s\rangle}(A) = \Phi_0(A) \subset \Phi_{\langle u\rangle\langle u\rangle}(A).$$

Corollary 11. *For every $A \in \mathcal{L}_2$,*

$$\bigcap_{\langle s\rangle \in \mathcal{S}} \Phi_{\langle s\rangle\langle s\rangle}(A) = \Phi_0(A).$$

Proposition 12. *Let $\langle s\rangle \in \mathcal{S}$. If $\Phi_{\langle s\rangle\langle s\rangle}(A) = \Phi_0(A)$ for every $A \in \mathcal{L}_2$ then $\langle s\rangle \in \mathcal{S}_+$.*

Proof. Let $\langle s\rangle \in \mathcal{S}_0$. By [3, Theorem 3] there exists $Y \subset \mathbb{R}$ such that 0 is not a density point of Y and

$$\lim_{n\to+\infty} \frac{\lambda_1(Y \cap (-1/s_n, 1/s_n))}{2/s_n} = 1.$$

Define

$$A := \bigcup_{y \in Y \cap [0;+\infty)} ((\{-y, y\} \times [-y, y]) \cup ([-y, y] \times \{-y, y\})).$$

By [3, Corollary 2.7] the set A cannot have 0 as its ordinary density point on the plane.

Analysis similar to that in [3, proof of Theorem 2.6] shows that $(0, 0) \in \Phi_{\langle s\rangle\langle s\rangle}(A)$. □

Summarizing, we have the following theorem.

Theorem 13. *Let $\langle s\rangle \in \mathcal{S}$. The set of ordinary density points of A is equal to $\Phi_{\langle s\rangle\langle s\rangle}(A)$ for every $A \in \mathcal{L}_2$ if and only if the sequence $\langle s\rangle$ belongs to $\mathcal{S}_+$.*

We are led to the following stronger version of Corollary 11.

Proposition 14. *For every $A \in \mathcal{L}_2$,*

$$\bigcap_{\langle s\rangle \in \mathcal{S}_0} \Phi_{\langle s\rangle\langle s\rangle}(A) = \Phi_0(A).$$

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PROOF. One inclusion comes from Corollary 10.

To show the second one, suppose that there exists a point $(x, y) \in \bigcap_{\langle s\rangle \in \mathcal{S}_0} \Phi_{\langle s\rangle\langle s\rangle}(A) \setminus \Phi_0(A)$. Then

$$\liminf_{h\to 0^+} \frac{\lambda_2(A \cap S((x, y), h))}{4h^2} < 1,$$

hence there exists a sequence $\langle s\rangle \in \mathcal{S}$ such that

$$\lim_{n\to+\infty} \frac{\lambda_2(A \cap S((x, y), 1/s_n))}{4/s_n^2} < 1.$$

We choose a subsequence $\langle t\rangle \subset \langle s\rangle$ such that $\langle t\rangle \in \mathcal{S}_0$. Thus

$$\lim_{n\to+\infty} \frac{\lambda_2(A \cap S((x, y), 1/t_n))}{4/t_n^2} < 1,$$

so $(x, y) \notin \Phi_{\langle s\rangle\langle s\rangle}(A)$, which is a contradiction. □

We wish to investigate whether we obtain something different by considering rectangles described by sequences.

PROPOSITION 15. *For every $\langle s\rangle, \langle t\rangle \in \mathcal{S}_+$ such that $\langle s\rangle$ reg $\langle t\rangle$, $\Phi_{\langle s\rangle\langle t\rangle}(A) \subset \Phi_0(A)$ for every $A \in \mathcal{L}_2$.*

PROOF. Let $A \in \mathcal{L}_2$, $\langle s\rangle, \langle t\rangle \in \mathcal{S}_+$, $\langle s\rangle$ reg $\langle t\rangle$. Put $k_n = \max(s_n, t_n)$ for every $n \in \mathbb{N}$. It suffices to prove that $\Phi_{\langle s\rangle\langle t\rangle}(A) \subset \Phi_{\langle k\rangle\langle k\rangle}(A)$, since Properties 4 and 3 show that $\langle k\rangle \in \mathcal{S}_+$ and $\Phi_{\langle k\rangle\langle k\rangle}(A) = \Phi_0(A)$, by Theorem 13.

Suppose that there exists a point $(x, y) \in \Phi_{\langle s\rangle\langle t\rangle}(A) \setminus \Phi_{\langle k\rangle\langle k\rangle}(A)$. Let $B := \mathbb{R}^2 \setminus A$. Then

$$\limsup_{n\to+\infty} \frac{\lambda_2(B \cap S((x, y), 1/k_n))}{4/k_n^2} > 0,$$

so there exist $\gamma > 0$ and a subsequence $\{k_{n_l}\}_{l\in\mathbb{N}}$ of $\{k_n\}$ such that

$$\lim_{l\to+\infty} \frac{\lambda_2(B \cap S((x, y), 1/k_{n_l}))}{4/k_{n_l}^2} = \gamma.$$

From this there exists $l_0 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l > l_0$,

$$\frac{\lambda_2(B \cap S((x, y), 1/k_{n_l}))}{4/k_{n_l}^2} > \frac{\gamma}{2}.$$

Hence

$$\frac{\lambda_2(B \cap R((x, y), 1/s_{n_l}, 1/t_{n_l}))}{4/(s_{n_l}t_{n_l})} \geq \frac{\lambda_2(B \cap S((x, y), 1/k_{n_l}))}{4/k_{n_l}^2} \cdot \frac{t_{n_l}}{k_{n_l}} \cdot \frac{s_{n_l}}{k_{n_l}} > \frac{\gamma}{2}\lambda^2 > 0,$$

and $(x, y) \notin \Phi_{\langle s\rangle\langle t\rangle}(A)$, which is a contradiction. □

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COROLLARY 16. *If* $\langle s\rangle, \langle t\rangle \in \mathcal{S}_+$ *and* $\langle s\rangle$ reg $\langle t\rangle$, *then* $\Phi_0(A) = \Phi_{\langle s\rangle\langle t\rangle}(A)$ *for every* $A \in \mathcal{L}_2$.

PROPOSITION 17. *Let* $\langle s\rangle, \langle t\rangle \in \mathcal{S}_+$ *or* $\langle s\rangle, \langle t\rangle \in \mathcal{S}_0$. *If* $\Phi_{\langle s\rangle\langle t\rangle}(A) = \Phi_0(A)$ *for every* $A \in \mathcal{L}_2$, *then* $\langle s\rangle$ reg $\langle t\rangle$.

PROOF. Let $\langle s\rangle, \langle t\rangle \in \mathcal{S}_+$ or $\langle s\rangle, \langle t\rangle \in \mathcal{S}_0$. Assume that $\langle s\rangle$ reg $\langle t\rangle$ fails, that is, for every $\alpha \in (0, 1)$ there exists $n \in \mathbb{N}$ such that $(s_n/t_n) \le \alpha$ or $(s_n/t_n) \ge (1/\alpha)$. Therefore there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\{s_{n_k}/t_{n_k}\}_{k\in\mathbb{N}}$ tends monotonically to $+\infty$ or to zero. We will assume that the second case holds, for the first case is analogous. We will assume additionally, by choosing a subsequence if necessary, that $s_{n_{k+1}} > 2s_{n_k}$.

Define a function $f : (0, 1/2s_{n_1}] \to \mathbb{R}$, where $f(1/2s_{n_k}) = 1/t_{n_k}$ for $n \in \mathbb{N}$ and f is linear and continuous on the intervals $[1/2s_{n_{k+1}}, 1/2s_{n_k}]$, $k \in \mathbb{N}$.

Since for every $x \in (1/2s_{n_{k+1}}, 1/2s_{n_k})$ the quotient $f(x)/x$ is between

$$\frac{f(1/2s_{n_k})}{1/2s_{n_k}} = \frac{2s_{n_k}}{t_{n_k}} \quad\text{and}\quad \frac{f(1/2s_{n_{k+1}})}{1/2s_{n_{k+1}}} = \frac{2s_{n_{k+1}}}{t_{n_{k+1}}},$$

it follows that

$$\lim_{x\to 0^+} \frac{f(x)}{x} = 0.$$

Set

$$A = [-1, 1]^2 \setminus \{(x, y) : x \in (0, 1/2s_{n_1}) \wedge y \in (0, f(x))\}.$$

Then

$$\frac{\lambda_2(A \cap [-h, h]^2)}{4h^2} \ge \frac{3h^2 + h(h - f(h))}{4h^2} = 1 - \frac{f(h)}{4h} \to 1$$

for $h \to 0^+$, so $(0, 0) \in \Phi_0(A)$. However,

$$\lambda_2((\mathbb{R}^2 \setminus A) \cap R((0, 0), 1/s_{n_k}, 1/t_{n_k})) \ge \frac{1}{2} \cdot \frac{1}{s_{n_k}} \cdot \frac{1}{t_{n_k}},$$

therefore $(0, 0) \notin \Phi_{\langle s\rangle\langle t\rangle}(A)$. Thus we have found a set A for which $\Phi_{\langle s\rangle\langle t\rangle}(A) \ne \Phi_0(A)$. □

Summarizing Corollary 16 and Proposition 17, we have the following theorem.

THEOREM 18. *Let* $\langle s\rangle, \langle t\rangle \in \mathcal{S}_+$. *The set of ordinary density points of* A *is equal to* $\Phi_{\langle s\rangle\langle t\rangle}(A)$ *for every* $A \in \mathcal{L}_2$ *if and only if* $\langle s\rangle$ reg $\langle t\rangle$.

PROPOSITION 19. *For every* $A \in \mathcal{L}_2$,

$$\bigcap_{\langle s\rangle,\langle t\rangle\in\mathcal{S}} \Phi_{\langle s\rangle\langle t\rangle}(A) = \Phi_s(A).$$

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PROOF. Let $A \in \mathcal{L}_2$. By the Heine definition of limit, each point of strong density of A belongs to $\Phi_{\langle s\rangle\langle t\rangle}(A)$ for every $\langle s\rangle, \langle t\rangle \in \mathcal{S}$.

Suppose that there exists a point (x, y) belonging to $\bigcap_{\langle s\rangle, \langle t\rangle\in\mathcal{S}} \Phi_{\langle s\rangle\langle t\rangle}(A)$ but which is not a strong density point of A. Hence there exist decreasing sequences $\{k_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ tending to 0 such that

$$\lim_{n\to+\infty} \frac{\lambda_2(A \cap R((x, y), h_n, k_n))}{4h_n k_n} < 1.$$

Then for $s_n = 1/h_n$, $t_n = 1/k_n$, $n \in \mathbb{N}$, we obtain $(x, y) \notin \Phi_{\langle s\rangle\langle t\rangle}(A)$, which is a contradiction. □

PROPOSITION 20. *For every $\langle s\rangle \in \mathcal{S}$ there exists $\langle t\rangle \in \mathcal{S}$ which is not regular to $\langle s\rangle$ and there exists a set $A \in \mathcal{L}_2$ such that the difference $\Phi_{\langle s\rangle\langle s\rangle}(A) \setminus \Phi_{\langle s\rangle\langle t\rangle}(A)$ is nonempty.*

PROOF. Let $\langle s\rangle \in \mathcal{S}$. Define $t_n := s_n^2$ for $n \in \mathbb{N}$. Then $\langle t\rangle$ is not regular to $\langle s\rangle$. Set $A := \{(x, y) : y > x^2 \vee y < -x^2\}$. It is clear that

$$(0, 0) \in \Phi(A) \subset \Phi_{\langle s\rangle\langle s\rangle}(A) \quad \text{and} \quad \frac{\lambda_2(A \cap R((0, 0), 1/s_n, 1/s_n^2))}{4/s_n^3} = \frac{2}{3},$$

so $(0, 0) \notin \Phi_{\langle s\rangle\langle t\rangle}(A)$. □

PROPOSITION 21. *For every $\langle s\rangle \in \mathcal{S}$ there exists $\langle t\rangle \in \mathcal{S}$ which is not regular to $\langle s\rangle$ and there exists a set $A \in \mathcal{L}_2$ such that the difference $\Phi_{\langle s\rangle\langle t\rangle}(A) \setminus \Phi_0(A)$ is nonempty.*

PROOF. Let $\langle s\rangle \in \mathcal{S}$. Define $t_n := n \cdot s_n$ for $n \in \mathbb{N}$. Then $\langle t\rangle$ is not regular to $\langle s\rangle$. Moreover, $\langle t\rangle \in \mathcal{S}_0$ if and only if $\langle s\rangle \in \mathcal{S}_0$. We choose a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}} \subset \{t_n\}_{n\in\mathbb{N}}$ such that $t_{n_{k+1}} > 2t_{n_k}$. Set

$$A := [-1, 1]^2 \setminus \bigcup_{k\in\mathbb{N}} [1/2t_{n_k}, 1/t_{n_k}]^2$$

and denote by B the complement of the set A.

We now consider two cases. If $1/t_n \in (1/2t_{n_k}, 1/t_{n_k})$ then

$$\begin{aligned}
\lambda_2\Big(B \cap R\Big((0, 0), \frac{1}{s_n}, \frac{1}{t_n}\Big)\Big) &\le \frac{1}{t_n} \cdot \frac{1}{2t_{n_k}} + \Big(\frac{1}{t_n} - \frac{1}{2t_{n_k}}\Big)^2 \\
&\quad + \Big(\frac{1}{t_{n_k}} - \frac{1}{t_{n_k}}\Big)\Big(\frac{1}{t_n} - \frac{1}{2t_{n_k}}\Big) \\
&\le \frac{1}{t_n} \cdot \frac{1}{2t_{n_k}} + \Big(\frac{1}{t_n} - \frac{1}{2t_{n_k}}\Big)^2 + \frac{1}{2t_{n_k}}\Big(\frac{1}{t_n} - \frac{1}{2t_{n_k}}\Big) \\
&= \Big(\frac{1}{t_n}\Big)^2.
\end{aligned}$$

If $1/t_n \in (1/t_{n_{k+1}}, 1/2t_{n_k})$ then $B \cap R((0, 0), 1/s_n, 1/t_n) \subset [0, 1/t_n]^2$.

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Therefore
$$\frac{\lambda_2(B \cap R((0,0), 1/s_n, 1/t_n))}{4/s_n t_n} \le \frac{(1/t_n)^2}{4/s_n t_n} = \frac{1}{4n},$$
so $(0,0) \in \Phi_{\langle s\rangle\langle t\rangle}(A)$.

The point $(0,0)$ is clearly not an ordinary density point of A since
$$\lambda_2(B \cap S((0,0), 1/t_{n_k})) \ge \frac{1}{4}(1/t_{n_k})^2.$$
This concludes the proof. □

After a slight modification of the last proof we can obtain the following proposition.

PROPOSITION 22. *For every $\langle s\rangle \in \mathcal{S}_0$ there exists $\langle t\rangle \in \mathcal{S}$, which is not regular to $\langle s\rangle$, and there exists a set $A \in \mathcal{L}_2$ such that the difference $\Phi_{\langle s\rangle\langle t\rangle}(A) \setminus \Phi_{\langle s\rangle\langle s\rangle}(A)$ is nonempty.*

PROPOSITION 23. *For every $\langle s\rangle \in \mathcal{S}_0$ there exists $\langle t\rangle \in \mathcal{S}$, such that $\langle s\rangle$ reg $\langle t\rangle$ and there exists a set $A \in \mathcal{L}_2$ such that the difference $\Phi_{\langle s\rangle\langle s\rangle}(A) \setminus \Phi_{\langle s\rangle\langle t\rangle}(A)$ is nonempty.*

PROOF. Let $\langle s\rangle \in \mathcal{S}_0$. Define $t_n := 2s_n$ for $n \in \mathbb{N}$. Then $\langle s\rangle$ reg $\langle t\rangle$. Let $\{s_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\langle s\rangle$ such that
$$\lim_{k\to+\infty} \frac{s_{n_k}}{s_{n_{k+1}}} = 0.$$
Set
$$B := \bigcup_{k\in\mathbb{N}} \left(\left(-\frac{1}{s_{n_{k+1}}}, \frac{1}{s_{n_{k+1}}} \right] \times \left(\left[-\frac{2}{s_{n_{k+1}}}, -\frac{1}{s_{n_{k+1}}} \right] \cup \left[\frac{1}{s_{n_{k+1}}}, \frac{2}{s_{n_{k+1}}} \right] \right) \right).$$

Let $\epsilon > 0$. There exists $k_0 \in \mathbb{N}$, such that for any $k > k_0$ we have $s_{n_k}/s_{n_{k+1}} < \sqrt{\epsilon/2}$. Set $k(n) := \min\{k \in \mathbb{N} : s_{n_k} \ge s_n\}$ and choose n_0 for which $k(n_0) > k_0$. Then for every $n > n_0$,
$$\frac{\lambda_2(B \cap S((0,0), 1/s_n))}{4/s_n^2} \le \frac{8(1/s_{n_{k(n)+1}})^2}{4(1/s_{n_{k(n)}})^2} = 2\left(\frac{s_{n_{k(n)}}}{s_{n_{k(n)+1}}}\right)^2 < \epsilon,$$
so, denoting by A the complement of B, we get $(0,0) \in \Phi_{\langle s\rangle\langle s\rangle}(A)$.

Since for every $k \in \mathbb{N}$,
$$\lambda_2(B \cap R((0,0), 1/s_{n_{k+1}}, 1/t_{n_{k+1}})) \ge 2 \cdot \frac{1}{s_{n_{k+1}}} \cdot \frac{1}{t_{n_{k+1}}},$$
then
$$\limsup_{n\to+\infty} \frac{\lambda_2(B \cap R((0,0), 1/s_n, 1/t_n))}{4/s_n t_n} > 0.$$
Therefore $(0,0) \notin \Phi_{\langle s\rangle\langle t\rangle}(A)$. □

PROPOSITION 24. *For every $\langle s\rangle \in \mathcal{S}_0$ there exists $\langle t\rangle \in \mathcal{S}$ such that $\langle s\rangle$ reg $\langle t\rangle$ and there exists a set $A \in \mathcal{L}_2$ such that $\Phi_{\langle s\rangle\langle t\rangle}(A) \setminus \Phi_{\langle s\rangle\langle s\rangle}(A) \ne \emptyset$.*

PROOF. Let $\langle s\rangle \in \mathcal{S}_0$. Define $t_n := (1/2)s_n$ for $n \in \mathbb{N}$. Then $\langle s\rangle$ reg $\langle t\rangle$. Let $\{s_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\langle s\rangle$ such that $\lim_{k\to+\infty} s_{n_k}/s_{n_k+1} = 0$.

Set

$$\mathcal{B} := \bigcup_{k\in\mathbb{N}}\left(\left[-\frac{1}{s_{n_k+1}}, \frac{1}{s_{n_k+1}}\right] \times \left(\left[-\frac{1}{s_{n_k+1}}, -\frac{1}{2s_{n_k+1}}\right] \cup \left[\frac{1}{2s_{n_k+1}}, \frac{1}{s_{n_k+1}}\right]\right)\right)$$

and let A be the complement of $\mathcal{B}$.

For every $k \in \mathbb{N}$,

$$\lambda_2(B \cap S((0, 0), 1/s_{n_k+1})) = 2/s_{n_k+1},$$

hence

$$\limsup_{n\to+\infty} \frac{\lambda_2(B \cap S((0, 0), 1/s_n))}{4/s_n^2} > 0,$$

so $(0, 0) \notin \Phi_{\langle s\rangle\langle t\rangle}(A)$.

Let $\epsilon > 0$. There exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$ we have $(s_{n_k}/s_{n_k+1}) < \sqrt{2\epsilon}$. Set $k(n) = \min\{k \in \mathbb{N} : s_{n_k} \geq s_n\}$ and choose n_0 for which $k(n_0) > k_0$. Then for every $n > n_0$,

$$\frac{\lambda_2(B \cap R((0, 0), 1/s_n, 1/t_n))}{4/s_n t_n} < \frac{4 \cdot (1/s_{n_{k(n)}+1})^2}{8/s_n^2} < \frac{1}{2}\left(\frac{s_{n(k)}}{s_{n_{k(n)}+1}}\right)^2 < \epsilon.$$

Therefore $(0, 0) \in \Phi_{\langle s\rangle\langle t\rangle}(A)$. □

We have considered connections between operators $\Phi_{\langle s\rangle\langle t\rangle}$ depending on sequences $\langle s\rangle, \langle t\rangle \in \mathcal{S}$. Now let us mention general results on such operators.

PROPOSITION 25. *For every $\langle s\rangle, \langle t\rangle \in \mathcal{S}$ and for every $A \in \mathcal{L}_2$ the set $\Phi_{\langle s\rangle\langle t\rangle}(A)$ is an $F_{\sigma\delta}$ set.*

PROOF. We first observe that

$$\Phi_{\langle s\rangle\langle t\rangle}(A) = \bigcap_{k=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\left\{(x, y) : \frac{\lambda_2(A \cap R((x, y), 1/s_n, 1/t_n))}{4/s_n t_n} \geq 1 - \frac{1}{k}\right\}$$

for every $\langle s\rangle, \langle t\rangle \in \mathcal{S}$ and every $A \in \mathcal{L}_2$.

Since for a fixed n the function

$$\frac{\lambda_2(A \cap R((x, y), 1/s_n, 1/t_n))}{4/s_n t_n}$$

is a continuous function of (x, y) the set

$$\bigcap_{n=m}^{\infty}\left\{(x, y) : \frac{\lambda_2(A \cap R((x, y), 1/s_n, 1/t_n))}{4/s_n t_n} \geq 1 - \frac{1}{k}\right\}$$

is closed, so $\Phi_{\langle s\rangle\langle t\rangle}(A)$ is an $F_{\sigma\delta}$ set. □

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THEOREM 26. *For every $\langle s\rangle, \langle t\rangle \in \mathcal{S}$ and for every $A, B \in \mathcal{L}_2$:*

(1) $\Phi_{\langle s\rangle\langle t\rangle}(\emptyset) = \emptyset$, $\Phi_{\langle s\rangle\langle t\rangle}(\mathbb{R}^2) = \mathbb{R}^2$;
(2) $\Phi_{\langle s\rangle\langle t\rangle}(A \cap B) = \Phi_{\langle s\rangle\langle t\rangle}(A) \cap \Phi_{\langle s\rangle\langle t\rangle}(B)$;
(3) $A \sim B \implies \Phi_{\langle s\rangle\langle t\rangle}(A) = \Phi_{\langle s\rangle\langle t\rangle}(B)$;
(4) $A \sim \Phi_{\langle s\rangle\langle t\rangle}(A)$,

where $A \sim B$ means that $\lambda_2(A \triangle B) = 0$.

PROOF. (1), (2) and (3) are obvious. (4) is a simple consequence of the inclusion $\Phi_s(A) \subset \Phi_{\langle s\rangle\langle t\rangle}(A)$ (Proposition 19) and the fact that the operator Φ_s is a lower density operator (which means it satisfies conditions (1)–(4)). □

We now recall results presented in [4]. Let $(X, \mathbb{S}, I)$ denote a measurable space, where $\mathbb{S}$ is a σ-algebra of subsets of X and I is a proper σ-ideal of $\mathbb{S}$-measurable sets. The space $(X, \mathbb{S}, I)$ is said to have the *hull property* if whenever $A \subset X$, there is a set $B \in \mathbb{S}$ such that $A \subset B$ and if $Z \in \mathbb{S}$ and $A \subset Z$, then $B \setminus Z \in I$.

For every lower density operator Φ on $\mathbb{S}$ let $\mathcal{T}_\Phi := \{A \in \mathbb{S} : A \subset \Phi(A)\}$.

THEOREM 27 [4]. *Let $(X, \mathbb{S}, I)$ be a measurable space having the hull property. Then for every lower density operator Φ the family $\mathcal{T}_\Phi$ is a topology on X.*

For every $\langle s\rangle, \langle t\rangle \in \mathcal{S}$ we define a family $\mathcal{T}_{\langle s\rangle\langle t\rangle} := \{A \in \mathcal{L}_2 : A \subset \Phi_{\langle s\rangle\langle t\rangle}(A)\}$. Since the assumptions of the last theorem are fulfilled $\mathcal{T}_{\langle s\rangle\langle t\rangle}$ is a topology on the plane.

For a deeper discussion of properties of this type of topology following from the properties of the operator, we refer the reader to [2].

We will need only one additional property of general lower density operators.

THEOREM 28. *Let Φ_1, Φ_2 be lower density operators in a measurable space $(X, \mathbb{S}, I)$. Then $\mathcal{T}_{\Phi_1} = \mathcal{T}_{\Phi_2}$ if and only if $\Phi_1 = \Phi_2$.*

PROOF. Sufficiency is obvious.

Suppose that $\mathcal{T}_{\Phi_1} = \mathcal{T}_{\Phi_2}$ but that there exists a set $A \in \mathbb{S}$ such that $\Phi_1(A) \setminus \Phi_2(A) = \emptyset$. Since Φ_1 is a lower density operator, $\Phi_1(A)$ belongs to $\mathcal{T}_{\Phi_1}$ and, by our supposition, belongs also to $\mathcal{T}_{\Phi_2}$, which means that $\Phi_1(A) \subset \Phi_2(\Phi_1(A))$, but $A \triangle \Phi_1(A) \in I$, so $\Phi_2(A) = \Phi_2(\Phi_1(A))$. Therefore $\Phi_1(A) \subset \Phi_2(A)$, which is a contradiction. □

By virtue of the above theorem and the properties shown earlier we get relations between topologies $\mathcal{T}_{\langle s\rangle\langle t\rangle}$, $\langle s\rangle, \langle t\rangle \in \mathcal{S}$.

THEOREM 29.

(1) $\bigcap_{\substack{\langle s\rangle, \langle t\rangle \in \mathcal{S} \\ \langle s\rangle \mathrm{reg} \langle t\rangle}} \mathcal{T}_{\langle s\rangle\langle t\rangle} = \mathcal{T}_0$, *where $\mathcal{T}_0$ denotes the ordinary density topology on the plane.*
(2) $\bigcap_{\langle s\rangle \in \mathcal{S}_0} \mathcal{T}_{\langle s\rangle\langle s\rangle} = \mathcal{T}_0$.
(3) *For every $\langle s\rangle \in \mathcal{S}$, $\mathcal{T}_{\langle s\rangle\langle s\rangle} = \mathcal{T}_0$ if and only if $\langle s\rangle \in \mathcal{S}_+$.*
(4) *For every $\langle s\rangle, \langle t\rangle \in \mathcal{S}_+ \mathcal{T}_{\langle s\rangle\langle t\rangle} = \mathcal{T}_0$ if and only if $\langle s\rangle$ reg $\langle t\rangle$.*

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(5) $\bigcap_{\langle s\rangle,\langle t\rangle\in\mathcal{S}} \mathcal{T}_{\langle s\rangle\langle t\rangle} = \mathcal{T}_s$, *where* $\mathcal{T}_s$ *denotes the strong density topology on the plane.*
(6) *For every* $\langle s\rangle \in \mathcal{S}$ *there exist* $\langle p\rangle, \langle t\rangle \in \mathcal{S}$ *which are not regular to* $\langle s\rangle$ *and such that* $\mathcal{T}_{\langle s\rangle\langle s\rangle} \setminus \mathcal{T}_{\langle s\rangle\langle t\rangle} \neq \emptyset$ *and* $\mathcal{T}_{\langle s\rangle\langle p\rangle} \setminus \mathcal{T}_{\langle s\rangle\langle s\rangle} \neq \emptyset$.
(7) *For every* $\langle s\rangle \in \mathcal{S}_0$ *there exist* $\langle p\rangle, \langle t\rangle \in \mathcal{S}$ *such that* $\langle t\rangle \operatorname{reg} \langle s\rangle$, $\langle p\rangle \operatorname{reg} \langle s\rangle$, $\mathcal{T}_{\langle s\rangle\langle s\rangle} \setminus \mathcal{T}_{\langle s\rangle\langle t\rangle} \neq \emptyset$ *and* $\mathcal{T}_{\langle s\rangle\langle p\rangle} \setminus \mathcal{T}_{\langle s\rangle\langle s\rangle} \neq \emptyset$.

Here are some natural properties of the $\mathcal{T}_{\langle s\rangle\langle t\rangle}$ topologies. Property (5) from the next theorem makes property (2) from the previous theorem more interesting since the topology $\mathcal{T}_0$ is invariant under similarity but, as we will see, $\mathcal{T}_{\langle s\rangle\langle s\rangle}$ for $\langle s\rangle \in \mathcal{S}_0$ is not.

We will use the following notation: for $A \in \mathbb{R}^2$ and $x, y \in \mathbb{R}$, write $A + (x, y)$ for $\{(a + x, b + y) : a, b \in A\}$, $-A$ for $\{(-a, -b) : a, b \in A\}$ and $(x, y) \cdot A$ for $\{(xa, yb) : a, b \in A\}$.

THEOREM 30. *For every* $A \in \mathcal{L}_2$ *and for every* $\langle s\rangle, \langle t\rangle \in \mathcal{S}$:

(1) *for every* $x, y \in \mathbb{R}$, *if* $A \in \mathcal{T}_{\langle s\rangle\langle t\rangle}$ *then* $A + (x, y) \in \mathcal{T}_{\langle s\rangle\langle t\rangle}$;
(2) *if* $A \in \mathcal{T}_{\langle s\rangle\langle t\rangle}$ *then* $-A \in \mathcal{T}_{\langle s\rangle\langle t\rangle}$;
(3) *for every* $m, p \in \mathbb{R}$, *such that* $|m| \geq 1$ *and* $|p| \geq 1$, *if* $A \in \mathcal{T}_{\langle s\rangle\langle t\rangle}$ *then* $(m, p) \cdot A \in \mathcal{T}_{\langle s\rangle\langle t\rangle}$;
(4) *for every* $A \in \mathcal{L}_2$ *and for every* $\langle s\rangle, \langle t\rangle \in \mathcal{S}_+$ *such that* $\langle s\rangle \operatorname{reg} \langle t\rangle$ *and for every* $m \in \mathbb{R} \setminus \{0\}$, *if* $A \in \mathcal{T}_{\langle s\rangle\langle t\rangle}$ *then* $(m, m) \cdot A \in \mathcal{T}_{\langle s\rangle\langle t\rangle}$;
(5) *for every* $\langle s\rangle \in \mathcal{S}_0$ *there exists a set* $A \in \mathcal{L}_2$, $A \in \mathcal{T}_{\langle s\rangle\langle s\rangle}$ *such that for every* $m \in \mathbb{R}$, $|m| < 1$, *the set* $(m, m) \cdot A$ *does not belong to* $\mathcal{T}_{\langle s\rangle\langle s\rangle}$.

PROOF. Properties (1)–(4) follow from the definition of the topology $\mathcal{T}_{\langle s\rangle\langle t\rangle}$ and Theorem 29 point (4). We give the proof only for (5).

Fix $\langle s\rangle \in \mathcal{S}_0$. Let $\{s_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\langle s\rangle$ such that

$$\lim_{n\to\infty} \frac{s_{n_k}}{s_{n_k+1}} = 0.$$

Let

$$X := \bigcup_{k=1}^{\infty} \left[\frac{1}{s_{n_k+1}}, \frac{1}{\sqrt{s_{n_k} s_{n_k+1}}}\right].$$

As was mentioned at the beginning, the one-dimensional version of a density topology with respect to a fixed sequence was considered in [1]. The definitions of the operator $\Phi_{\langle s\rangle}$ and the topology $\mathcal{T}_{\langle s\rangle}$ for the fixed sequence $\langle s\rangle$ are analogous to those in the two-dimensional case so we omit them.

The set $Y := (\mathbb{R} \setminus X) \cup \{0\}$ belongs to $\mathcal{T}_{\langle s\rangle}$ (see [1, proof of Theorem 3]).

Define

$$A := \bigcup_{y\in Y\setminus\mathbb{R}_-} ((\{-y, y\} \times [-y, y]) \cup ([-y, y] \times \{-y, y\})).$$

An analysis similar to that in [3, proof of Theorem 2.6] shows that $A \in \mathcal{T}_{\langle s\rangle\langle s\rangle}$.

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For $m = 0$ it is obvious that $(m, m) \cdot A \notin \mathcal{T}_{\langle s\rangle\langle s\rangle}$.

Without loss of generality we assume now that $m \in (0, 1)$.

Let k_0 be a positive integer such that $\sqrt{s_{n_k}/s_{n_k+1}} < m$ for $k > k_0$. Then the set $m \cdot Y_m$, where

$$Y_m := (\mathbb{R} \setminus X_m) \cup \{0\} \quad \text{and} \quad X_m := \bigcup_{k=k_0}^{\infty} \left[\frac{1}{s_{n_k+1}}, \frac{1}{\sqrt{s_{n_k} s_{n_k+1}}}\right],$$

does not belong to $\mathcal{T}_{\langle s\rangle}$ (see [1, proof of Theorem 4]) and, again following ideas from [3, proof of Theorem 2.6], we get that the set

$$(m, m) \cdot \left[\bigcup_{y \in Y_m \setminus \mathbb{R}_-} ((\{-y, y\} \times [-y, y]) \cup ([-y, y] \times \{-y, y\}))\right]$$

does not belong to $\mathcal{T}_{\langle s\rangle\langle s\rangle}$, so neither does the set $(m, m) \cdot A$. □

The next theorem expresses the connection between $\mathcal{T}_{\langle s\rangle\langle t\rangle}$ and the product topology $\mathcal{T}_{\langle s\rangle} \times \mathcal{T}_{\langle t\rangle}$.

THEOREM 31. *For every $\langle s\rangle, \langle t\rangle \in \mathcal{S}$ the product topology $\mathcal{T}_{\langle s\rangle} \times \mathcal{T}_{\langle t\rangle}$ is contained in $\mathcal{T}_{\langle s\rangle\langle t\rangle}$.*

PROOF. Let $\langle s\rangle, \langle t\rangle \in \mathcal{S}$ and let $E \in \mathcal{T}_{\langle s\rangle} \times \mathcal{T}_{\langle t\rangle}$. Fix any point $(x_0, y_0) \in E$ and a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of the sequence of all natural numbers. Define $R_k := R((x_0, y_0), 1/s_{n_k}, 1/t_{n_k})$.

Since $E \in \mathcal{T}_{\langle s\rangle} \times \mathcal{T}_{\langle t\rangle}$, there exist sets $A, B \subset \mathbb{R}$, such that $A \times B \subset E$, $x_0 \in A \subset \Phi_{\langle s\rangle}(A)$ and $y_0 \in B \subset \Phi_{\langle t\rangle}(B)$. Therefore for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every natural k, $k > k_0$,

$$\frac{\lambda_1((-1/s_{n_k}, 1/s_{n_k}) \setminus A)}{2/s_{n_k}} < \frac{\epsilon}{2} \quad \text{and} \quad \frac{\lambda_1((-1/t_{n_k}, 1/t_{n_k}) \setminus B)}{2/s_{n_k}} < \frac{\epsilon}{2}.$$

Since

$$\begin{aligned} P_k \setminus E \subset (((-1/s_{n_k}, 1/s_{n_k}) \setminus A) \times (-1/t_{n_k}, 1/t_{n_k})) \\ \cup((-1/s_{n_k}, 1/s_{n_k}) \times (-1/t_{n_k}, 1/t_{n_k}) \setminus B), \end{aligned}$$

it follows that

$$\lambda_2(R_k \setminus E) \le \frac{\epsilon}{2} \cdot \frac{2}{s_{n_k}} \cdot \frac{2}{t_{n_k}} + \frac{2}{s_{n_k}} \cdot \frac{\epsilon}{2} \cdot \frac{2}{t_{n_k}} = \epsilon \cdot \lambda_2(R_k).$$

Hence $(x_0, y_0) \in \Phi_{\langle s\rangle\langle t\rangle}(E)$. □

THEOREM 32. *For every $\langle s\rangle \in \mathcal{S}_0$ and $\langle t\rangle \in \mathcal{S}$ the topologies $\mathcal{T}_{\langle s\rangle} \times \mathcal{T}_{\langle t\rangle}$ and $\mathcal{T}_{\langle s\rangle\langle t\rangle}$ are different.*

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PROOF. Let $\langle s\rangle \in \mathcal{S}_0$ and $\langle t\rangle \in \mathcal{S}$. Then $\mathcal{T}_{\langle s\rangle} \subset \mathcal{T}_{\langle t\rangle}$ (see [1]). Define $G := (\mathbb{R}^2 \setminus \Delta) \cup \{(0, 0)\}$, where $\Delta = \{(x, x) : x \in \mathbb{R}\}$. Then $G \in \mathcal{T}_{\langle s\rangle\langle t\rangle}$ is a set of full measure. If $G \in \mathcal{T}_{\langle s\rangle} \times \mathcal{T}_{\langle t\rangle}$ then there exist sets $A \in \mathcal{T}_{\langle s\rangle}$ and $B \in \mathcal{T}_{\langle t\rangle}$ such that $(0, 0) \in A \times B \subset G$. Let $C := A \cap B$. Then C is nonempty and C is open in $\mathcal{T}_{\langle t\rangle}$, so $C \setminus \{0\}$ cannot be empty. Then $(C \times C \setminus \{(0, 0\}) \cap \Delta \neq \emptyset$, but $C \times C \subset G$, which is impossible, so $G \in \mathcal{T}_{\langle s\rangle\langle t\rangle} \setminus \mathcal{T}_{\langle s\rangle} \times \mathcal{T}_{\langle t\rangle}$. □

For the sake of simplicity we have presented all results in $\mathbb{R}^2$ but they can be easily generalized to Euclidean spaces of dimension higher than two. In $\mathbb{R}^m$ we consider sets of m sequences from the family $\mathcal{S}$: $\{\langle s^p\rangle : p \in \{1, \ldots, m\}\}$. For such a set we can define a density operator which for measurable set $A \subset \mathbb{R}^m$ is the set of all points $(x_1, \ldots, x_m)$ for which

$$\lim_{n\to+\infty} \frac{\lambda_m(A \cap ((x_1 - 1/s_n^1, x_1 + 1/s_n^1) \times \cdots \times (x_m - 1/s_n^m, x_m + 1/s_n^m)))}{2^m/s_n^1 \cdots s_n^m} = 1.$$

Following Saks (see [5]), a set of sequences $\{\langle s^p\rangle : p \in \{1, \ldots, m\}\}$ will be called regular if there exists a positive number α such that

$$\frac{\min_{p\in\{1,\ldots,m\}} s_n^p}{\max_{p\in\{1,\ldots,m\}} s_n^p} > \alpha \quad \text{for } n \in \mathbb{N}.$$

Having defined these notions we can prove theorems analogous to the two-dimensional case. The same line of reasoning applies to higher-dimensional versions.

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