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# Invariant Means on a Class of von Neumann Algebras Related to Ultraspherical Hypergroups II

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*Abstract.* Let $K$ be an ultraspherical hypergroup associated with a locally compact group $G$ and a spherical projector $\pi$ and let $\mathrm{VN}(K)$ denote the dual of the Fourier algebra $A(K)$ corresponding to $K$. In this note, we show that the set of invariant means on $\mathrm{VN}(K)$ is singleton if and only if $K$ is discrete. Here $K$ need not be second countable. We also study invariant means on the dual of the Fourier algebra $A_0(K)$, the closure of $A(K)$ in the cb-multiplier norm. Finally, we consider generalized translations and generalized invariant means.

## Introduction

Let $G$ be a locally compact group and let $A(G)$ and $\mathrm{VN}(G)$ denote the Fourier algebra and its Banach space dual (von Neumann algebra), respectively. Let $K$ be an ultraspherical hypergroup associated with the locally compact group $G$ and a spherical projector $\pi$. Let $A(K)$ denote the Fourier algebra corresponding to the hypergroup $K$ and let $VN(K)$ be its Banach space dual.

In [17], the author introduced and studied the notion of invariant means on $\mathrm{VN}(K)$. One of the main results of that paper is that, if $K$ is second countable, then a unique invariant mean exists if and only if $K$ is discrete. For the case of locally compact groups, this result is due to Renaud [16]. Renaud's result was extended to the case of general locally compact groups by Lau and Losert [12]. In Section 1, we shall imitate the proof and ideas of [12], to generalize the result of [17] to a general ultraspherical hypergroup.

Let $\mathrm{TIM}(\widehat{K})$ denote the set of all invariant means on $VN(K)$. In [17], the proof of Theorem 5.6 contains the fact that $r_K^{**}(\mathrm{TIM}(\widehat{K})) = \mathrm{TIM}(\widehat{H})$, where $H$ is an open subhypergroup. Theorem 1.10 of this paper gives the same result for a general open subhypergroup.

Forrest and Miao [9] studied invariant means on the algebra $A_0(G)$. Here $A_0(G)$ denotes the closure of $A(G)$ with the cb-multiplier norm. In Section 2 we also define the algebra $A_0(K)$ for an ultraspherical hypergroup and study invariant means on its

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dual. The main result of this section states that the set of invariant means on the dual of $A_0(K)$ coincides with the set of invariant means on $\mathrm{VN}(K)$.

M. Y.-H. Cheng introduced generalized translations and generalized invariant means on $\mathrm{VN}(G)$ [1]. In the same spirit, we also consider generalized translations and generalized invariant means on $\mathrm{VN}(K)$. This is done in Section 3, where we prove some analogues of Theorem 1.10 and Theorem 2.4.

We shall follow the notations as in [17]. For any undefined notations or definitions, we refer the reader to [15, 17].

## 1 Invariant Means on $\mathrm{VN}(K)$

The author proved that for a second countable ultraspherical hypergroup $K$, the space $\mathrm{VN}(K)$ admits a unique topological invariant mean if and only if $K$ is discrete [17]. The aim of this section is to extend this to the context of general ultraspherical hypergroups. We begin with a theorem that is an analogue of [10, Theorem 8.7].

***Theorem 1.1*** *Let $K$ be a $\sigma$-compact ultraspherical hypergroup. Then for every countable family $\{\widetilde{U}_n\}$ of neighbourhoods of $\{\dot{e}\}$, there exists a compact subhypergroup $H$ of $K$ such that*

(i) $H \subseteq \bigcap_{n=1}^{\infty} \widetilde{U}_n$,

(ii) *$K//H$ is a second countable ultraspherical hypergroup.*

**Proof** Let $\widetilde{V}_0$ be a neighbourhood of $\{\dot{e}\}$ in $K$ with compact closure. Construct a sequence $\{\widetilde{V}_n\}$ of symmetric neighbourhoods of $\{\dot{e}\}$ in $K$ such that $\widetilde{\widetilde{V}_n}^2 \subset \widetilde{V}_{n-1} \cap \widetilde{U_n}$ and $\overline{\widetilde{V}_n} \subset \widetilde{V}_{n-1}$. Consider the family $\{V_n\}$ of neighbourhoods of $\{e\}$ in $G$, where $V_n = p^{-1}(\widetilde{V}_n)$. As $K$ is $\sigma$-compact, $G$ is also $\sigma$-compact and hence by [10, Theorem 8.7], there exists a compact normal subgroup $N$ of $G$ such that $N \subseteq \bigcap_{n=1}^{\infty} V_n$ and $G/N$ is second countable. As $N \subseteq \bigcap_{n=1}^{\infty} V_n$, it follows that $p(N) \subseteq \bigcap_{n=1}^{\infty} \widetilde{V}_n$. Further, as in [10, Theorem 5.6], one can show that $\bigcap_{n=1}^{\infty} \widetilde{V}_n$ is a closed subhypergroup of $K$ containing the symmetric set $p(N)$. Let $H = \bigcap_{n=1}^{\infty} \widetilde{V}_n$. Then $H$ is compact, as $H \subset \overline{\widetilde{V}_0}$. Also $H = \bigcap_{n=1}^{\infty} \widetilde{V}_n \subseteq \bigcap_{n=1}^{\infty} \widetilde{U}_n$. As $H$ is a subhypergroup of $K$, by [3, Corollary 3.2], it follows that $p^{-1}(H)$ is a compact subgroup of $G$. Now $G//p^{-1}(H)$ is second countable, being the continuous image of $G/N$ under the map $gN \mapsto p^{-1}(H)gp^{-1}(H)$. Notice that by [11, Theorem 14.4A] and the fact that the positive definite functions separate points, it follows that $p(p^{-1}(H)gp^{-1}(H)) = Hp(g)H$, for all $g \in G$. Hence, if q denotes the canonical quotient map from $K$ onto $K//H$, the map $p^{-1}(H)gp^{-1}(H) \mapsto \mathrm{q}(p(g))$, from $G//p^{-1}(H)$ to $K//H$ is well defined. Further, it is onto and continuous. Thus, $K//H$ is second countable. As $p(p^{-1}(H)gp^{-1}(H)) = Hp(g)H$, for all $g \in G$, it follows that $\pi \circ \pi_{p^{-1}(H)} = \pi_{p^{-1}(H)} \circ \pi$. Hence, $\pi \circ \pi_{p^{-1}(H)}$ is a spherical projector. Thus, the pair $(G, \pi \circ \pi_{p^{-1}(H)})$ defines an ultraspherical hypergroup. Let $K_{p^{-1}(H)}$ denote this ultraspherical hypergroup. Further, the hypergroup $K_{p^{-1}(H)}$ is isomorphic to the hypergroup $K//H$. Hence the proof. ■

***Remark 1.2*** Although the above theorem extends [10, Theorem 8.7], our proof makes use of it.

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***Lemma 1.3*** *Let $K$ be a $\sigma$-compact ultraspherical hypergroup. Let $H$ be a second countable closed subhypergroup of $K$. Then there exists a compact subhypergroup $H'$ of $K$ such that*

(i) $H \cap H' = \{\dot{e}\}$,
(ii) *$K//H'$ is a second countable ultraspherical hypergroup.*

**Proof** Let $\{\widetilde{U}_n\}$ be a neighbourhood basis of $\{\dot{e}\}$ in $H$. Let $\widetilde{V}_n$ be neighbourhoods of $\{\dot{e}\}$ in $K$ such that $\widetilde{U}_n = H \cap \widetilde{V}_n$. By Theorem 1.1, there exists a compact subhypergroup $H'$ of $K$ such that $K//H'$ is a second countable ultraspherical hypergroup and $H' \subseteq \bigcap_{n=1}^{\infty} \widetilde{V}_n$. Further, it is clear that $H \cap H' = \{\dot{e}\}$. ■

***Lemma 1.4*** *Let $K$ be an ultraspherical hypergroup and let $H$ be a compact subhypergroup of $K$ with $H \neq \{\dot{e}\}$. Then there exists a compact subhypergroup $H'$ of $K$ such that*

(i) $H' \subseteq H$,
(ii) *$H//H'$ is a non-trivial second countable hypergroup.*

**Proof** If $H$ is second countable, choose a proper subhypergroup $H'$ of $H$. This will satisfy our requirements. If $H$ is not second countable, let $\widetilde{U}$ be a neighbourhood of $\dot{e}$ in $H$. Then by Lemma 1.3, there exists a compact subhypergroup $H'$ of $K$ satisfying our requirements. ■

***Proposition 1.5*** *Let $K$ be an ultraspherical hypergroup which is not second countable. Then there exists a limit ordinal $\lambda$ and a decreasing family $\{H_\alpha\}_{\alpha<\lambda}$ of subhypergroups of $K$ such that*

(i) $H_0 = K$,
(ii) *$H_\alpha$ is compact for $\alpha > 0$,*
(iii) $\bigcap_{\alpha<\lambda} H_\alpha = \{\dot{e}\}$,
(iv) *$H_\alpha // H_{\alpha+1}$ is a non-discrete second countable ultraspherical hypergroup for $\alpha < \lambda$.*

**Proof** By Lemma 1.4 and transfinite induction, one obtains a decreasing family $\{H_\alpha\}_{\alpha\le\mu}$ such that

(i) $H_0 = K$,
(ii) $H_\alpha$ is a compact subhypergroup for $\alpha > 0$,
(iii) $H_\mu = \{\dot{e}\}$,
(iv) $\bigcap_{\alpha<\lambda} H_\alpha = H_\lambda$ for every limit ordinal $\lambda \le \mu$,
(v) $H_\alpha // H_{\alpha+1}$ is second countable.

Without loss of generality, one can even assume that $\mu$ is minimal among all such families. Now the fact that $\mu$ is a limit ordinal follows exactly as in [12, Lemma 4.8]. In the group case, the proof of this uses [12, Lemma 4.6], while for the ultraspherical hypergroup case one will have to use Lemma 1.3. Now by passing to an appropriate subfamily, we get the desired family. ■

The following is an analogue of [12, Lemma 4.9].

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***Lemma 1.6*** *Let* $\{H_\alpha\}_{\alpha<\lambda}$ *be the family given by Proposition 1.5. For* $\alpha > 0$*, let* $m_\alpha$ *be a topologically invariant mean on* $\mathrm{VN}(K//H_\alpha)$*. If* $m$ *is a weak*-cluster point of* $\{m_\alpha\}_{\alpha<\lambda}$*, then* $m$ *is a topologically invariant mean on* $\mathrm{VN}(K)$*.*

**Proof** Let $\varphi \in A(K//H_\alpha)$. Then $\varphi.m_\alpha = \varphi(\dot{e})m_\alpha$. Hence $\varphi.m = \varphi(\dot{e})m$. By the Stone–Weierstrass theorem, $\bigcup_{\alpha<\lambda} L^2(K//H_\alpha)$ is dense in $L^2(K)$ and hence it follows that $\bigcup_\alpha < \lambda A(K//H_\alpha)$ is dense in $A(K)$ and consequently, $\varphi.m = \varphi(\dot{e})m$ for all $\varphi \in A(K)$. ∎

***Theorem 1.7*** *Let* $K$ *be an ultraspherical hypergroup. If* $\mathrm{VN}(K)$ *admits a unique topological invariant mean, then* $K$ *is discrete.*

**Proof** Notice that if $K$ is second countable, then by [17, Theorem 5.9], we are done. So let us assume that $K$ is not separable. We now claim that the topological invariant mean on $\mathrm{VN}(K)$ is not unique. In order to prove this claim, without loss of generality, let us assume that $K$ is $\sigma$-compact. Let $\{H_\alpha\}_{\alpha<\lambda}$ and $m_\alpha$ be as in Lemma 1.6. Suppose to the contrary that the topological invariant mean on $\mathrm{VN}(K)$ is unique. The net $\{m_\alpha\}_{\alpha<\lambda}$ would be w*-convergent. Let $P_\alpha \in \mathrm{VN}(K)$ be the central projection defined by convolution with the Haar measure of $H_\alpha$. The remaining proof is similar to the proof of [12, Theorem 4.10]. Hence we omit the details. ∎

In the next corollary, we would like to single out the special case when the ultraspherical hypergroup is the double coset hypergroup.

***Corollary 1.8*** *Let* $G$ *be a locally compact group containing a compact subgroup* $H$*. Then* $\mathrm{VN}(G//H)$ *admits a unique topological invariant mean if and only if* $H$ *is open.*

**Proof** The proof of this follows from Theorem 1.7 and [11, Theorem 7.1B]. ∎

For the notations and terminologies used in the next corollary we shall refer to [2, 7].

***Corollary 1.9*** *Let* $G$ *be a locally compact group containing a compact subgroup* $H$*. Then* $\mathrm{VN}(G/H)$ *admits a unique topological invariant mean if and only if* $H$ *is open.*

**Proof** If $G$ is second countable, then this is proved in [2, Proposition 5.22, Proposition 5.24]. If $G$ is not second countable, then it follows form Corollary 1.8. ∎

In [17], the author proved that $r_K^{**}(\mathrm{TIM}(\widehat{K})) = \mathrm{TIM}(\widehat{H})$, under the assumption that $H$ is open. Our next theorem generalizes this to an arbitrary closed subhypergroup. The proof of this is an adaptation of the proof given in [1] for the case of locally compact groups.

***Theorem 1.10*** *Let* $H$ *be a closed subhypergroup of* $K$*. Then*

$$r_K^{**}(\mathrm{TIM}(\widehat{K})) = \mathrm{TIM}(\widehat{H}).$$

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**Proof** (a) Let $m \in r_K^{**}(\mathrm{TIM}(\widehat{K}))$. Let $\widetilde{m} \in \mathrm{TIM}(\widehat{K})$ such that $r_K^{**}(\widetilde{m}) = m$. Let $\psi \in A(H)$ and $T \in \mathrm{VN}(H)$. Choose $\varphi \in A(K)$ such that $r_K(\varphi) = \psi$. Then $\langle \psi.T, m\rangle = \psi(\dot{e})\langle T, m\rangle$ follows exactly as in [17, Theorem 5.6]. Thus $r_K^{**}(\mathrm{TIM}(\widehat{K})) \subseteq \mathrm{TIM}(\widehat{H})$.

(b) Let $m \in \mathrm{TIM}(\widehat{H})$. We now claim that there exists a mean $\widetilde{m}$ on $\mathrm{VN}(K)$ such that $r_K(\widetilde{m}) = m$. Since $m \in \mathrm{TIM}(\widehat{H})$, there exists a net $\{\varphi_\alpha\} \subset M_{A(H)}$ converging to $m$ in the weak*-topology. As $r_K(M_{A(K)}) = M_{A(H)}$, there exists a net $\widetilde{\varphi}_\alpha \subset M_{A(K)}$ such that $r_K(\widetilde{\varphi}_\alpha) = \varphi_\alpha$. By the Banach–Alaoglu theorem, $\{\widetilde{\varphi}_\alpha\}$ has a subnet $\{\widetilde{\varphi}_{\alpha_\beta}\}$ converging weak* to say $\widetilde{m}$. As $r_K^{**}$ is weak*-weak*-continuous, it follows that $r_K^{**}(\widetilde{m}) = m$.

(c) Let $m \in \mathrm{TIM}(\widehat{H})$ and let $\widetilde{m}$ be the mean (provided by (ii)) such that $r_K^{**}(\widetilde{m}) = m$. We now claim that for any $\varphi \in M_{A(K)}$, $r_K^{**}(\varphi.\widetilde{m}) = m$. Indeed, for any $T \in \mathrm{VN}(H)$, we have

$$\begin{aligned}\langle T, r_K^{**}(\varphi.\widetilde{m})\rangle &= \langle r_K^*(T), \varphi.\widetilde{m}\rangle = \langle \varphi.r_K^*(T), \widetilde{m}\rangle \\ &= \langle r_K^*(r_K(\varphi).T), \widetilde{m}\rangle = \langle r_K(\varphi).T, r_K^{**}(\widetilde{m})\rangle \\ &= \langle r_K(\varphi).T, m\rangle = \langle T, m\rangle.\end{aligned}$$

Hence the claim.

(d) We now prove the remaining inclusion. Let $m \in \mathrm{TIM}(\widetilde{H})$. Let

$$\mathcal{C} = \{\widetilde{m} : \widetilde{m} \text{ is a mean on } \mathrm{VN}(K) \text{ and } r_K^{**}(\widetilde{m}) = m\}.$$

By (b) it follows that $\mathcal{C}$ is non-empty. It is clear that $\mathcal{C}$ is weak*-compact. By (c), it follows that for any $\varphi \in M_{A(K)}$, the mapping $\widetilde{m} \mapsto \varphi.\widetilde{m}$ maps $\mathcal{C}$ to itself. Thus the remaining proof follows from the Markov–Kakutani fixed point theorem. ∎

## 2 cb-multipliers and Invariant Means

In this section, we define a new algebra, denoted by $A_0(K)$, and study the invariant means on the dual of this algebra. Let $MA(K)$ denote the Banach algebra of all bounded multipliers of $A(K)$ [14]. That is, the space of all continuous functions $\psi$ on $K$ such that $\psi\varphi \in A(K)$ for every $\varphi \in A(K)$. For each $\psi \in MA(K), \varphi \in A(K)$, let $m_\psi(\varphi) = \psi\varphi$. As $A(K)$ being the dual of the von Neumann algebra $\mathrm{VN}(K)$, it makes sense to speak of completely bounded multipliers [4]. We call $\psi$ a completely bounded multiplier if the map $m_\psi$ is completely bounded. Let $\|\psi\|_{\mathrm{cb}}$ denote the completely bounded norm of $m_\psi$. Let $M_{\mathrm{cb}}A(K)$ denote the Banach algebra of all completely bounded multipliers of $A(K)$. Then $B(K) \subset M_{\mathrm{cb}}(A(K))$.

We will denote by $A_0(K)$ the closure of $A(K)$ in the cb-multiplier norm. We now proceed to study the invariant means on $A_0(K)$.

***Lemma 2.1*** (i) *For $\varphi \in A(K)$ and $T \in \mathrm{VN}(K)$, $\varphi.T \in \iota^*(A_0(K)^*)$.*
(ii) *For $\varphi \in A_0(K)$ and $T \in A_0(K)^*$, $\iota^*(\varphi.T) = \varphi.\iota^*(T)$.*

**Proof** (i) Define $f_{\varphi,T}: A_0(K) \to \mathbb{C}$ as $f_{\varphi,T}(\psi) = \langle \varphi\psi, T\rangle$. It is clear that $f_{\varphi,T}$ defines a bounded linear functional on $A_0(K)$ and $\|f_{\varphi,T}\| \le \|\varphi\|_{A(G)}\|T\|_{\mathrm{VN}(G)}$. Further, notice that $f_{\varphi,T}$, when restricted to $A(K)$, equals $\varphi.T$ and hence it follows that $\varphi.T = \iota^*(f_{\varphi,T})$.

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(ii) For any $\psi \in A(K)$, we have

$$\begin{aligned}\langle \psi, \iota^*(\varphi.T)\rangle &= \langle \iota(\psi), \varphi.T\rangle = \langle \varphi\iota(\psi), T\rangle \\ &= \langle \iota(\varphi\psi), T\rangle = \langle \varphi\psi, \iota^*(T)\rangle \\ &= \langle \psi, \varphi.\iota^*(T)\rangle. \end{aligned}$$ ∎

***Definition 2.2*** (i) A linear functional $m$ on $A_0(K)^*$ is called a *mean* if $\|m\| = m(I) = 1$, where $I$ denotes the identity in $A_0(K)^*$.

(ii) A mean $m$ on $A_0(K)^*$ is said to be topologically invariant if $\varphi.m = \varphi(\dot{e})m$ for all $\varphi \in A_0(K)$ *i.e.*, $\langle T, \varphi.m\rangle = \langle \varphi.T, m\rangle = \varphi(\dot{e})\langle T, m\rangle$ $\forall$ $T \in A_0(K)^*$, $\forall$ $\varphi \in A_0(K)$.

***Proposition 2.3*** $\iota^{**}(\mathrm{TIM}(\widehat{K})) \subseteq \mathrm{TIM}_0(\widehat{K})$.

**Proof** Let $m \in \mathrm{TIM}(\widehat{K})$. Let $\varphi \in A_0(K)$ and $T \in A_0(K)^*$. Since $\varphi \in A_0(K)$, there exists a sequence $\{\varphi_n\} \subset A(K)$ such that $\varphi_n \to \varphi$ in the cb-multiplier norm. Hence $\varphi_n(\dot{e}) \to \varphi(\dot{e})$. Further, for any $\psi \in A_0(K)$,

$$\begin{aligned}|\langle \psi, \varphi_n.T - \varphi.T\rangle| &= |\langle \psi, (\varphi_n - \varphi).T\rangle| = |\langle (\varphi_n - \varphi)\psi, T\rangle| \\ &\le \|\varphi_n - \varphi\|_{A_0(K)}\|\psi\|_{A_0(K)}\|T\|_{A_0(K)^*}.\end{aligned}$$

Therefore, it follows that $\varphi_n.T \to \varphi.T$ in $A_0(K)^*$. Thus

$$\begin{aligned}2\langle \varphi.T, \iota^{**}(m)\rangle &= \langle \lim_{n\to\infty} \varphi_n.T, \iota^{**}(m)\rangle = \lim n \to \infty\langle \varphi_n.T, \iota^{**}(m)\rangle \\ &= \lim n \to \infty\langle \iota^*(\varphi_n.T), m\rangle \\ &= \lim n \to \infty\langle \varphi_n.\iota^*(T), m\rangle \qquad \text{(by Lemma 2.1)} \\ &= \lim n \to \infty\varphi_n(\dot{e})\langle \iota^*(T), m\rangle = \varphi(\dot{e})\langle \iota^*(T), m\rangle \\ &= \varphi(\dot{e})\langle T, \iota^{**}(m)\rangle.\end{aligned}$$ ∎

***Theorem 2.4*** *Let $K$ be an ultraspherical hypergroup. Then*

$$\iota^{**}\colon \mathrm{TIM}(\widehat{K}) \to \mathrm{TIM}_0(\widehat{K})$$

*is a bijection.*

**Proof** We first claim that $\iota^{**}\colon \mathrm{TIM}(\widehat{K}) \to \mathrm{TIM}_0(\widehat{K})$ is injective. In fact, let $m_1, m_2 \in \mathrm{TIM}(\widehat{K})$ with $m_1 \neq m_2$. Then there exists $T \in \mathrm{VN}(K)$ such that $m_1(T) \neq m_2(T)$. Choose $\varphi_0 \in A(K)$ with $\varphi_0(\dot{e}) = 1$. Then

$$\langle m_1, \varphi_0.T\rangle = \langle m_1, T\rangle \neq \langle m_2, T\rangle = \langle m_2, \varphi_0.T\rangle.$$

By Lemma 2.1, $\varphi.T \in A_0(K)^*$. Thus, we have,

$$\begin{aligned}\langle \varphi_0.T, \iota^{**}(m_1)\rangle &= \langle \iota^*(\varphi_0.T), m_1\rangle = \langle \varphi_0.T, m_1\rangle \qquad \text{(by Lemma 2.1)} \\ &\neq \langle \varphi_0.T, m_2\rangle = \langle \iota^*(\varphi_0.T), m_2\rangle \qquad \text{(by Lemma 2.1)} \\ &= \langle \varphi_0.T, \iota^{**}(m_2)\rangle.\end{aligned}$$

To show that $\iota^{**}\colon \mathrm{TIM}(\widehat{K}) \to \mathrm{TIM}_0(\widehat{K})$ is surjective, let $m \in \mathrm{TIM}_0(\widehat{K})$. Define $\widetilde{m} \in \mathrm{VN}(K)^*$ as $\langle T, \widetilde{m}\rangle := \langle \varphi.T, m\rangle$, for all $T \in \mathrm{VN}(K)$, where $\varphi \in A(K)$ with

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$\varphi_0(\dot{e}) = 1$. Notice that $\widetilde{m}$ is independent of the choice of $\varphi_0$ and hence it follows that $\widetilde{m} \in \mathrm{TIM}_0(\widehat{K})$. Now for any $T \in A_0(K)^*$, we have

$$\langle T, \iota^{**}(\widetilde{m})\rangle = \langle \iota^*(T), \widetilde{m}\rangle = \langle \varphi_0.\iota^*(T), m\rangle = \langle \iota^*(T), m\rangle = \langle T, m\rangle.$$

Thus $\iota^{**}(\widetilde{m}) = m$. ■

***Lemma 2.5*** *Let H be a closed subhypergroup. Then the restriction map $r_K$ is a contraction from $A_0(K)$ into $A_0(H)$.*

**Proof** The proof of this is exactly the same as in the group case. See [8, Lemma 1]. ■

***Corollary 2.6*** *Let H be a closed subhypergroup of K. Then*

$$r_K^{**}(\mathrm{TIM}_0(\widehat{K})) = \mathrm{TIM}_0(\widehat{H}).$$

**Proof** This follows from Theorem 1.10 and Theorem 2.4. ■

## 3 Generalized Translation and Generalized Invariant Means

Let $K$ be an ultraspherical hypergroup and let $K^*$ denote the set of all extreme points of $M_{B(K)}$. For $\varphi \in K^*$, define $L_\varphi\colon A(K) \to A(K)$ as $L_\varphi(\psi) = \varphi\psi$. The adjoint of $L_\varphi$ is called the generalized translation on $\mathrm{VN}(K)$ with respect to $\varphi \in K^*$. The map $L_\varphi$ is called the *generalized translation*.

We say that a subset $E$ of $A(K)$ is said to be $K^*$-invariant if $L_\varphi(E) \subseteq E$ for all $\varphi \in K^*$.

In the case of a locally compact group $G$, we know that a closed subspace $I$ of $L^1(G)$ is an ideal if and only if it is translation invariant [5, Proposition 2.43]. The same result is also true for hypergroups [13, Corollary 1]. The following is the corresponding analogue for $A(K)$. For the case of locally compact groups, see [1, Proposition 6.3]. As the proof of this is exactly the same as [1, Proposition 6.3], we omit the proof.

***Theorem 3.1*** *Let I be a closed subalgebra of $A(K)$. Suppose that for any $\varphi \in A(K)$, $\varphi \in \overline{\varphi A(K)}$. Then I is an ideal if and only if I is $K^*$-invariant.*

Similarly, a subset $F$ of $\mathrm{VN}(K)$ is said to be $K^*$-invariant if $L_\varphi^*(F) \subseteq F$ for all $\varphi \in K^*$.

***Theorem 3.2*** *Let $0 \neq T \in \mathrm{VN}(K)$. Then the following are equivalent.*

(i) *$L_\varphi^*(T) = T$ for all $\varphi \in K^*$.*
(ii) *$\varphi.T = T$ for all $\varphi \in M_{B(K)}$.*
(iii) *$\varphi.T = T$ for all $\varphi \in M_{A(K)}$.*
(iv) *$T = c\lambda(\dot{e})$ for $0 \neq c \in \mathbb{C}$.*

**Proof** (i) ⇒ (ii) is true, as $co(K^*)$ is dense in $M_{B(K)}$ with respect to the strict topology.

(ii) ⇒ (iii) is obvious, as $M_{A(K)} \subset M_{B(K)}$.

(iii) $\Rightarrow$ (iv). By the properties of the support of a linear functional, for each $\varphi \in M_{A(K)}$, it follows that $\text{supp}(T) \subset \text{supp}(\varphi)$. Since functions in $M_{A(K)}$ separate points, $\text{supp}(T) = \{\dot{e}\}$ and hence $T = c\lambda(\dot{e})$ for some non-zero scalar $c$.

(iv) $\Rightarrow$ (i). Let $\psi \in A(K)$. Then

$$\begin{aligned}\langle \psi, L_\varphi^*(T) \rangle &= \langle L_\varphi(\psi), T \rangle = \langle \varphi\psi, T \rangle = \langle \varphi\psi, c\lambda(\dot{e}) \rangle \\ &= c\langle \varphi\psi, \lambda(\dot{e}) \rangle = c\varphi(\dot{e})\psi(\dot{e}) = c\psi(\dot{e}) \\ &= \langle \psi, c\lambda(\dot{e}) \rangle \end{aligned}$$

■

***Definition 3.3*** A $K^*$*-invariant mean* (or *generalized invariant mean*) on $\text{VN}(K)$ is a mean $m$ on $\text{VN}(K)$ such that $m(L_\varphi^*(T)) = m(T)$ for all $\varphi \in K^*$.

We denote by $\text{TIM}(K^*)$ the set of all $K^*$-invariant means on $\text{VN}(K)$. As $A(K)$ is an ideal in $B(K)$, it follows that $\text{TIM}(\widehat{K}) \subseteq \text{TIM}(K^*)$. Thus by [17, Theorem 3.5], $\text{TIM}(K^*)$ is always a non-empty subset of $\text{VN}(K)^*$.

For any $m \in \text{TIM}(K^*)$, let

$$B_m(K) = \{\varphi \in B(K) : m(\varphi.T) = \varphi(\dot{e})m(T) \forall\ T \in \text{VN}(K)\},$$
$$B_{\text{TIM}}(K) = \bigcap_{m \in \text{TIM}(K^*)} B_m(K).$$

Also, let $B_{K^*}(K)$ denote the closure of the span of $K^*$ in $B(K)$.

***Lemma 3.4*** (i) *$B_m(K)$ is a closed subalgebra of $B(K)$.*
(ii) $m \in \text{TIM}(K^*)$ *if and only if* $B_{K^*}(K) \subseteq B_m(K)$.
(iii) $m \in \text{TIM}(\widehat{K})$ *if and only if* $A(K) \subseteq B_m(K)$.
(iv) $\text{TIM}(K^*) = \text{TIM}(\widehat{K})$ *if and only if* $A(K) \subseteq B_{\text{TIM}}(K)$.

The next theorem is an analogue of Theorem 1.10. As the proof is same as the proof of Theorem 1.10, we omit it. Here some extra conditions are needed because of the previous lemma.

***Theorem 3.5*** (i) *If* $r_K(B_{K^*}(K)) \subseteq B_{H^*}(H)$, *then* $r_K^{**}(\text{TIM}(K^*)) \supseteq \text{TIM}(H^*)$.
(ii) *If* $r_K(B_{K^*}(K)) \supseteq B_{H^*}(H)$, *then* $r_K^{**}(\text{TIM}(K^*)) \subseteq \text{TIM}(H^*)$.

The following theorem is an analogue of Theorem 2.4.

***Theorem 3.6*** *Let $H$ be a closed subhypergroup of an ultraspherical hypergroup $K$ such that $r_K(B_{K^*}(K)) \subseteq B_{H^*}(H)$. Then*
(i) $r_K(B_{\text{TIM}}(K)) \subseteq B_{\text{TIM}}(H)$.
(ii) *If* $\text{TIM}(K^*) = \text{TIM}(\widehat{K})$, *then* $\text{TIM}(H^*) = \text{TIM}(\widehat{H})$.

**Proof** (i) Let $\varphi \in B_{\text{TIM}}(K)$. Then $r_K(\varphi) \in B(H)$. For any $m \in \text{TIM}(H^*)$ and $S \in \text{VN}(H)$, we have

$$\begin{aligned}\langle r_K(\varphi).S, r_K^{**}(m) \rangle &= \langle r_K^*(r_K(\varphi).S), m \rangle = \langle \varphi.r_K^*(S), m \rangle \\ &= \varphi(\dot{e})\langle r_K^*(S), m \rangle = \varphi(\dot{e})\langle S, r_K^{**}(m) \rangle.\end{aligned}$$

Thus (i) follows.

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(ii) This follows from the hypothesis and (i). ∎

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