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THE ALPERIN WEIGHT CONJECTURE AND DADE'S CONJECTURE FOR THE SIMPLE GROUP Fi_{24}'

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Abstract

We classify the radical p-subgroups and chains of the Fischer simple group Fi_{24}' and then verify the Alperin weight conjecture and the Uno reductive conjecture for Fi_{24}'.

1. *Introduction*

Let G be a finite group, p a prime and B a p-block of G. Alperin [1] conjectured that the number of B-weights equals the number of irreducible Brauer characters of B. Dade [11] generalized the Knörr-Robinson version of the Alperin weight conjecture and presented his ordinary conjecture exhibiting the number of ordinary irreducible characters of a fixed defect in B in terms of an alternating sum of related values for p-blocks of some p-local subgroups of G. Dade [12] announced that his final conjecture needs only to be verified for finite non-abelian simple groups; in addition, if a finite group has a cyclic outer automorphism group, then the projective invariant conjecture is equivalent to the reductive conjecture.

Dade's reductive conjecture [12] has now been verified for all of the sporadic simple groups except the Fischer simple group Fi_{24}', the Baby Monster $\mathbb{B}$ for $p = 2$, and the Monster $\mathbb{M}$. Recently, Isaacs and Navarro [15] proposed a new conjecture which is a refinement of the Alperin-McKay conjecture, and Uno [21] raised an alternating sum version of the conjecture which is a refinement of the Dade conjecture [12]. In this paper, we use the local strategy of [4] and [5] to verify Alperin's conjecture and Uno's reductive conjectures for Fi_{24}'.

The paper is organized as follows. In Section 2, we fix notation and state the conjectures and two lemmas. In Section 3, we recall our modified local strategy and explain how we applied it to determine the radical subgroups of Fi_{24}'. In Section 4, we classify the radical subgroups of Fi_{24}' up to conjugacy and verify the Alperin weight conjecture. In Section 5, we do some cancellations in the alternating sum of Dade's conjecture when $p = 2$ or 3, and then determine radical chains (up to conjugacy) and their local structures. In Section 6, we verify Uno's invariant conjecture for Fi_{24}' and finally we verify Uno's projective conjecture for $3.\mathrm{Fi}_{24}'$. Two Appendices record degrees of irreducible characters of chain normalizers.

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2. *The conjectures and lemmas*

Let R be a p-subgroup of a finite group G and let $O_p(N(R))$ be the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Then R is *radical* if $O_p(N(R)) = R$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G and by $\mathrm{Blk}(G)$ the set of p-blocks. Let $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a B-*weight* if $\mathrm{d}(\varphi) = 0$ and $B(\varphi)^G = B$ (in the sense of Brauer), where $\mathrm{d}(\varphi) = \log_p(|G|_p) - \log_p(\varphi(1)_p)$ is the p-defect of φ and $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{2.1}$$

of G, define $|C| = n$, the k-th subchain $C_k : P_0 < P_1 < \cdots < P_k$, and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n). \tag{2.2}$$

The chain C is *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$ for $1 \leqslant k \leqslant n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G. Let $H \leqslant G, B \in \mathrm{Blk}(G)$, $\mathrm{Blk}(H, B) = \{b \in \mathrm{Blk}(H) : b^G = B\}$, and let $D(B)$ be a defect group of B. We define the p-local rank of B as follows:

$$plr(B) = \max\{|C| : C \in \mathcal{R},\ C : P_0 < \cdots < P_n \leqslant D(B),\ \mathrm{Blk}(N(C), B) \neq \emptyset\}.$$

Note that if there is a block b of $N_G(C)$ such that $b^G = B$, then the final subgroup P_n of C is G-conjugate to a subgroup of $D(B)$, so we may suppose $P_n \leqslant D(B)$. Thus our definition is equivalent to that given in [2, p. 370].

Let Z be a cyclic group and $\hat{G} = Z.G$ a central extension of Z by G, and $C \in \mathcal{R}(G)$. Denote by $N_{\hat{G}}(C)$ the preimage $\eta^{-1}(N(C))$ of $N(C)$ in $\hat{G}$, where η is the natural group homomorphism from $\hat{G}$ onto G with kernel Z. Let ρ be a faithful linear character of Z and $\hat{B}$ a block of $\hat{G}$ covering the block $B(\rho)$ of Z containing ρ. Denote by $\mathrm{Irr}(N_{\hat{G}}(C), \hat{B}, d, \rho)$ the set of irreducible characters ψ of $N_{\hat{G}}(C)$ such that ψ lies over ρ, $\mathrm{d}(\psi) = d$ and $B(\psi)^{\hat{G}} = \hat{B}$ and set $\mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \rho) = |\mathrm{Irr}(N_{\hat{G}}(C), \hat{B}, d, \rho)|$.

Dade's Projective Conjecture [12]. If $O_p(G) = 1$ and $\hat{B}$ is a p-block of $\hat{G}$ covering $B(\rho)$ with defect group $D(\hat{B}) \neq O_p(Z)$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \rho) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

Let $\hat{H}$ be a subgroup of a finite group $\hat{G}$, $\varphi \in \mathrm{Irr}(\hat{H})$ and let $r(\varphi) = r_p(\varphi)$ be the integer $0 < r(\varphi) \leqslant (p-1)$ such that the p'-part $(|\hat{H}|/\varphi(1))_{p'}$ of $|\hat{H}|/\varphi(1)$ satisfies

$$\left(\frac{|\hat{H}|}{\varphi(1)}\right)_{p'} \equiv r(\varphi) \mod p.$$

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Given an integer $1 \leqslant r \leqslant (p-1)$, let $\mathrm{Irr}(\hat{H},[r])$ be the subset of $\mathrm{Irr}(\hat{H})$ consisting of characters φ such that $r(\varphi) \equiv \pm r \bmod p$, and let $\mathrm{Irr}(\hat{H},\hat{B},d,\rho,[r]) = \mathrm{Irr}(\hat{H},\hat{B},d,\rho) \cap \mathrm{Irr}(\hat{H},[r])$ and $\mathrm{k}(\hat{H},\hat{B},d,\rho,[r]) = |\mathrm{Irr}(\hat{H},\hat{B},d,\rho,[r])|$.

Suppose $Z = 1$ and let $\hat{B} = B \in \mathrm{Blk}(G)$ with a defect group $D = D(B)$ and the Brauer correspondent $b \in \mathrm{Blk}(N_G(D))$. Then $\mathrm{k}(N(D),B,d(B),[r])$ is the number of characters $\varphi \in \mathrm{Irr}(b)$ such that φ has height 0 and $r(\varphi) \equiv \pm r \bmod p$, where $\mathrm{d}(B)$ is the defect of B.

Isaacs-Navarro Conjecture [15, Conjecture B]. *In the notation above,*

$$\mathrm{k}(G,B,\mathrm{d}(B),[r]) = \mathrm{k}(N(D),B,\mathrm{d}(B),[r]).$$

The following refinement of Dade's conjecture is due to Uno.

Uno's Projective Conjecture [21, Conjecture 3.2]. *If $O_p(G) = 1$ and if $D(\hat{B}) \neq O_p(Z)$, then for all integers $d \geqslant 0$, faithful $\rho \in \mathrm{Irr}(Z)$ and $1 \leqslant r \leqslant (p-1)$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_{\hat{G}}(C),\hat{B},d,\rho,[r]) = 0. \tag{2.3}$$

If $Z = 1$, then the projective conjecture is called the ordinary conjecture.

Note that if $p = 2$ or 3, then Uno's conjecture is equivalent to Dade's conjecture.

If $\hat{E}$ is an extension of $\hat{G}$ centralizing Z and $N_{\hat{E}}(C,\psi)$ is the stabilizer of $(N_{\hat{G}}(C),\psi)$ in $\hat{E}$, then $N_{\hat{E}/\hat{G}}(C,\psi) = N_{\hat{E}}(C,\psi)/N_{\hat{G}}(C)$ is a subgroup of $\hat{E}/\hat{G}$. For $\hat{U} \leqslant \hat{E}/\hat{G}$, denote by $\mathrm{k}(N_{\hat{G}}(C),\hat{B},d,\hat{U},\rho,[r])$ the number of characters ψ in $\mathrm{Irr}(N_{\hat{G}}(C),\hat{B},d,\rho,[r])$ such that $N_{\hat{E}/\hat{G}}(C,\psi) = \hat{U}$. In the notation above, Uno's projective invariant conjecture is stated as follows.

Uno's Projective Invariant Conjecture. If $O_p(G) = 1$ and $\hat{B}$ is a p-block of $\hat{G}$ covering $B(\rho)$ with $D(\hat{B}) \neq O_p(Z)$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_{\hat{G}}(C),\hat{B},d,\hat{U},\rho,[r]) = 0. \tag{2.4}$$

In addition, if $\hat{E}/\hat{G}$ is cyclic and $u = |\hat{U}|$, then we set

$$\mathrm{k}(N_{\hat{G}}(C),\hat{B},d,u,\rho,[r]) = \mathrm{k}(N_{\hat{G}}(C),\hat{B},d,\hat{U},\rho,[r]).$$

In particular, if $Z = 1$ and ρ is the trivial character of Z, then $\hat{G} = G$ and $\hat{B}$ is a block B of G; we set $U = \hat{U}$ and

$$\mathrm{k}(N(C),B,d,U,[r]) = \mathrm{k}(N_{\hat{G}}(C),\hat{B},d,\hat{U},\rho,[r]).$$

Then the Projective Invariant Conjecture is equivalent to the Invariant Conjecture.

Uno's Invariant Conjecture. If $O_p(G) = 1$ and B is a p-block of G with defect $\mathrm{d}(B) > 0$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N(C),B,d,U,[r]) = 0. \tag{2.5}$$

The following lemma follows by [6, Lemma 7.1].

Lemma 2.1. *Let G be a finite group, $B \in \mathrm{Blk}(G)$ with $plr(B) = 2$ and abelian defect group $D = D(B)$. Let $O_p(G) \neq R < D$ be radical, $b \in Blk(N_G(R))$ with $b^G = B$. Then*

$$\mathrm{k}(N_G(R) \cap N_G(D), b, d, \rho, [r]) = \mathrm{k}(N_G(R), b, d, \rho, [r]).$$

In Section 5, we will use the following lemma, whose proof is straight-forward. Note that it is more important to pair orbits of chains, rather than actual chains.

Lemma 2.2. *Let $\sigma : O_p(G) < P_1 < \ldots < P_{m-1} < Q = P_m < P_{m+1} < \ldots < P_\ell$ be a fixed radical p-chain of a finite group G, where $1 \leqslant m < \ell$. Suppose*

$$\sigma' : O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell$$

is also a radical p-chain such that $N_G(\sigma) = N_G(\sigma')$ and $N_E(\sigma) = N_E(\sigma')$, where E is an extension of G. Let $\mathcal{R}^-(\sigma, Q)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C whose $(\ell-1)$-th subchain $C_{\ell-1}$ is conjugate to σ' in G, and $\mathcal{R}^0(\sigma, Q)$ the subfamily of $\mathcal{R}(G)$ consisting of chains C whose ℓ-th subchain C_ℓ is conjugate to σ in G. Then the map g sending any $O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell < \ldots$ in $\mathcal{R}^-(\sigma, Q)$ to $O_p(G) < P_1 < \ldots < P_{m-1} < Q < P_{m+1} < \ldots < P_\ell < \ldots$ induces a bijection, denoted again by g, from $\mathcal{R}^-(\sigma, Q)$ onto $\mathcal{R}^0(\sigma, Q)$. Moreover, for any C in $\mathcal{R}^-(\sigma, Q)$, we have $|C| = |g(C)| - 1$, $N_G(C) = N_G(g(C))$ and $N_E(C) = N_E(g(C))$.

3. *The modified local strategy*

The maximal subgroups of Fi_{24}' were classified by Wilson [**22**]. Using this classification, we know that each radical 2- or 3-subgroup R of Fi_{24}' is radical in one of the 14 maximal subgroups M of Fi_{24}' and further that $N_{\mathrm{Fi}_{24}'}(R) = N_M(R)$.

In [**4**] and [**5**], a modified local subgroup strategy was developed to classify the radical subgroups R. We review this method here. Suppose M is a subgroup of G such that $N_M(R) = N_G(R)$.

Step (1). We first consider the case where M is a p-local subgroup. Let $Q = O_p(M)$, so that $Q \leqslant R$. Choose a subgroup X of M. We explicitly compute the coset action of M on the cosets of X in M; we obtain a group W representing this action, a group homomorphism f from M to W, and the kernel K of f. For a suitable X, we have $K = Q$ and the degree of the action of W on the cosets is usually much smaller than that of M. We can now directly classify the radical p-subgroup classes of W, and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M.

Step (2). Now consider the case where M is not p-local. We may be able to find its radical p-subgroup classes directly. Alternatively, we find a (maximal) subgroup L of M such that $N_L(R) = N_M(R)$ for each radical subgroup R of M. If L is p-local, then we apply Step (1) to L. If L is not p-local, we can replace M by L and repeat Step (2).

Steps (1) and (2) constitute the *modified local strategy*. After applying the strategy, possible fusions among the resulting list of radical subgroups can be decided readily by testing whether the subgroups in the list are pairwise Fi_{24}'-conjugate.

In investigating the conjectures for Fi_{24}', we used its minimal degree representation as a permutation group on 306936 points. Its maximal subgroups were constructed using the details supplied in [**9**] and the black-box algorithms of Wilson

[23]. We also made extensive use of the algorithm described in [10] to construct random elements, and the procedures described in [4] and [5] for deciding the conjectures.

In investigating the projective conjecture for $3.\mathrm{Fi}_{24}'$, we used its minimal degree representation as a permutation group on 920808 points. Both representations are available from the ATLAS of Finite Group Representations [24].

The computations reported in this paper were carried out using various versions of MAGMA [7]. The construction of the character tables of various chain normalisers posed significant practical problems. Many could not be computed directly using the MAGMA implementation of the algorithm of Schneider [18]. Instead, we used a new and more powerful algorithm developed by Unger [19].

4. *Radical subgroups and weights of* Fi_{24}'

Let $\mathcal{R}_0(G,p)$ be a set of representatives for conjugacy classes of radical p-subgroups of G. For $H, K \leqslant G$, we write $H \leqslant_G K$ if $x^{-1}Hx \leqslant K$; and write $H \in_G \mathcal{R}_0(G,p)$ if $x^{-1}Hx \in \mathcal{R}_0(G,p)$ for some $x \in G$. We shall follow the notation of [9]. In particular, if p is odd, then $p^{1+2\gamma}$ or $p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p; if δ is $+$ or $-$, then $2_\delta^{1+2\gamma}$ is an extra-special group of order $2^{1+2\gamma}$ with type δ. If X and Y are groups, we use $X.Y$ and $X:Y$ to denote an extension and a split extension of X by Y, respectively. Given a positive integer n, we use E_{p^n} or simply p^n to denote the elementary abelian group of order p^n, $\mathbb{Z}_n$ or simply n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

Let G be the simple Fischer group Fi_{24}'. Then

$$|G| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29,$$

and we may suppose $p \in \{2,3,5,7\}$, since both conjectures hold for a block with a cyclic defect group by [11, Theorem 9.1] and [3, Theorem 5.2].

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H and by $\mathrm{d}(H)$ the number $\log_p(|H|)$. Given $R \in \mathcal{R}_0(G,p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then (c.f. (4.1) of [4])

$$\mathcal{W}(B_0) = \sum_R |\mathrm{Irr}^0(N/C(R)R)|, \tag{4.1}$$

where R runs over the set $\mathcal{R}_0(G,p)$ such that $\mathrm{d}(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by MAGMA, and so we find $|\mathrm{Irr}^0(N/C(R)R)|$.

PROPOSITION 4.1. *Let $G = \mathrm{Fi}_{24}'$ and $E = \mathrm{Aut}(G) = G.2 = \mathrm{Fi}_{24}$. Then the non-trivial radical p-subgroups R of G (up to conjugacy) and their local structures are given in Tables 1 and 2 according as $p \geqslant 3$ or $p = 2$, where $S \in \mathrm{Syl}_2(G)$ is a Sylow 2-subgroup, H^* denotes a subgroup of G such that $H^* \simeq H$ and $H^* \neq_G H$. Moreover, $N_E(R) = N.2$ for all radical p-subgroups R except when $p = 7$ and $R = 7^2$, in which case $R^\tau = (7^2)^*$ for some $\tau \in E \backslash G$.*

Proof. Case (1). Suppose $p = 7$, so that by [9, p. 207], 7_+^{1+2} is a Sylow subgroup of G and 7_+^{1+2} has 19 subgroup classes. Thus the radical 7-subgroups can be determined easily.

R	$C(R)$	$N_G(R)$	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
7	$7 \times A_7$	$7{:}6 \times A_7$	
7^2	7^2	$7^2{:}2L_2(7).2$	2
$(7^2)^*$	7^2	$7^2{:}2L_2(7).2$	2
7_+^{1+2}	7	$7_+^{1+2}{:}(S_3 \times 6)$	18
5	$5 \times A_9$	$(5{:}2 \times A_9).2$	
5^2	$5^2 \times A_4$	$(5^2{:}4A_4 \times A_4).2$	16
3	$3 \times O_8^+(3){:}3$	$(3 \times O_8^+(3){:}3){:}2$	
3^2	$3^2 \times G_2(3)$	$(3^2{:}2 \times G_2(3)){:}2$	
3^7	3^7	$3^7.O_7(3)$	1
3_+^{1+10}	3	$3_+^{1+10}.U_5(2){:}2$	1
$3_+^{1+10}.3$	3	$3_+^{1+10}.3.U_4(2){:}2$	2
$3^3.3^4.3^3.3^3$	3^3	$3^3.3^4.3^3.3^3.(L_3(3) \times 2)$	2
$3^2.3^4.3^8$	3^2	$3^2.3^4.3^8.(A_5 \times 2A_4){:}2$	1
$3_+^{1+10}.3^4$	3	$3_+^{1+10}.3^4.(S_5 \times 2)$	2
$3_+^{1+10}.(3 \times 3_+^{1+2})$	3	$3_+^{1+10}.(3 \times 3_+^{1+2}).(2 \times 2S_4)$	4
$3^2.3^4.3^8.3$	3^2	$3^2.3^4.3^8.3.(2 \times 2A_4).2$	4
$3^2.3^4.3^8.3^2$	3	$3^2.3^4.3^8.3^2.2^3$	8

Table 1: Non-trivial radical p-subgroups of Fi_{24}' with $p \geqslant 3$

Case (2). Suppose $p = 5$, so that G has a unique class of elements x of order 5, $C(x) = 5 \times A_9 \leqslant (A_5 \times A_9).2$ and $N(\langle x\rangle) = (5{:}2 \times A_9).2$. Thus $5^2 \in \mathrm{Syl}_5(G)$, $C(5^2) = C_{A_5 \times A_9}(5^2) = 5^2 \times A_4$ and $N(5^2) = (5^2{:}\,4A_4 \times A_4).2$.

Case (3). Suppose $p = 3$, $i \in \{1, \dots, 6\}$, and M_i is a maximal 3-local subgroup of G where $M_1 = N(3A) \simeq (3 \times O_8^+(3){:}3){:}2$, and $M_2 = N(3^7) \simeq 3^7.O_7(3)$, and $M_3 = N(3B) \simeq 3_+^{1+10}{:}U_5(2){:}2$, and $M_4 = N(3B^2) \simeq 3^2.3^4.3^8{:}(A_5 \times 2A_4){:}2$, and $M_5 \simeq (3^2{:}2 \times G_2(3)).2$ and $M_6 = N(3B^3) = 3^3.3^4.3^3.3^3{:}(L_3(3) \times 2)$. By [22, Theorem B], we may suppose a 3-local subgroup R of G is a subgroup of some M_i with $N_G(R) = N_{M_i}(R)$. We apply the modified local strategy to each M_i.

Case (3.1) We may take

$$\mathcal{R}_0(M_3, 3) = \{3_+^{1+10}, 3_+^{1+10}.3, 3_+^{1+10}.3^4, 3_+^{1+10}.(3 \times 3_+^{1+2}), 3^2.3^4.3^8.3^2\} \qquad (4.2)$$

and moreover, $N(R) = N_{M_3}(R)$ and $N_E(R) = N(R).2$ for each $R \in \mathcal{R}_0(M_3, 3)$. We may suppose $\mathcal{R}_0(M_3, 3) \subseteq \mathcal{R}_0(G, 3)$.

Case (3.2) We may take

$$\mathcal{R}_0(M_4, 3) = \{3^2.3^4.3^8, 3_+^{1+10}.3^4, 3^2.3^4.3^8.3, 3^2.3^4.3^8.3^2\}, \qquad (4.3)$$

and moreover, $N(R) = N_{M_4}(R)$ and $N_E(R) = N(R).2$ for $R \in \mathcal{R}_0(M_4, 3)$. We may suppose $\mathcal{R}_0(M_4, 3) \subseteq \mathcal{R}_0(G, 3)$.

Case (3.3) We may take

$$\mathcal{R}_0(M_5, 3) = \{3^2, 3^4 \times 3_+^{1+2}, 3^2 \times (3^2 \times 3_+^{1+2}).3\} \qquad (4.4)$$

and moreover, $N(R) \neq N_{M_5}(R)$ and $N_E(R) = N(R).2$ for $R \in \mathcal{R}_0(M_5, 3) \setminus \{3^2\}$.

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In addition, $C(3^4 \times 3_+^{1+2}) = 3^5, C(3^2 \times (3^2 \times 3_+^{1+2}).3) = 3^4$, and

$$N_{M_5}(R) = \begin{cases} 3^2{:}2 \times (3^2 \times 3_+^{1+2}){:}2S_4 & \text{if } R = 3^4 \times 3_+^{1+2}, \\ (3^2{:}2 \times (3^2 \times 3_+^{1+2}){:}3.2^2).2 & \text{if } R = 3^2 \times (3^2 \times 3_+^{1+2}).3. \end{cases}$$

Case (3.4) We may take

$$\mathcal{R}_0(M_6, 3) = \{3^3.3^4.3^3.3^3, 3^2.3^4.3^8.3, 3_+^{1+10}(3 \times 3_+^{1+2}), 3^2.3^4.3^8.3^2\} \tag{4.5}$$

and moreover, $N(R) = N_{M_6}(R)$ and $N_E(R) = N(R).2$ for $R \in \mathcal{R}_0(M_6, 3)$. We may suppose $\mathcal{R}_0(M_6, 3) \subseteq \mathcal{R}_0(G, 3)$.

Case (3.5) We may take

$$\mathcal{R}_0(M_2, 3) = \{3^7, 3_+^{1+10}.3, 3^7.3^{3+3}, 3^7.3_+^{1+6}, 3^7.3_+^{1+6}.3, 3^7.3^5.3^3, 3^7.3^5.3_+^{1+2}, 3^2.3^4.3^8.3^2\}$$

and moreover, $N(R) \neq N_{M_2}(R)$ and $N_E(R) = N(R).2$ for $R \in \mathcal{R}_0(M_2, 3) \setminus \{3^7, 3_+^{1+10}.3\}$, $C(3^7.3^5.3^3) = C(3^7.3^5.3_+^{1+2}) = 3$, $C(3^7.3_+^{1+6}) = C(3^7.3_+^{1+6}.3) = 3^2$, $C(3^7.3^{3+3}) = 3^3$ and

$$N_{M_2}(R) = \begin{cases} 3^7.3^{3+3}.L_3(3) & \text{if } R = 3^7.3^{3+3}, \\ 3^7.3_+^{1+6}.(2A_4 \times A_4).2 & \text{if } R = 3^7.3_+^{1+6}, \\ 3^7.3_+^{1+6}3.2S_4 & \text{if } R = 3^7.3_+^{1+6}.3, \\ 3^7.3^5.3^3.(S_4 \times 2) & \text{if } R = 3^7.3^5.3^3, \\ 3^7.3^5.3_+^{1+2}.2S_4 & \text{if } R = 3^7.3^5.3_+^{1+2}, \\ 3^2.3^4.3^8.3^2.2^2 & \text{if } R = 3^2.3^4.3^8.3^2. \end{cases}$$

Case (3.6) Let R be a radical subgroup of $M_1 = (3 \times O_8^+(3){:}3){:}2$ and $Q = R \cap H$, where $H = 3 \times O_8^+(3)$ is a normal subgroup of M_1. Then Q is a radical subgroup of H and Q is normal in $N_{M_1}(R)$. In particular, $N_{M_1}(R) \leqslant N_{M_1}(Q)$ and R is radical in $N_{M_1}(Q)$.

Let $L_1 \simeq (3 \times 3^6{:}L_4(2)){:}2$, $L_2 \simeq (3 \times 3_+^{1+8}{:}2(A_4 \times A_4 \times A_4)2{:}3){:}2$ and $L_3 = 3^2{:}2 \times G_2(3)$ be p-local subgroups of M_1, so that we may suppose $Q \in \mathcal{R}_0(L_i \cap H, 3)$ (cf. [**9**, p. 140]). In addition, if $Q \leqslant \mathcal{R}_0(L_i \cap H, 3)$ for $i = 2, 3$, then $N_{M_1}(R) \leqslant L_i$ and so $R \in \mathcal{R}_0(L_i, 3)$ with $N(R) = N_{L_i}(R) = N_{M_1}(R)$.

We may take

$$\mathcal{R}_0(L_1, 3) = \{3^7, 3^7{:}3^3, 3^7{:}3^4, 3^7{:}3^3.3^2, 3^7{:}3^4.3, 3^7{:}3^4.3^2\}, \tag{4.6}$$

and moreover, $N(R) \neq N_{M_1}(R) \neq N_{L_1}(R)$ for $R \in \mathcal{R}_0(L_1, 3) \setminus \{3^7, 3^7{:}3^4\}$, $N(R) \neq N_{M_1}(R) = N_{L_1}(R)$ for $R = 3^7$ or $3^7{:}3^4.3^2$. In addition, $N_{M_1.2}(R) = N_{N_{M_1.2}(L_1)}(R)$ for $R \in \mathcal{R}_0(L_1, 3)$, $C(3^7) = 3^7$, $C(3^7{:}3^3) = 3^4$, $C(3^7{:}3^4) = C(3^7{:}3^3.3^2) = C(3^7{:}3^4.3^2) = 3^2$, $C(3^7{:}3^4.3) = 3^3$, and

$$N_{L_1}(R) = \begin{cases} 3^7.3^3.L_3(3) & \text{if } R = 3^7.3^3, \\ 3^7.3^4.2(A_4 \times A_4).2.2 & \text{if } R = 3^7.3^4, \\ 3^7.3^3.3^2.2S_4 & \text{if } R = 3^7.3^3.3^2, \\ 3^7.3^4.3.2S_4.2 & \text{if } R = 3^7.3^4.3, \\ 3^7.3^4.3^2.2^3 & \text{if } R = 3^7.3^4.3^2. \end{cases}$$

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R	$C(R)$	N	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
2	$2.\mathrm{Fi}_{22}.2$	$2.\mathrm{Fi}_{22}.2$	
2^2	$2^2.U_6(2)$	$2^2.U_6(2).S_3$	
$(2^2)^*$	$2^2 \times O_8^+(2){:}3$	$(A_4 \times O_8^+(2){:}3){:}2$	
D_8	$2 \times S_6(2)$	$D_8 \times S_6(2)$	
2^8	2^8	$2^8.O_8^-(2)$	1
2^{11}	2^{11}	$2^{11}.M_{24}$	0
2_+^{1+12}	2	$2_+^{1+12}.3U_4(3){:}2$	2
$2_+^{1+12}.2$	2	$2_+^{1+12}.2U_4(2)$	1
2^{6+8}	2^6	$2^{6+8}.(A_8 \times S_3)$	1
$2^{11}.2^4$	2^6	$2^{11}.2^4.A_8$	1
2^{3+12}	2^3	$2^{3+12}.(A_6 \times L_3(2))$	2
$2^{11}.2^6$	2	$2^{11}.2^6.3S_6$	1
$2^3.2^6.2^8$	2^3	$2^3.2^6.2^8.(S_3 \times L_3(2))$	1
$2_+^{1+12}.2^4$	2	$2_+^{1+12}.2^4.(A_6 \times S_3)$	2
$2^{3+12}.2^2$	2^2	$2^{3+12}.2^2(A_6 \times S_3)$	2
$2^3.2^3.2^6.2^5$	2^3	$2^3.2^3.2^6.2^5.(S_3 \times L_3(2))$	1
$2^{6+8}.2^3$	2^3	$2^{6+8}.2^3(L_3(2) \times S_3)$	1
$2^{11}.2^3.2^4$	2^3	$2^{11}.2^3.2^4.L_3(2)$	1
$2^{11}.2^{1+6}$	2^3	$2^{11}.2^{1+6}.L_3(2)$	1
$2_+^{1+12}.2.2^4$	2	$2_+^{1+12}.2.2^4.(S_3 \times S_3 \times S_3)$	1
$2_+^{1+12}.2^4.2$	2	$2_+^{1+12}.2^4.2.A_6$	2
$2^{11}.2^2.2^6$	2^2	$2^{11}.2^2.2^6.(S_3 \times S_3)$	1
$2^{11}.2^4.2^4$	2	$2^{11}.2^4.2^4.(S_3 \times S_3)$	1
$2^{1+12}.2.2^4.2$	2	$2^{1+12}.2.2^4.2.(S_3 \times S_3)$	1
$2^{1+12}.2^2.2^4$	2	$2^{1+12}.2^2.2^4.(S_3 \times S_3)$	1
$2^{6+8}.2^4.2$	2	$2^{6+8}.2^4.2.(S_3 \times S_3)$	1
$2^{6+8}.2^3.2^2$	2^2	$2^{6+8}.2^3.2^2(S_3 \times S_3)$	1
$2^{6+8}.2^4.2^2$	2	$2^{6+8}.2^4.2^2.S_3$	1
$2^{6+8}.2^3.2^3$	2^2	$2^{6+8}.2^3.2^3.S_3$	1
$2^{11}.2^2.2^3.2^4$	2	$2^{11}.2^2 2^3.2^4.S_3$	1
$2^{1+12}.2^4.2^2.2$	2	$2^{1+12}.2^4.2^2.2.S_3$	1
$2^{3+12}.(D_8 \times 2^2)$	2	$2^{3+12}.(D_8 \times 2^2).S_3$	1
S	2	S	1

Table 2: Non-trivial radical 2-subgroups of Fi_{24}'

Suppose $R \in \mathcal{R}_0(M_1, 3)$ such that $Q = R \cap H \in \mathcal{R}_0(L_1, 3)$. Then

$$N_{M_1}(R) = \begin{cases} 3^7.3^3.(L_3(3) \times 2) & \text{if } Q = 3^7.3^3, \\ 3^7.3^4.2(A_4 \times A_4).2.2 & \text{if } Q = 3^7.3^4, \\ 3^7.3^3.3^2.2S_4.2 & \text{if } Q = 3^7.3^3.3^2, \\ 3^5.3^4.3^4.2S_4.2 & \text{if } Q = 3^7.3^4.3, \\ (3 \times 3_+^{1+8}).3^3.3.2^3 & \text{if } Q = 3^7.3^4.3^2. \end{cases}$$

We may take

$$\mathcal{R}_0(L_2, 3) = \{3 \times 3_+^{1+8}, 3^7.3^4, (3 \times 3_+^{1+8}).3, (3 \times 3_+^{1+8}).3^2, (3 \times 3_+^{1+8}).3^3.3\}$$

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and then $N(R) \neq N_{M_1}(R) = N_{L_2}(R)$, $N_{M_1.2}(R) = N_{N_{M_1.2}(L_2)}(R)$ for all $R \in \mathcal{R}_0(L_2, 3)$. In addition, $C(R) = 3^2$ for $R \in \mathcal{R}_0(L_2, 3)$ and

$$N_{L_2}(R) = \begin{cases} (3 \times 3_+^{1+8}).3.2S_4.2 & \text{if } R = (3 \times 3_+^{1+8}).3, \\ (3 \times 3_+^{1+8}).3^2.2S_4.2 & \text{if } R = (3 \times 3_+^{1+8}).3^2, \\ (3 \times 3_+^{1+8}).3^3.3.2^3 & \text{if } R = (3 \times 3_+^{1+8}).3^3.3. \end{cases}$$

We may take

$$\mathcal{R}_0(L_3, 3) = \{3^2, 3^4 \times 3_+^{1+2}, (3^4 \times 3_+^{1+2})^*, 3^2 \times (3^2 \times 3_+^{1+2}).3\}, \tag{4.7}$$

and $N(R) \neq N_{M_1}(R) \neq N_{L_3}(R)$, $N_{M_1.2}(R) = N_{N_{M_1.2}(L_3)}(R)$ for $R \in \mathcal{R}_0(L_3, 3) \backslash \{3^2\}$.

It follows that $\mathcal{R}_0(M_1, 3) =$

$$\{3, 3^2, 3^7, 3 \times 3_+^{1+8}, 3^7.3^3, 3^7{:}3^4, (3 \times 3_+^{1+8}).3, (3 \times 3_+^{1+8}).3^2, 3^5.3^4.3^4, (3 \times 3_+^{1+8}).3^3.3\},$$

and $N(R) \neq N_{M_1}(R)$ for $R \in \mathcal{R}_0(M_3, 3) \backslash \{3\}$.

Case (4) For $1 \leqslant i \leqslant 8$, let M_i be the maximal 2-local subgroups of Fi_{24}' such that $M_1 = N(2A) \simeq 2.\mathrm{Fi}_{22}{:}2$, $M_2 = N(2^{11}) \simeq 2^{11}.M_{24}$, $M_3 = N(2A^2) \simeq 2^2.U_6(2){:}S_3$, $M_4 = N(2B) \simeq 2_+^{1+12}.3.U_4(3){:}2$, $M_5 = N(2A^2) \simeq (A_4 \times O_8^+(2){:}3){:}2$, $M_6 = N(2B^3) \simeq 2^{3+12}.(A_6 \times L_3(2))$, $M_7 = N(2^6) = 2^{6+8}.(A_8 \times S_3)$ and $M_8 = N(2^8) \simeq 2^8{:}O_8^-(2)$.

If R is a non-trivial radical 2-subgroup of $G = \mathrm{Fi}_{24}'$, then we may suppose $R \in \mathcal{R}_0(M_i, 2)$ such that $N(R) = N_{M_i}(R)$ for some $i = 1, \ldots, 8$.

Case (4.1) We may take

$$\begin{aligned}\mathcal{R}_0(M_2, 2) = \{&2^{11}, 2^{11}.2^4, 2^{11}.2^6, 2^3.2^6.2^8, 2^{11}.2^3.2^4, 2^{11}.2_+^{1+6}, 2^{11}.2^2.2^6, 2^{11}.2^4.2^4, \\ &2_+^{1+12}.2.2^4.2, 2^{6+8}.2^4.2^2, 2^{6+8}.2^3.2^3, 2^{11}.2^2.2^3.2^4, 2_+^{1+12}.2^4.2^2.2, S\}\end{aligned}$$

and moreover, $N(R) = N_{M_2}(R)$ for each $R \in \mathcal{R}_0(M_2, 2)$, so that we may suppose $\mathcal{R}_0(M_2, 2) \subseteq \mathcal{R}_0(G, 2)$.

Case (4.2) Let $K_1 \simeq (2^2 \times 2_+^{1+8}).U_4(2){:}S_3$, $K_2 \simeq 2^{11}.L_3(4){:}S_3$, $K_3 \simeq 2^{6+8}.(S_3 \times S_5).3$ and $K_4 \simeq D_8 \times S_6(2)$ be the maximal p-local subgroups of M_3 (cf. [9, p. 115]). Then we may suppose each radical subgroup R of M_3 is a subgroup of $\mathcal{R}_0(K_i, 3)$ for some i with $N(R) = N_{K_i}(R) = N_{M_3}(R)$.

The radical subgroups $R \in \mathcal{R}_0(K_1, 2)$ and their local structures are given in Table 3 and moreover, $N(R) \neq N_{M_3}(R) = N_{K_1}(R)$, $N_{M_3.2}(R) = N_{N_{M_3.2}(K_1)}(R)$ for each $R \in \mathcal{R}_0(K_1, 2)$.

We may take

$$\begin{aligned}\mathcal{R}_0(K_2, 2) = \{&2^{11}, 2^{11}.2, (2^2 \times 2_+^{1+8}).2^4, 2^{11}.2^4, 2^{6+8}.2^2, (2^2 \times 2_+^{1+8}).2^4.2, \\ &(2^2 \times 2_+^{1+8}).2^4.2^2, (2^2 \times 2_+^{1+8}).2^4.2^2.2\}\end{aligned}$$

and moreover, $N(R) \neq N_{M_3}(R) = N_{K_2}(R)$, $N_{M_3.2}(R) = N_{N_{M_3.2}(K_2)}(R)$ for each $R \in \mathcal{R}_0(K_2, 2)$. In addition, $C(2^{11}) = 2^{11}$, $C(2^{11}.2) = 2^7$, $C(2^{11}.2^4) = 2^6$, $C(2^{6+8}.2^2) = 2^4$ and

$$N_{K_2}(R) = \begin{cases} 2^{11}.2.L_2(7) & \text{if } R = 2^{11}.2, \\ 2^{11}.2^4.A_5.S_3 & \text{if } R = 2^{11}.2^4, \\ 2^{6+8}.2^2.S_3 & \text{if } R = 2^{6+8}.2^2. \end{cases}$$

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R	$C(R)$	$N_{K_1}(R)$
$2^2 \times 2_+^{1+8}$	2^3	$(2^2 \times 2_+^{1+8}).U_4(2){:}S_3$
$(2^2 \times 2_+^{1+8}).2$	2^2	$(2^2 \times 2_+^{1+8}).2S_6$
$(2^2 \times 2_+^{1+8}).2^4$	2^3	$(2^2 \times 2_+^{1+8}).2^4{:}A_5{:}S_3$
$(2^2 \times 2_+^{1+8}).2.2^4$	2^3	$(2^2 \times 2_+^{1+8}).2.2^4.3^2.2.S_3$
$(2^2 \times 2_+^{1+8}).2^4.2$	2^2	$(2^2 \times 2_+^{1+8}).2^4.2.S_3$
$(2^2 \times 2_+^{1+8}).2^4.2^2$	2^3	$(2^2 \times 2_+^{1+8}).2^4.2^2.(3 \times S_3)$
$(2^2 \times 2_+^{1+8}).2.2^4.2$	2^2	$(2^2 \times 2_+^{1+8}).2.2^4.2.S_3$
$(2^2 \times 2_+^{1+8}).2^4.2^2.2$	2^2	$(2^2 \times 2_+^{1+8}).2^4.2^2.2$

Table 3: Radical 2-subgroups of $(2^2 \times 2_+^{1+8}).U_4(2){:}S_3$

We may take

$$\mathcal{R}_0(K_3, 2) = \{2^{6+8}, 2^{11}.2^4, 2^{6+8}.2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{6+8}.2^2, (2^2 \times 2_+^{1+8}).2^4.2^2, (2^2 \times 2_+^{1+8}).2.2^4.2, (2^2 \times 2_+^{1+8}).2^4.2^2.2\}$$

and moreover, $N(R) \neq N_{M_3}(R) = N_{K_3}(R)$, $N_{M_3.2}(R) = N_{N_{M_3.2}(K_3)}(R)$ for each $R \in \mathcal{R}_0(K_3, 2)$. In addition, $C(2^{6+8}) = 2^6$, $C(2^{6+8}.2) = 2^4$, and $N_{K_3}(2^{6+8}.2) = 2^{6+8}.2.(S_3 \times S_3)$.

The radical subgroups $R \in \mathcal{R}_0(K_4, 2)$ and their local structures are given in Table 4 and moreover, $N(R) \neq N_{M_3}(R) \neq N_{K_4}(R)$, $N_{M_3.2}(R) = N_{N_{M_3.2}(K_4)}(R)$ for each $R \in \mathcal{R}_0(K_4, 2) \setminus \{D_8\}$.

R	$C(R)$	$N_{K_4}(R)$
D_8	$2 \times S_6(2)$	$D_8 \times S_6(2)$
$D_8 \times 2^5$	2^6	$D_8 \times 2^5{:}S_6$
$D_8 \times 2^6$	2^7	$D_8 \times 2^6{:}L_3(2)$
$D_8 \times 2^3.2^4$	2^4	$D_8 \times 2^3.2^4.(S_3 \times S_3)$
$D_8 \times 2^3.2^2.2^3$	2^4	$D_8 \times 2^3.2^2.2^3.S_3$
$D_8 \times 2^3.2^5$	2^4	$D_8 \times 2^3.2^5.S_3$
$D_8 \times 2^2.2^3.2^3.S_3$	2^3	$D_8 \times 2^2.2^3.2^3.S_3$
$D_8 \times 2^3.2^5.2$	2^2	$D_8 \times 2^3.2^5.2$

Table 4: Radical 2-subgroups of $D_8 \times S_6(2)$

It follows that we may take

$$\begin{aligned}\mathcal{R}_0(M_3, 2) = \{&2^2, D_8, 2^{11}, 2^2 \times 2_+^{1+8}, 2^{11}.2, (2^2 \times 2_+^{1+8}).2, 2^{6+8}, 2^{11}.2^4,\\ &(2^2 \times 2_+^{1+8}).2^4, 2^{6+8}.2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{6+8}.2^2,\\ &(2^2 \times 2_+^{1+8}).2^4.2, (2^2 \times 2_+^{1+8}).2.2^4.2,\\ &(2^2 \times 2_+^{1+8}).2^4.2^2, (2^2 \times 2_+^{1+8}).2^4.2^2.2\}\end{aligned}$$

and moreover, $N(R) \neq N_{M_3}(R)$ for each $R \in \mathcal{R}_0(M_3, 2) \setminus \{2^2, D_8\}$.

Case (4.3) We may take

$$\mathcal{R}_0(M_4, 2) = \{2^{1+12}, 2^{1+12}.2, 2^{1+12}.2^4, 2^{11}.2^6, 2_+^{1+12}.2.2^4, 2_+^{1+12}.2^4.2,$$

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$$2_+^{1+12}.2.2^4.2, 2_+^{1+12}.2^2.2^4, 2^{11}.2^4.2^4, 2^{6+8}.2^4.2, 2^{6+8}.2^4.2^2,$$
$$2^{3+12}.(D_8 \times 2^2), 2^{11}.2^2.2^3.2^4, 2_+^{1+12}.2^4.2^2.2, S\}$$

and moreover, $N(R) = N_{M_4}(R)$ for each $R \in \mathcal{R}_0(M_4, 2)$, so that we may suppose $\mathcal{R}_0(M_4, 2) \subseteq \mathcal{R}_0(G, 2)$.

Case (4.4) The radical subgroups $R \in \mathcal{R}_0(M_5, 2)$ and their local structures are given in Table 5 and moreover, $N(R) \neq N_{M_5}(R)$, $N_{M_5.2}(R) = N_{M_5}(R).2$ for each $R \in \mathcal{R}_0(M_5, 2) \setminus \{(2^2)^*, D_8\}$ and $N(R) = N_{M_5}(R)$ for $R \in \{(2^2)^*, D_8\}$.

R	$C(R)$	$N_{M_5}(R)$
$(2^2)^*$	$2^2 \times O_8^+(2){:}3$	$(A_4 \times O_8^+(2){:}3){:}2$
D_8	$2 \times S_6(2)$	$D_8 \times S_6(2)$
2^8	2^8	$(A_4 \times 2^6{:}A_8){:}2$
$2^8.2$	2^6	$2^8.2.S_6$
$2^8.2^3$	2^5	$2^8.2^3.3(L_3(2) \times S_3)$
$2^3.2^8$	2^3	$2^3.2^8.3.(S_3 \times S_3 \times S_3).S_3$
$2^8.2^4$	2^3	$2^8.2^4.(S_3 \times S_3).S_3$
$2^3.2^8.2$	2^2	$2^3.2^8.2.(S_3 \times S_3)$
$2^4.2^4.2^4$	2^4	$2^4.2^4.2^4.L_3(2)$
$2^4.2^3.2^6$	2^4	$2^4.2^3.2^6.(S_3 \times S_3).3$
$2^8.2^2.2^3$	2^2	$2^8.2^2.2^3.S_3$
$2^3.2^8.2^2$	2^3	$2^3.2^8.2^2.(S_3 \times S_3)$
$2^8.2^3.2^2.2$	2^3	$2^8.2^3.2^2.2.S_3$
$2^3.2^8.2^3$	2^3	$2^3.2^8.2^3.S_3$
$2^3.2^8.2^2.2$	2^2	$2^3.2^8.2^2.2.S_3$
$2^8.2^4.2^3$	2^2	$2^8.2^4.2^3$

Table 5: Radical 2-subgroups of $(A_4 \times O_8^+(2){:}3){:}2$

Case (4.5) We may take

$$\mathcal{R}_0(M_6, 2) = \{2^{3+12}, 2^3.2^6.2^8, 2^{3+12}.2^2, 2^3.2^3.2^6.2^5, 2^{1+12}.2^4, 2^{11}.2^3.2^4,$$
$$2^{1+12}.2^4.2, 2^{6+8}.2^3.2^2, 2^{6+8}.2^4.2, 2^{11}.2^4.2^4, 2^{11}.2^2.2^6, 2^{11}.2^2.2^3.2^4,$$
$$2_+^{1+12}.2^4.2^2.2, 2^{6+8}.2^3.2^3, 2^{3+12}.(D_8 \times 2), S\}$$

and moreover, $N(R) = N_{M_6}(R)$ for each $R \in \mathcal{R}_0(M_6, 2)$.

Case (4.6) We may take

$$\mathcal{R}_0(M_7, 2) = \{2^{6+8}, 2^{11}.2^4, 2^{6+8}.2^3, 2^3.2^3.2^6.2^5, 2^{11}.2^{1+6}, 2^{11}.2^3.2^4,$$
$$2^{1+12}.2.2^4, 2^{6+8}.2^4.2, 2^{1+12}.2.2^4.2, 2^{1+12}.2^2.2^4, 2^{6+8}.2^3.2^2,$$
$$2^{6+8}.2^4.2^2, 2^{1+12}.(D_8 \times 2^2), 2^{6+8}.2^3.2^3, 2^{11}.2^2.2^3.2^4, S\}$$

and moreover, $N(R) = N_{M_7}(R)$ for each $R \in \mathcal{R}_0(M_7, 2)$.

Case (4.7) We may take $\mathcal{R}_0(M_8, 2) =$

$$\{2^8, 2^{1+12}.2, 2^8.2^{3+6}, 2^8.2^{1+8}, 2^8.2^6.2^4, 2^8.2^6.2.2^4, 2^8.2^{3+6}.2^2, 2^8.2^{3+6}.D_8\}$$

and moreover, $N(R) \neq N_{M_8}(R)$ for each $R \in \mathcal{R}_0(M_8, 2) \setminus \{2^8, 2^{1+12}.2\}$, $N(2^{1+12}.2) = N_{M_8}(2^{1+12}.2)$, $C(2^8{:}2^{3+6}) = 2^3$, $C(2^8.2^6.2^4) = C(2^8.2^6.2.2^4) = C(2^8.2^{3+6}.D_8) = 2,$

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$C(2^8.2^{1+8}) = 2^2 = C(2^8{:}2^{3+6}.2^2)$ and

$$N_{M_8}(R) = \begin{cases} 2^8.2^{3+6}.(L_3(2)\times 3) & \text{if } R = 2^8.2^{3+6}, \\ 2^8.2^{1+8}.(S_3\times A_5) & \text{if } R = 2^8.2^{1+8}, \\ 2^8.2^6.2^4.A_5 & \text{if } R = 2^8.2^6.2^4, \\ 2^8.2^6.2.2^4.3^2.2 & \text{if } R = 2^8.2^6.2.2^4, \\ 2^8.2^{3+6}.2^2.(S_3\times 3) & \text{if } R = 2^8.2^{3+6}.2^2, \\ 2^8.2^{3+6}.D_8.3 & \text{if } R = 2^8.2^{3+6}.D_8. \end{cases}$$

Case (4.8) Let

$$\begin{array}{ll} T_1 \simeq 2^2.U_6(2){:}2, & T_2 \simeq 2^{11}.M_{22}{:}2, \\ T_3 \simeq 2^8.S_6(2), & T_4 \simeq 2^2.2^{10}.U_4(2).2, \\ T_5 \simeq 2^{6+8}.(S_3\times S_6), & T_6 \simeq (2^2\times O_8^+(2){:}3){:}2 \end{array}$$

be the maximal 2-local subgroups of $M_1 = 2.\mathrm{Fi}_{22}{:}2$. Then we may suppose each radical subgroup of M_1 is a subgroup of $\mathcal{R}_0(T_i,3)$ for some i with $N(R) = N_{T_i}(R) = N_{M_1}(R)$.

The radical subgroups $R\in\mathcal{R}_0(T_1,2)$ and their local structures are given in Table 6 and moreover, $N(R)\neq N_{M_1}(R)\neq N_{T_1}(R)$ for each $R\in\mathcal{R}_0(T_1,2)\backslash\{2^2,D_8\}$.

R	$C(R)$	$N_{T_1}(R)$
2^2	$2^2.U_6(2)$	$2^2.U_6(2){:}2$
D_8	$2\times S_6(2)$	$D_8\times S_6(2)$
2^{11}	2^{11}	$2^{11}.L_3(4){:}2$
$2^2\times 2_+^{1+8}$	2^3	$(2^2\times 2_+^{1+8}).U_4(2).2$
$2^{11}.2$	2^7	$2^{11}.2.L_2(7)$
$(2^2\times 2_+^{1+8}).2$	2^2	$(2^2\times 2_+^{1+8}).2S_6$
2^{6+8}	2^6	$2^{6+8}.(S_3\times S_5)$
$2^{11}.2^4$	2^6	$2^{11}.2^4.S_5$
$(2^2\times 2_+^{1+8}).2^4$	2^3	$(2^2\times 2_+^{1+8}).2^4.S_5$
$2^{6+8}.2$	2^4	$2^{6+8}.2.(S_3\times S_3)$
$(2^2\times 2_+^{1+8}).2.2^4$	2^3	$(2^2\times 2_+^{1+8}).2.2^4.(S_3\times S_3)$
$(2^2\times 2_+^{1+8}).2^4.2$	2^2	$(2^2\times 2_+^{1+8}).2^4.2^2.S_3$
$2^{6+8}.2^2$	2^4	$2^{6+8}.2^2.S_3$
$(2^2\times 2_+^{1+8}).2.2^4.2$	2^2	$(2^2\times 2_+^{1+8}).2.2^4.2.S_3$
$(2^2\times 2_+^{1+8}).2^4.2^2$	2^3	$(2^2\times 2_+^{1+8}).2^4.2^2.S_3$
$(2^2\times 2_+^{1+8}).2^4.D_8$	2^2	$(2^2\times 2_+^{1+8}).2^4.D_8$

Table 6: Radical 2-subgroups of $2^2.U_6(2){:}2$

We may take $\mathcal{R}_0(T_2,2) =$

$$\{2^{11}, 2^{11}.2^4, 2^{11}.2^3.2, 2^{11}.2^4.2, 2^{11}.2^2.2^3.2, 2^{6+8}.2^3.2, 2^{11}.2^2.2^4.2, 2^{6+8}.2^3.2^2\}$$

and moreover, $N(R)\neq N_{M_1}(R) = N_{T_2}(R)$ for each $R\in\mathcal{R}_0(T_2,2)$.

We may take

$$\mathcal{R}_0(T_3,2) = \{2^8, 2^8.2^5, 2^8.2^6, 2^8.2^3.2^4, 2^8.2^5.2^3, 2^8.2^3.2^4.2, 2^2.2^3.2^5.2^6, 2^8.2^5.2^3.2\}$$

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and moreover, $N(R) \neq N_{M_1}(R) \neq N_{T_3}(R)$ for each $R \in \mathcal{R}_0(T_3,2)\backslash\{2^8, 2^8.2^5\}$ and $N(R) \neq N_{M_1}(R) = N_{T_3}(R)$ for $R \in \{2^8, 2^8.2^5\}$. In addition, $C(2^8.2^6) = 2^4$, $C(2^8.2^3.2^4) = C(2^8.2^3.2^4.2) = 2^3$, $C(2^8.2^5.2^3) = C(2^2.2^3.2^5.2^6) = C(2^8.2^5.2^3.2) = 2^2$, and

$$N_{T_3}(R) = \begin{cases} 2^8.2^6.L_3(2) & \text{if } R = 2^8.2^6, \\ 2^8.2^3.2^4.(S_3 \times S_3) & \text{if } R = 2^8.2^3.2^4, \\ 2^8.2^5.2^3.S_3 & \text{if } R = 2^8.2^5.2^3, \\ 2^8.2^3.2^4.2.S_3 & \text{if } R = 2^8.2^3.2^4.2, \\ 2^2.2^3.2^5.2^6.S_3 & \text{if } R = 2^2.2^3.2^5.2^6, \\ 2^8.2^5.2^3.2 & \text{if } R = 2^8.2^5.2^3.2. \end{cases}$$

We may take $\mathcal{R}_0(T_4,2) =$

$$\{2^2.2^{10}, 2^8.2^5, 2^{11}.2^4.2, 2^2.2^{10}.2.2^4, 2^{11}.2^2.2^3.2, 2^{11}.2^2.2^4.2, 2^{6+8}.D_8.2, 2^{6+8}.2^3.2^2\}$$

and moreover, $N(R) \neq N_{M_1}(R) = N_{T_4}(R)$ for each $R \in \mathcal{R}_0(T_4,2)$.

The radical subgroups $R \in \mathcal{R}_0(T_6,2)$ and their local structures are given in Table 7. Moreover, $N(R) \neq N_{M_1}(R) \neq N_{T_6}(R)$ for each $R \in \mathcal{R}_0(T_6,2)\backslash\{2^2, D_8\}$.

R	$C(R)$	$N_{T_6}(R)$
$(2^2)^*$	$2^2 \times O_8^+(2){:}3$	$(2^2 \times O_8^+(2){:}3){:}2$
D_8	$2 \times S_6(2)$	$D_8 \times S_6(2)$
2^8	2^8	$2^8.A_8{:}2$
$2^8.2$	2^6	$2^8.2.S_6$
$2^3.2^8$	2^3	$2^3.2^8.3(S_3 \times S_3 \times S_3).2$
$2^8.2^4$	2^3	$2^8.2^4.(S_3 \times S_3).2$
$2^4.2^4.2^4$	2^4	$2^4.2^4.2^4.L_3(2)$
$2^3.2^8.2$	2^2	$2^3.2^8.2.(S_3 \times S_3)$
$2^4.2^3.2^6$	2^4	$2^4.2^3.2^6.(S_3 \times S_3)$
$2^8.2^2.2^3$	2^2	$2^8.2^2.2^3.S_3$
$2^3.2^8.2^3$	2^3	$2^3.2^8.2^3.S_3$
$2^8.2^3.2^2.2$	2^3	$2^8.2^3.2^2.2.S_3$
$2^3.2^8.2^2.2$	2^2	$2^3.2^8.2^2.2.S_3$
$2^8.2^4.2^3$	2^2	$2^8.2^4.2^3$

Table 7: Radical 2-subgroups of $(2^2 \times O_8^+(2){:}3){:}2$

We may take $\mathcal{R}_0(T_5,2) =$

$$\{2^{6+8}, 2^{11}.2^4, 2^{6+8}.2^3, 2^2.2^{10}.2.2^4, 2^{11}.2^2.2^4.2, 2^{6+8}.D_8.2, 2^{6+8}.2^3.2, 2^{6+8}.2^3.2^2\}$$

and moreover, $N(R) \neq N_{M_1}(R) = N_{T_5}(R)$ for each $R \in \mathcal{R}_0(T_5,2)$.

It follows that the radical subgroups of M_1 and their local structures are given in Table 8. Moreover, $N(R) \neq N_{M_1}(R)$ for each $R \in \mathcal{R}_0(M_1,2)\backslash\{2, D_8\}$.

This completes the classification of radical 2-subgroups of G. The centralizers and normalizers of $R \in \mathcal{R}_0(G,2)$ are given by MAGMA. □

LEMMA 4.2. *Let* $G = \mathrm{Fi}_{24}'$, *and let* $\mathrm{Blk}^0(G,p)$ *be the set of* p*-blocks with a non-trivial defect group and* $\mathrm{Irr}^+(G)$ *the characters of* $\mathrm{Irr}(G)$ *with positive* p*-defect. If a defect group* $D(B)$ *of* B *is cyclic and* $p \neq 2$, *then* $\mathrm{Irr}(B)$ *is given by* [**14**, p. 352].

R	$C(R)$	$N_{M_1}(R)$
2	$2.\mathrm{Fi}_{22}.2$	$2.\mathrm{Fi}_{22}.2$
2^2	$2^2.U_6(2)$	$2^2.U_6(2).2$
$(2^2)^*$	$2^2 \times O_8^+(2){:}3$	$(2^2 \times O_8^+(2){:}3){:}2$
D_8	$2 \times S_6(2)$	$D_8 \times S_6(2)$
2^8	2^8	$2^8.S_6(2)$
2^{11}	2^{11}	$2^{11}.M_{22}{:}2$
$2^2.2^{10}$	2^2	$2^2.2^{10}.U_4(2).2$
$2^8.2^5$	2^2	$2^8.2^5.S_6$
2^{6+8}	2^6	$2^{6+8}.(S_6 \times S_3)$
$2^{11}.2^4$	2^6	$2^{11}.2^4.S_6$
$2^{11}.2^3.2$	2^4	$2^{11}2^3.2.L_3(2)$
$2^{11}.2^4.2$	2^2	$2^{11}.2^4.2.S_5$
$2^{11}.2^2.2^3.2$	2^2	$2^{11}.2^2.2^3.2.S_3$
$2^2.2^{10}.2.2^4$	2^2	$2^2.2^{10}.2.2^4.(S_3 \times S_3)$
$2^{6+8}.2^3$	2^3	$2^{6+6}.2^3.(S_3 \times S_3)$
$2^{6+8}.2^3.2$	2^3	$2^{6+6}.2^3.2S_3$
$2^{11}.2^2.2^4.2$	2^2	$2^{11}.2^2.2^4.2.S_3$
$2^{6+8}.D_8.2$	2^2	$2^{6+8}.D_8.2.S_3$
$2^{6+8}.2^3.2^2$	2^2	$2^{6+8}.2^3.2^2$

Table 8: Radical 2-subgroups of $2.\mathrm{Fi}_{22}{:}2$

(a) *If* $p = 7$, *then* $\mathrm{Blk}^0(G,p) = \{B_i \mid 0 \leqslant i \leqslant 4\}$ *such that* $D(B_0) \simeq 7_+^{1+2}$ *and* $D(B_i) \simeq 7$ *for* $i \geqslant 1$, *where* $B_0 = B_0(G)$ *is the principal block of* G. *In addition,* $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\backslash(\cup_{i=1}^4 \mathrm{Irr}(B_i))$, $\ell(B_i) = 6$ *for* $1 \leqslant i \leqslant 4$ *and* $\ell(B_0) = 22$.

(b) *If* $p = 5$, *then* $\mathrm{Blk}^0(G,p) = \{B_i \mid 0 \leqslant i \leqslant 5\}$ *such that* $D(B_i) \simeq 5^2$ *and* $D(B_j) \simeq 5$ *with* $0 \leqslant i \leqslant 2$ *and* $3 \leqslant j \leqslant 5$. *In the notation of* [9, p. 200], $\mathrm{Irr}(B_1) =$

$$\{\chi_k : k \in \{4,5,13,17,19,20,21,23,25,31,33,51,54,55,67,71,79,85,94,103\}\}$$

$\mathrm{Irr}(B_2) = \{\chi_k : k \in \{2,3,16,38,46,47,52,64,65,77,78,82,88,91,92,104\}\}$, *and*
$\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\backslash(\cup_{i=1}^4 \mathrm{Irr}(B_i))$. *Moreover,* $\ell(B_i) = 4$ *for* $3 \leqslant i \leqslant 4$, $\ell(B_5) = 2$, $\ell(B_2) = 14$, *and* $\ell(B_1) = \ell(B_0) = 16$.

(c) *If* $p = 3$, *then* $\mathrm{Blk}^0(G,2) = \{B_0, B_1\}$ *such that* $D(B_1) \simeq 3^2$. *In the notation of* [9, p. 200],

$$\mathrm{Irr}(B_1) = \{\chi_k : k \in \{59,76,88,89,91,92\}\}$$

and $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\backslash\mathrm{Irr}(B_1)$. *Moreover,* $\ell(B_1) = 4$ *and* $\ell(B_0) = 25$.

(d) *If* $p = 2$, *then* $\mathrm{Blk}^0(G,2) = \{B_0, B_1, B_2\}$ *such that* $D(B_1) \simeq (2^2)^*$ *and* $D(B_2) \simeq D_8$. *In the notation of* [9, p. 184],

$$\mathrm{Irr}(B_1) = \{\chi_{35}, \chi_{64}, \chi_{65}, \chi_{79}\},$$

$\mathrm{Irr}(B_2) = \{\chi_{54}, \chi_{71}, \chi_{84}, \chi_{95}, \chi_{104}\}$ *and* $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\backslash(\mathrm{Irr}(B_1) \cup \mathrm{Irr}(B_2))$. *Moreover,* $\ell(B_1) = \ell(B_2) = 3$ *and* $\ell(B_0) = 33$.

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In addition, let $D = D_8$, 3_+^{1+2} *or* 5^2 *be a subgroup of the covering group* $\hat{G} = 3.\mathrm{Fi}_{24}'$, *and let* $\mathrm{Irr}(Z(3.\mathrm{Fi}_{24}')) = \{1, \zeta, \zeta'\}$. *If* $B(\zeta)$ *is the block of* $Z(3.\mathrm{Fi}_{24}')$ *containing* ζ, *then* $3.\mathrm{Fi}_{24}'$ *has a unique block* $\hat{B}_1$ *such that* $\hat{B}_1$ *covers* $B(\zeta)$ *and* $D(\hat{B}_1) \simeq D$. *Let* $\hat{B}_1'$ *be the complex dual of* $\hat{B}_1$, *so that* $\hat{B}_1'$ *covers* $B(\zeta')$. *In the notation of* [9, p. 204], *if* $\mathrm{Irr}(\hat{B}_1, \zeta)$ *denotes the characters of* $\mathrm{Irr}(\hat{B}_1)$ *covering* ζ, *then*

$$\mathrm{Irr}(\hat{B}_1, \zeta) \cup \mathrm{Irr}(\hat{B}_1', \zeta') = \begin{cases} \{\chi_j, \chi_{j+1} : j \in \{203, 205, 215, 217, 239\}\} & \text{if } p = 2, \\ \{\chi_j, \chi_{j+1} : j \in \{167, 169, 237, 247\}\} & \text{if } p = 3, \end{cases}$$

and $\mathrm{Irr}(\hat{B}_1, \zeta) \cup \mathrm{Irr}(\hat{B}_1', \zeta') = \{\chi_j, \chi_{j+1} : j \in \{113, 121, 123, 127, 129, 135, 151, 161, 175, 189, 199, 203, 213, 217, 219, 237, 245, 247, 251, 255\}\}$ *when* $p = 5$.

Moreover, if D *is a Sylow* p*-subgroup of* $\hat{G}$, *then* $\hat{G}$ *has a block* $\hat{B}$ *covering* $B(\zeta)$ *with* $D(\hat{B}) \simeq D$. *Let* $\hat{B}'$ *be the complex dual of* $\hat{B}$, *so that* $\hat{B}'$ *covers* $B(\zeta')$. *Let*

$$\Omega = \mathrm{Irr}(\hat{B}_1, \zeta) \cup \mathrm{Irr}(\hat{B}_1', \zeta') \cup \Big(\bigcup_{B'} \mathrm{Irr}(B')\Big),$$

where B' *runs over the blocks of* $\hat{G}$ *with cyclic defect group* p *given by* [14, p. 352]. *Then*

$$\mathrm{Irr}(\hat{B}, \zeta) \cup \mathrm{Irr}(\hat{B}', \zeta') = \mathrm{Irr}^+(\hat{G}) \setminus (\Omega \cup \mathrm{Irr}^+(G)).$$

Proof. If $B \in \mathrm{Blk}(G, p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character θ and $N(\theta)/C(D)D$ is a p'-group, where $N(\theta)$ is the stabilizer of θ in $N(D)$. If p is odd, then by [14, p. 352], we may suppose D is non-cyclic, so by Proposition 4.1, $D \in_G \{5^2, 3^2, 2, 2^2, (2^2)^*, D_8\}$.

If $D = 5^2$, then $C(D) = 5^2 \times A_4$ and $N(D) = (5^2{:}4A_4 \times A_4).2$, so that $N(D)$ has 2 orbits on $\mathrm{Irr}^0(C(D)/D)$ and G has 2 blocks with $D = 5^2$.

If $D = 3^2$, then $C(D) = 3^2 \times G_2(3)$ and $N(D) = (3^2{:}2 \times G_2(3)).2$, so that $N(D)$ has one orbit on $\mathrm{Irr}^0(C(D)/D)$ and G has one block with $D = 3^2$.

If $D = 2$, the $C(D) = N(D) = 2.\mathrm{Fi}_{22}.2$ and $|\mathrm{Irr}^0(C(D)/D)| = 0$.

If $D = 2^2$, then $C(D) = 2^2.U_6(2)$ and $N(D) = 2^2.U_6(2).S_3$, so that the only character of $\mathrm{Irr}^0(C(D)/D)$ is stabilized by a 2-element in $N(D)\backslash C(D)$. In particular, G has no block with $D = 2^2$.

If $D = (2^2)^*$, then $C(D) = 2^2 \times O_8^+(2){:}3$ and $N(D) = (A_4 \times O_8^+(2){:}3){:}2$. Now $\mathrm{Irr}^0(C(D)/D)$ consists of two characters θ with $N(\theta)/C(D)$ a p'-group, and both lying in an orbit of $N(D)$. Thus G has one block with $D = (2^2)^*$.

If $D = D_8$, then $C(D) = 2 \times S_6(2)$ and $N(D) = D_8 \times S_6(2)$, so that $N(D)$ has one orbit on $\mathrm{Irr}^0(C(D)/D)$ and G has one block with $D = D_8$.

Using the method of central characters, $\mathrm{Irr}(B)$ is as above. If $D(B)$ is cyclic or isomorphic to 2^2 or D_8, then $\ell(B)$ is the number of B-weights (see [11] and [20]).

Suppose $p = 5$ and $B = B_0, B_1, B_2$, so that $D(B) =_G 5^2$ and the non-trivial elements of $D(B_1)$ consists of $5A$-elements. By [16, Theorem 4.13],

$$\mathrm{k}(B_i) = \ell(B_i) + \sum_{b_i} \ell(b_i),$$

where b_i runs over blocks of $C_G(5A) = 5 \times A_9$ such that $b_i^G = B_i$. Thus $b_i = B_0(5) \times b_i'$ for some $b_i' \in \mathrm{Blk}(A_9)$ with $D(b_i') = 5$ and $\ell(b_i) = \ell(b_i')$. Since $D(b_i')$ is cyclic, it follows that $\ell(b_i')$ is the number of b_i'-weights. Now $N_{A_9}(D(b_i')) \simeq (5{:}2 \times A_4).2$, so for each i there exists a unique such block b_i' of A_9, and $\ell(b_0') = 4 = \ell(b_1')$, $\ell(b_2') = 2$. It follows that $\ell(B_0) = \ell(B_1) = 16$ and $\ell(B_2) = 14$.

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Suppose $p = 3$ and $B = B_1$, so that $D(B) =_G 3^2$. The non-trivial elements of $D(B)$ lie in $3A$ and $3E$, $C_G(3A) = 3 \times O_8^+(3){:}3$ and $C_G(3E) = 3^2 \times G_2(3)$. But $D(B) \cap O_8^+(3) = 1$ and $C_{O_8^+(3):3}(D(B) \cap O_8^+(3){:}3) = 3 \times G_2(3)$, so a similar argument to above shows that $\ell(B) = 6 - 1 - 1 = 4$.

If $\ell_p(G)$ is the number of p-regular G-conjugacy classes, then $\ell_7(G) = 95$, $\ell_5(G) = 94$, $\ell_3(G) = 30$ and $\ell_2(G) = 41$. Thus $\ell(B_0)$ can be calculated by the following equation due to Brauer:

$$\ell_p(G) = \bigcup_{B \in \mathrm{Blk}^0(G,p)} \ell(B) + |\mathrm{Irr}^0(G)|,$$

where $|\mathrm{Irr}^0(G)| = 49, 38, 1$ or 2 when $p = 7, 5, 3$ or 2.

Since $C_{\hat{G}}(5^2) = 3 \times 5^2 \times A_4$, it follows that $\hat{G}$ has 7 blocks with defect group 5^2 and 3 of these are blocks of G. Similarly, since $C_{\hat{G}}(D_8) = 6 \times S_6(2)$, it follows that $\hat{G}$ has 3 blocks with defect group D_8. Using the method of central characters, $\mathrm{Irr}(\hat{B})$ is as stated for $\hat{B} \in \mathrm{Blk}(\hat{G})$. □

Theorem 4.3. *Let $G = \mathrm{Fi}_{24}'$ and let B be a p-block of G with a non-cyclic defect group. Then the number of B-weights is the number of irreducible Brauer characters of B.*

Proof. If $B = B_0$, then the proof follows by Proposition 4.1, Lemma 4.2 and (4.1). Suppose $B \neq B_0$.

If $p = 2$, then $B = B_1$ or B_2, and $D(B) \simeq 2^2$ or D_8, so that the weight conjecture for B follows by Uno [**20**].

Suppose $p = 3$ and $B = B_1$, so that $D = D(B_1) = 3^2$. Since $C(D) \simeq 3^2 \times G_2(3)$ and $N(D) \simeq (3^2{:}2 \times G_2(3)){:}2$, it follows that B has 4 weights and so $\ell(B) = \mathcal{W}(B)$.

Suppose $p = 5$ and $B = B_1$ or B_2, so that $D = D(B) = 5^2$. Thus $C(D) = 5^2 \times A_4$ and $N(D) = (5^2{:}4A_4 \times A_4).2$. Since $|\mathrm{Irr}(N(D)/D)| = 46$, G has 46 weights of the form (D, φ) for $\varphi \in \mathrm{Irr}(N(D)/D)$. If $B = B_1$, then the canonical character θ of the root block b of B has an extension to $N(D)$, so that B has 16 weights, since $|\mathrm{Irr}(N(D)/C(D))| = 16$. Thus $\mathcal{W}(B_2) = 46 - 16 - 16 = 14 = \ell(B_2)$. □

5. *Radical chains of* Fi_{24}'

Let $G = \mathrm{Fi}_{24}'$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$. In this section, we do some cancellations in the alternating sum of Uno's conjecture. First we list some radical p-chains $C(i)$ with their normalizers, then reduce the proof of the conjecture to the subfamily $\mathcal{R}^0(G)$ of $\mathcal{R}(G)$ consisting of the union of G-orbits of all $C(i)$. The subgroups of the p-chains in Tables 10 – 13 are given either by Tables 1 and 2 or in the proofs of Lemmas 4.1 and 5.1. If p is 5 or 7, then the radical p-chains are given in Table 9.

Lemma 5.1. *Let $G = \mathrm{Fi}_{24}'$, $E = G.2$ and let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that*

$$\mathcal{R}^0(G)/G = \begin{cases} \{C(i) : 1 \leqslant i \leqslant 36\} & \text{with } C(i) \text{ given in Table 10 if } p = 3, \\ \{C(i) : 1 \leqslant i \leqslant 88\} & \text{with } C(i) \text{ given in Tables 11– 13 if } p = 2, \end{cases}$$

Then

$$\sum_{C\in\mathcal{R}(G)/G}(-1)^{|C|}\mathrm{k}(N(C),B,d,u,[r]) = \sum_{C\in\mathcal{R}^0(G)/G}(-1)^{|C|}\mathrm{k}(N(C),B,d,u,[r]) \tag{5.1}$$

for all integers $d,r,u \geqslant 0$. *If* $p=3$ *or* 2, *then* $N_E(C(i)) = N(C(i)).2$ *for each* i.

C		$N(C)$	$N_E(C)$
$C(1)$	1	Fi_{24}'	Fi_{24}
$C(2)$	$1<5$	$(5{:}2\times A_9).2$	$N(C).2$
$C(3)$	$1<5<5^2$	$(5^2\times A_4).4^2$	$N(C).2$
$C(4)$	$1<5^2$	$(5^2{:}4A_4\times A_4).2$	$N(C).2$
$C(1)$	1	Fi_{24}'	Fi_{24}
$C(2)$	$1<7$	$7:6\times A_7$	$N(C).2$
$C(3)$	$1<7<7^2$	$7^2.(6\times 3)$	$N(C).2$
$C(4)$	$1<7^2$	$7^2{:}2SL_2(7).2$	$N(C)$
$C(5)$	$1<7^2<7_+^{1+2}$	$7^{1+2}{:}(6\times 2)$	$N(C)$
$C(6)$	$1<(7^2)^*$	$7^2{:}2SL_2(7).2$	$N(C)$
$C(7)$	$1<(7^2)^*<7_+^{1+2}$	$7^{1+2}{:}(6\times 2)$	$N(C)$
$C(8)$	$1<7^{1+2}$	$7^{1+2}{:}(S_3\times 6)$	$N(C).2$

Table 9: Radical p-chains of Fi_{24}' with $p=5,7$

Proof. Throughout, C and C' will denote radical chains $C: 1<P_1<\cdots<P_n$ and

$$C': 1<P_1'<\ldots<P_m'. \tag{5.2}$$

Thus $C\in\mathcal{R}(G)$ and P_1 is a radical subgroup of G. We may suppose $P_1\in\mathcal{R}_0(G,p)$. Sometimes we also denote by $g(C')$ a radical chain of G corresponding to C' such that $N(C')=N(g(C'))$ and $N_E(C')=N_E(g(C'))$.

Case (1) Suppose $p=3$, and $R\in\mathcal{R}_0(M_3,3)\backslash\{3^{1+10}\}$ given by (4.2). Let $\sigma(R)$: $1<Q=3^{1+10}<R$ and $\sigma(R)': 1<R$. Then $\sigma(R)$ and $\sigma(R)'$ satisfy the conditions of Lemma 2.2, so there is a bijection g from $\mathcal{R}^-(\sigma(R),3^{1+10})$ onto $\mathcal{R}^0(\sigma(R)',3^{1+10})$ such that $N(C')=N(g(C'))$, $N_E(C')=N_E(g(C'))$ and $|C'|=|g(C')|-1$ for each $C'\in\mathcal{R}^-(\sigma(R),3^{1+10})$. Thus

$$\mathrm{k}(N(C'),B,d,u,[r]) = \mathrm{k}(N(g(C')),B,d,u,[r]), \tag{5.3}$$

and we may suppose

$$C\notin \bigcup_{R\in\mathcal{R}_0(M_3,3)\backslash\{3^{1+10}\}} (\mathcal{R}^-(\sigma(R),3^{1+10})\cup\mathcal{R}^0(\sigma(R),3^{1+10})). \tag{5.4}$$

Thus we may suppose $P_1\notin\{3^{1+10}.3, 3^{1+10}.3^4, 3^{1+10}.(3\times 3^{1+2}), 3^2.3^4.3^8.3^2\}$, and if $P_1=3^{1+10}$, then $C=_G C(28)$. We may suppose

$$P_1\in\{3,3^2,3^7,3^3.3^4.3^3.3^3,3^2.3^4.3^8,3^2.3^4.3^8.3\}\subseteq\mathcal{R}_0(G,3).$$

Let $C': 1<3^2.3^4.3^8<3^2.3^4.3^8.3^2$ and $g(C'): 1<3^2.3^4.3^8<3_+^{1+10}.3^4<3^2.3^4.3^8.3^2$, where $3^2.3^4.3^8.3^2, 3_+^{1+10}.3^4\in\mathcal{R}_0(M_4,3)$. Then $N(C')=N(g(C'))$ and $N_E(C')=N_E(g(C'))$ and we may suppose $C\neq_G C', g(C')$.

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C		$N(C)$
$C(1)$	1	Fi_{24}'
$C(2)$	$1<3$	$(3\times O_8^+(3){:}3){:}2$
$C(3)$	$1<3<3^2$	$3^2{:}2\times G_2(3)$
$C(4)$	$1<3<3^2<3^4\times 3_+^{1+2}$	$3^2{:}2\times(3^2\times 3_+^{1+2}){:}2S_4$
$C(5)$	$1<3<3^7$	$3^7{:}L_4(3){:}2$
$C(6)$	$1<3<3^7<3^7{:}3^3$	$3^7{:}3^3.L_3(3)$
$C(7)$	$1<3<3^7<3^7{:}3^3<3^7{:}3^4.3$	$3^7{:}3^4.3.2S_4$
$C(8)$	$1<3<3^7<3^7{:}3^4$	$3^7{:}3^4{:}2(A_4\times A_4).2.2$
$C(9)$	$1<3<3^7<3^7{:}3^4<3^7.3^3.3^2$	$3^7.3^3.3^2.2S_4$
$C(10)$	$1<3<3^7<3^7{:}3^4<3^7.3^3.3^2<3^7.3^4.3^2$	$3^7.3^4.3^2.2^2$
$C(11)$	$1<3<3^7<3^7{:}3^4.3<3^7.3^4.3^2$	$3^7.3^4.3^2.2^3$
$C(12)$	$1<3<3^7<3^7{:}3^4.3$	$3^7{:}3^4.3.2S_4.2$
$C(13)$	$1<3<3\times 3_+^{1+8}$	$(3\times 3_+^{1+8})2(A_4\times A_4\times A_4).2{:}3){:}2$
$C(14)$	$1<3<3^7.3^3<3^3.3^3.3^6$	$3^3.3^3.3^6.(2\times 2S_4)$
$C(15)$	$1<3<3^7.3^3$	$3^7.3^3.(2\times L_3(3))$
$C(16)$	$1<3<3^7.3^3<3^2.3^2.3^4.3^4$	$3^2.3^2.3^4.3^4.(2\times 2S_4)$
$C(17)$	$1<3<3^7.3^3<3^3.3^3.3^6<3^7.3^4.3^2$	$3^7.3^4.3^2.2^3$
$C(18)$	$1<3<3^5.3^4.3^4<(3\times 3_+^{1+8}).3^3.3$	$(3\times 3_+^{1+8}).3^3.3.2^3$
$C(19)$	$1<3<3^5.3^4.3^4$	$3^5.3^4.3^4.2S_4.2$
$C(20)$	$1<3^7$	$3^7.O_7(3)$
$C(21)$	$1<3^7<3_+^{1+10}.3$	$3_+^{1+10}.3.U_4(2){:}2$
$C(22)$	$1<3^7<3^7.3^{3+3}<3^7.3^5.3_+^{1+2}$	$3^7.3^5.3_+^{1+2}.2S_4$
$C(23)$	$1<3^7<3^7.3^{3+3}$	$3^7.3^{3+3}.L_3(3)$
$C(24)$	$1<3^7<3^7.3_+^{1+6}<3^7.3_+^{1+6}.3$	$3^7.3_+^{1+6}.3.2S_4$
$C(25)$	$1<3^7<3^7.3_+^{1+6}$	$3^7.3_+^{1+6}.(2A_4\times A_4).2$
$C(26)$	$1<3^7<3^7.3_+^{1+6}<3^7.3^5.3^3$	$3^7.3^5.3^3.(S_4\times 2)$
$C(27)$	$1<3^7<3^7.3_+^{1+6}<3^7.3_+^{1+6}.3<3^2.3^4.3^8.3^2$	$3^2.3^4.3^8.3^2.2^2$
$C(28)$	$1<3_+^{1+10}$	$3_+^{1+10}.U_5(2){:}2$
$C(29)$	$1<3^2.3^4.3^8<3_+^{1+10}.3^4$	$3_+^{1+10}.3^4.(2\times A_5){:}2$
$C(30)$	$1<3^2.3^4.3^8$	$3^2.3^4.3^8.(2A_4\times A_5){:}2$
$C(31)$	$1<3^2<3^2\times(3^2\times 3_+^{1+2}).3$	$(3^2{:}2\times(3^2\times 3_+^{1+2}).3.2^2).2$
$C(32)$	$1<3^2$	$(3^2{:}2\times G_2(3)){:}2$
$C(33)$	$1<3^3.3^4.3^3.3^3<3^2.3^4.3^8.3$	$3^2.3^4.3^8.3.(2S_4\times 2)$
$C(34)$	$1<3^3.3^4.3^3.3^3$	$3^3.3^4.3^3.3^3{:}(L_3(3)\times 2)$
$C(35)$	$1<3^3.3^4.3^3.3^3<3_+^{1+10}.(3\times 3_+^{1+2})$	$3_+^{1+10}.(3\times 3_+^{1+2}).(2S_4\times 2)$
$C(36)$	$1<3^3.3^4.3^3.3^3<3^2.3^4.3^8.3<3^2.3^4.3^8.3^2$	$3^2.3^4.3^8.3^2.2^3$

Table 10: Some radical 3-chains of Fi_{24}'

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Let $\sigma : 1 < Q = 3^2.3^4.3^8 < 3^2.3^4.3^8.3$, so that $\sigma' : 1 < 3^2.3^4.3^8.3$. A similar proof to above shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 3^2.3^4.3^8) \cup \mathcal{R}^0(\sigma, 3^2.3^4.3^8)). \tag{5.5}$$

In particular, we may suppose $P_1 \neq_G 3^2.3^4.3^8.3$ and moreover, if $P_1 = 3^2.3^4.3^8$, then $C \in_G \{C(29), C(30)\}$.

Let $\sigma : 1 < Q = 3 < 3^2 < 3^4 \times 3^{1+2}$, so that $\sigma' : 1 < 3^2 < 3^4 \times 3^{1+2}$. A similar proof to above shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 3) \cup \mathcal{R}^0(\sigma, 3).$$

In particular, we may suppose $P_3 \neq_G 3^4 \times 3^{1+2}$ when $P_1 = 3$ and $P_2 = 3^2$, and moreover, if $P_1 = 3^2$, then $C \in_G \{C(31), C(32)\}$.

Let $C' : 1 < 3^3.3^4.3^3.3^3 < 3^2.3^4.3^8.3^2$ and $g(C') : 1 < 3^3.3^4.3^3.3^3 < 3^{1+10}.(3 \times 3^{1+2}) < 3^2.3^4.3^8.3^2$. Then $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C'))$ and we may suppose $C \neq_G C', g(C')$. Thus if $P_1 = 3^3.3^4.3^3.3^3$, then $C \in_G \{C(33), C(34), C(35), C(36)\}$.

Suppose $P_1 = 3^7 \in \mathcal{R}_0(M_2, 3)$. Applying the Borel-Tits Theorem [**8**] (see [**17**, p. 361]) to $O_7(3)$, it follows that $C \in_G \{C(j) : 20 \leqslant j \leqslant 27\}$.

Finally, suppose $P_1 = 3 \in \mathcal{R}_0(M_1, 3)$. Let $R \in \mathcal{R}_0(L_2, 3)\backslash\{3 \times 3^{1+8}\} \subseteq \mathcal{R}_0(M_1, 3)$, and $\sigma(R) : 1 < 3 < Q = 3 \times 3^{1+8} < R$, so that $\sigma(R)' : 1 < 3 < R$. A similar proof to above shows that (5.4) holds with 3^{1+10} replaced by $3 \times 3^{1+8}$ and $\mathcal{R}_0(M_3, 3)\backslash\{3^{1+10}\}$ by $\mathcal{R}_0(L_2, 3) \setminus \{3 \times 3^{1+8}\}$.

In particular, we may suppose $P_2 \notin \mathcal{R}_0(L_2, 3)\backslash\{3 \times 3^{1+8}\}$ and moreover, if $P_2 = 3 \times 3^{1+8}$, then $C =_G C(13)$.

We may suppose $P_2 \in \{3^2, 3^7, 3^7.3^3, 3^5.3^4.3^4\}$ and if $P_2 = 3^2$, then $P_3 \neq_G 3^4 \times 3^{1+2}$. Let $C' : 1 < 3 < 3^2 < (3^4 \times 3^{1+2})^* < (3^4 \times 3^{1+2}).3$ and $g(C') : 1 < 3 < 3^2 < (3^4 \times 3^{1+2}).3$. Then $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C'))$ and we may suppose $C \neq_G C', g(C')$. Thus if $P_1 = 3$ and $P_2 = 3^2$, then $C \in_G \{C(3), C(4)\}$.

If $P_2 = 3^5.3^4.3^4$, then $N_{M_1}(P_2) = 3^5.3^4.3^4.2S_4.2$, so that $C \in_G \{C(18), C(19)\}$.

If $P_2 = 3^7.3^3 \in \mathcal{R}_0(M_1, 3)$, then $L_0 := N_{M_1}(P_2) = 3^7.3^3.(2 \times L_3(3))$ and we may take

$$\mathcal{R}_0(L_0, 3) = \{3^7.3^3, 3^3.3^3.3^6, 3^2.3^2.3^4.3^4, 3^7.3^4.3^2\},$$

and $N_{L_0}(3^3.3^3.3^6) = 3^3.3^3.3^6.(2S_4 \times 2)$, $N_{L_0}(3^2.3^2.3^4.3^4) = 3^2.3^2.3^4.3^4.(2S_4 \times 2)$ and
$N_{L_0}(3^7.3^4.3^2) = 3^7.3^4.3^2.2^3$.

Let $C' : 1 < 3 < 3^7.3^3 < 3^2.3^2.3^4.3^4 < 3^7.3^4.3^2$ and $g(C') : 1 < 3 < 3^7 < 3^7.3^3 < 3^7.3^4.3^2$. Then $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C'))$ and we may suppose $C \neq_G C', g(C')$. It follows that if $P_1 = 3$ and $P_2 = 3^7.3^3$, then $C \in_G \{C(i) : 14 \leqslant i \leqslant 17\}$.

Suppose $P_2 = 3^7$, so that $L_1 = N_{M_1}(3^7)$. If $3^7.3^3 \in \mathcal{R}_0(L_1, 3)$, then $L_4 := N_{L_1}(3^7.3^3)$
$= 3^7.3^3.L_3(3)$ and we may take

$$\mathcal{R}_0(L_4, 3) = \{3^7.3^3, 3^7.3^4.3, 3^7.3^3.3^2, 3^7.3^4.3^2\},$$

and $N_{L_4}(3^7.3^3.3^2) = N_{L_1}(3^7.3^3.3^2)$, $N_{L_4}(3^7.3^4.3) = 3^7.3^4.3.2S_4 \neq N_{L_1}(3^7.3^4.3)$ and
$N_{L_4}(3^7.3^4.3^2) = 3^7.3^4.3^2.2^2$.

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Let $\sigma : 1 < 3 < 3^7 < Q = 3^7.3^3 < 3^7.3^3.3^2$, so that $\sigma' : 1 < 3 < 3^7 < 3^7.3^3.3^2$. A similar proof to above shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $3^7.3^3$. Let $C' : 1 < 3 < 3^7 < 3^7.3^3 < 3^7.3^4.3 < 3^7.3^4.3^2$ and $g(C') : 1 < 3 < 3^7 < 3^7.3^3 < 3^7.3^4.3^2$. Then $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C'))$ and we may suppose $C \neq_G C', g(C')$. Thus we may suppose $P_3 \neq_G 3^7.3^3.3^2$ and if $P_3 = 3^7.3^4$, then $P_4 \neq_G 3^7.3^3.3^2$.

If $3^7.3^4 \in \mathcal{R}_0(L_1, 3)$, then $L_5 := N_{L_1}(3^7.3^4) = 3^7.3^4.2.(A_4 \times A_4).2.2$ and we may take

$$\mathcal{R}_0(L_5, 3) = \{3^7.3^4, 3^7.3^3.3^2, 3^7.3^4.3^2\},$$

and $N_{L_5}(R) = N_{L_1}(R)$ and $N_{L_1.2}(R) = N_{N_{L_1.2}(L_5)}(R)$ for each $R \in \mathcal{R}_0(L_5, 3)$.

Let $C' : 1 < 3 < 3^7 < 3^7.3^4.3^2$ and $g(C') : 1 < 3 < 3^7 < 3^7.3^4 < 3^7.3^4.3^2$. Then $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C'))$ and we may suppose $C \neq_G C', g(C')$. It follows that if $P_1 = 3$ and $P_2 = 3^7$, then $C \in_G \{C(i) : 5 \leqslant i \leqslant 12\}$.

Case (2) Let $R \in \mathcal{R}_0(M_2, 2)\backslash\{2^{11}\}$, and $\sigma(R) : 1 < Q = 2^{11} < R$, so that $\sigma(R) : 1 < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by 2^{11} and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(M_2, 2) \setminus \{2^{11}\}$. Thus if $P_1 = 2^{11}$, then $C =_G C(42)$ and we may assume that $P_1 \notin \mathcal{R}_0(M_2, 2)$.

Case (2.1) Let $\sigma : 1 < Q = 2 < 2^2 < D_8$, so that $\sigma' : 1 < 2^2 < D_8$. A similar proof to above shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by 2. Thus if $P_1 = 2^2$, then $P_2 \neq_G D_8$, and if $P_1 = 2$ and $P_2 = 2^2$, then $P_3 \neq_G D_8$.

Let $R \in \mathcal{R}_0(K_2, 2)\backslash\{2^{11}\}$, and $\sigma(R) : 1 < 2^2 < Q = 2^{11} < R$, so that $\sigma(R)' : 1 < 2^2 < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with $\mathcal{R}_0(M_3, 2) \setminus \{3^{1+10}\}$ replaced by $\mathcal{R}_0(K_2, 2)\backslash\{2^{11}\}$.

Let $\mathcal{V} = \{(2 \times 2_+^{1+8}).2, (2 \times 2_+^{1+8}).2.2^4, (2 \times 2_+^{1+8}).2.2^4.2\} \subseteq \mathcal{R}_0(K_1, 2)$, $R \in \mathcal{V}$ and let $\sigma(R) : 1 < 2^2 < Q = 2 \times 2_+^{1+8} < R$, so that $\sigma(R)' : 1 < 2^2 < R$. A similar proof to Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2 \times 2_+^{1+8}$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{V}$. Thus if $P_1 = 2^2$, then $P_2 \notin_G \mathcal{V}$ and in addition, if $P_2 = 2 \times 2_+^{1+8}$, then $P_3 \notin_G \mathcal{V}$.

If $V_1 = (2 \times 2_+^{1+8}){:}2^4.A_5{:}S_3 \leqslant K_1$, then we may take $\mathcal{R}_0(V_1, 2) =$

$$\{(2 \times 2_+^{1+8}){:}2^4, (2 \times 2_+^{1+8}){:}2^4.2, (2 \times 2_+^{1+8}){:}2^4.2^2, (2 \times 2_+^{1+8}){:}2^4.2^2.2\} \subseteq \mathcal{R}_0(K_1, 2),$$

so that $\mathcal{R}_0(K_1, 2) = \mathcal{V} \cup \mathcal{R}_0(V_1, 2)$. In addition, $N_{K_1}(R) = N_{V_1}(R)$, $N_{K_1.2}(R) = N_{N_{K_1.2}(V_1)}(R)$ for each $R \in \mathcal{R}_0(V_1, 2)$.

Let $R \in \mathcal{R}_0(V_1, 2) \setminus \{(2 \times 2_+^{1+8}){:}2^4\}$ and $\sigma(R) : 1 < 2^2 < 2 \times 2_+^{1+8} < Q = (2 \times 2_+^{1+8}){:}2^4 < R$, so that $\sigma(R)' : 1 < 2^2 < 2 \times 2_+^{1+8} < R$. We may suppose (5.4) holds with 3^{1+10} replaced by $(2 \times 2_+^{1+8}){:}2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(V_1, 2) \setminus \{(2 \times 2_+^{1+8}){:}2^4\}$.

Let $\sigma : 1 < 2^2 < Q = 2^{6+8} < 2^{6+8}.2$, so that $\sigma' : 1 < 2^2 < 2^{6+8}.2$. A similar proof to above shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by 2^{6+8}. Thus if $P_1 = 2^2$, then $P_2 \neq_G 2^{6+8}.2$.

If $V_2 = (2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3).3 \leqslant K_3$, then we may take $\mathcal{R}_0(V_2, 2) =$

$$\{(2 \times 2_+^{1+8}).2.2^4, (2 \times 2_+^{1+8})2^4.2^2, (2 \times 2_+^{1+8})2.2^4.2, (2 \times 2_+^{1+8}){:}2^4.2^2.2\} \subseteq \mathcal{R}_0(K_3, 2).$$

In addition, $N_{K_3}(R) = N_{V_2}(R)$, $N_{K_3.2}(R) = N_{N_{K_3.2}(V_2)}(R)$ for each $R \in \mathcal{R}_0(V_2, 2)$.

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C		$N(C)$
$C(1)$	1	Fi_{24}'
$C(2)$	$1 < 2$	$2.\mathrm{Fi}_{22}.2$
$C(3)$	$1 < 2 < 2^2$	$2^2.U_6(2){:}2$
$C(4)$	$1 < 2 < 2^2 < 2^{11}$	$2^{11}.L_3(4){:}2$
$C(5)$	$1 < 2 < 2^2 < 2^2 \times 2_+^{1+8} < (2^2 \times 2_+^{1+8}){:}2^4$	$(2^2 \times 2_+^{1+8}){:}2^4.S_5$
$C(6)$	$1 < 2 < 2^2 < 2^2 \times 2_+^{1+8}$	$(2^2 \times 2_+^{1+8}).U_4(2){:}2$
$C(7)$	$1 < 2 < 2^2 < 2^{6+8} < 2^{11}.2^4$	$2^{11}.2^4.S_5$
$C(8)$	$1 < 2 < 2^2 < 2^{6+8}$	$2^{6+8}.(S_3 \times S_5)$
$C(9)$	$1 < 2 < 2^2 < 2^{6+8} < (2 \times 2_+^{1+8}).2.2^4$	$(2 \times 2_+^{1+8}).2.2^4(S_3 \times S_3)$
$C(10)$	$1 < 2 < 2^2 < 2^{6+8} < (2 \times 2_+^{1+8}).2.2^4 < (2 \times 2_+^{1+8}).2^4.2^2$	$(2 \times 2_+^{1+8}).2^4.2^2.S_3$
$C(11)$	$1 < 2 < (2^2)^*$	$(2^2 \times O_8^+(2){:}3){:}2$
$C(12)$	$1 < 2 < (2^2)^* < 2^8$	$2^8{:}S_8$
$C(13)$	$1 < 2 < (2^2)^* < 2^8 < 2^8.2^3$	$2^8.2^3.L_3(2)$
$C(14)$	$1 < 2 < (2^2)^* < 2^8 < 2^8.2^3 < 2^8.2^3.2^2$	$2^8.2^3.2^2.S_3$
$C(15)$	$1 < 2 < (2^2)^* < 2^3.2^8 < 2^8.2^4$	$2^8.2^4.(S_3 \times S_3).2$
$C(16)$	$1 < 2 < (2^2)^* < 2^3.2^8 < 2^8.2^4 < 2^3.2^2.2^4.2^4$	$2^3.2^2.2^4.2^4.S_3$
$C(17)$	$1 < 2 < (2^2)^* < 2^3.2^8 < 2^8.2^4 < 2^3.2^2.2^4.2^4 < 2^8.2^4.2^2$	$2^8.2^4.2^2$
$C(18)$	$1 < 2 < (2^2)^* < 2^3.2^8$	$2^3.2^8.3.(S_3 \times S_3 \times S_3).2$
$C(19)$	$1 < 2 < (2^2)^* < 2^4.2^4.2^4 < 2^4.2^4.2^4.2^2$	$2^4.2^4.2^4.2^2.S_3$
$C(20)$	$1 < 2 < (2^2)^* < 2^4.2^4.2^4$	$2^4.2^4.2^4.L_3(2)$
$C(21)$	$1 < 2 < (2^2)^* < 2^4.2^4.2^4 < 2^3.2^8.D_8$	$2^3.2^8.D_8.S_3$
$C(22)$	$1 < 2 < (2^2)^* < 2^4.2^4.2^4 < 2^4.2^4.2^4.2^2 < 2^3.2^8.D_8.2$	$2^3.2^8.D_8.2$
$C(23)$	$1 < 2 < (2^2)^* < 2^4.2^3.2^6 < 2^4.2^3.2^6.2$	$2^4.2^3.2^6.2.S_3$
$C(24)$	$1 < 2 < (2^2)^* < 2^4.2^3.2^6$	$2^4.2^3.2^6.(S_3 \times S_3)$
$C(25)$	$1 < 2 < (2^2)^* < 2^4.2^3.2^6 < 2^4.2^4.2^4.2^2$	$2^4.2^4.2^4.2^2.S_3$
$C(26)$	$1 < 2 < (2^2)^* < 2^4.2^3.2^6 < 2^4.2^4.2^4.2^2 < 2^3.2^8.D_8.2$	$2^3.2^8.D_8.2$
$C(27)$	$1 < 2 < 2^8$	$2^8.S_6(2)$
$C(28)$	$1 < 2 < 2^8 < 2^8.2^5$	$2^8.2^5.S_6$
$C(29)$	$1 < 2 < 2^8 < 2^8.2^6 < 2^2.2^3.2^5.2^6$	$2^2.2^3.2^5.2^6.S_3$
$C(30)$	$1 < 2 < 2^8 < 2^8.2^6$	$2^8.2^6.L_3(2)$

Table 11: Some radical 2-chains of Fi_{24}'

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C		$N(C)$
$C(31)$	$1 < 2 < 2^8 < 2^8.2^3.2^4 < 2^8.2^5.2^3$	$2^8.2^5.2^3.S_3$
$C(32)$	$1 < 2 < 2^8 < 2^8.2^3.2^4$	$2^8.2^3.2^4.(S_3 \times S_3)$
$C(33)$	$1 < 2 < 2^8 < 2^8.2^3.2^4 < 2^8.2^3.2^4.2$	$2^8.2^3.2^4.2.S_3$
$C(34)$	$1 < 2 < 2^8 < 2^8.2^3.2^4 < 2^8.2^5.2^3 < 2^8.2^5.2^3.2$	$2^8.2^5.2^3.2$
$C(35)$	$1 < 2 < 2^{11}$	$2^{11}.M_{22}{:}2$
$C(36)$	$1 < 2 < 2^2.2^{10} < 2^{11}.2^4.2$	$2^{11}.2^4.2.S_5$
$C(37)$	$1 < 2 < 2^2.2^{10}$	$2^2.2^{10}.U_4(2).2$
$C(38)$	$1 < 2 < 2^{6+8} < 2^{11}.2^4$	$2^{11}.2^4.S_6$
$C(39)$	$1 < 2 < 2^{6+8}$	$2^{6+8}.(S_3 \times S_6)$
$C(40)$	$1 < 2 < 2^{6+8} < 2^2.2^{10}.2.2^4$	$2^2.2^{10}.2.2^4.(S_3 \times S_3)$
$C(41)$	$1 < 2 < 2^{6+8} < 2^2.2^{10}.2.2^4 < 2^{11}.2^2.2^4.2$	$2^{11}.2^2.2^4.2.S_3$
$C(42)$	$1 < 2^{11}$	$2^{11}.M_{24}$
$C(43)$	$1 < 2^2 < 2^{11}$	$2^{11}.L_3(4).S_3$
$C(44)$	$1 < 2^2$	$2^2.U_6(2).S_3$
$C(45)$	$1 < 2^2 < 2^2 \times 2_+^{1+8}$	$(2^2 \times 2_+^{1+8}).U_4(2){:}S_3$
$C(46)$	$1 < 2^2 < 2^2 \times 2_+^{1+8} < (2^2 \times 2_+^{1+8}){:}2^4$	$(2^2 \times 2_+^{1+8}){:}2^4{:}A_5{:}S_3$
$C(47)$	$1 < 2^2 < 2^{6+8}$	$2^{6+8}.(S_3 \times S_5).3$
$C(48)$	$1 < 2^2 < 2^{6+8} < (2^2 \times 2_+^{1+8}).2.2^4$	$(2^2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3).3$
$C(49)$	$1 < 2^2 < 2^{6+8} < 2^{11}.2^4 < (2^2 \times 2_+^{1+8}).2^4.2^2$	$(2^2 \times 2_+^{1+8}).2^4.2^2.S_3.3$
$C(50)$	$1 < 2^2 < 2^{6+8} < 2^{11}.2^4$	$2^{11}.2^4.A_5.S_3$
$C(51)$	$1 < 2_+^{1+12} < 2^{11}.2^6$	$2^{11}.2^6.(3 \times A_6).2$
$C(52)$	$1 < 2_+^{1+12}$	$2_+^{1+12}.3.U_4(3){:}2$
$C(53)$	$1 < (2^2)^* < 2^3.2^8$	$2^3.2^8.3.(S_3 \times S_3 \times S_3){:}S_3$
$C(54)$	$1 < (2^2)^*$	$(A_4 \times O_8^+(2){:}3){:}2$
$C(55)$	$1 < (2^2)^* < 2^8.2^3$	$2^8.2^3.(L_3(2) \times S_3)$
$C(56)$	$1 < (2^2)^* < 2^8.2^3 < 2^3.2^8.2^2$	$2^3.2^8.2^2.(S_3 \times S_3)$
$C(57)$	$1 < (2^2)^* < 2^8.2^3 < 2^3.2^8.2^2 < 2^3.2^8.2^3$	$2^3.2^8.2^3.S_3$
$C(58)$	$1 < (2^2)^* < 2^8.2^3 < 2^4.2^3.2^6$	$2^4.2^3.2^6.(S_3 \times S_3)$
$C(59)$	$1 < (2^2)^* < 2^4.2^3.2^6$	$2^4.2^3.2^6.(S_3 \times S_3).3$
$C(60)$	$1 < (2^2)^* < 2^4.2^3.2^6 < 2^3.2^8.2^3$	$2^3.2^8.2^3.3^2.2$

Table 12: Some radical 2-chains of Fi_{24}'

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C		$N(C)$
$C(61)$	$1 < (2^2)^* < 2^8$	$(A_4 \times 2^6{:}A_8).2$
$C(62)$	$1 < (2^2)^* < 2^8 < 2^8.2^3$	$2^8.2^3(L_3(2) \times 3)$
$C(63)$	$1 < (2^2)^* < 2^8 < 2^8.2^3 < 2^4.2^3.2^6$	$2^4.2^3.2^6.(S_3 \times 3)$
$C(64)$	$1 < (2^2)^* < 2^8 < 2^4.2^3.2^6$	$2^4.2^3.2^6.(S_3 \times S_3)$
$C(65)$	$1 < (2^2)^* < 2^8 < 2^4.2^3.2^6 < 2^3.2^8.2^3$	$2^3.2^8.2^3.S_3$
$C(66)$	$1 < (2^2)^* < 2^8 < 2^8.2^4$	$2^8.2^4.(S_3 \times S_3).S_3$
$C(67)$	$1 < (2^2)^* < 2^8 < 2^8.2^4 < 2^3.2^8.2^2$	$2^3.2^8.2^2.(S_3 \times 3)$
$C(68)$	$1 < (2^2)^* < 2^8 < 2^8.2^4 < 2^3.2^8.2^2 < 2^3.2^8.2^2.2$	$2^3.2^8.2^2.6$
$C(69)$	$1 < 2^{3+12} < 2^3.2^6.2^8$	$2^3.2^6.2^8.(S_3 \times L_3(2))$
$C(70)$	$1 < 2^{3+12}$	$2^{3+12}.(A_6 \times L_3(2))$
$C(71)$	$1 < 2^{3+12} < 2_+^{1+12}.2^4$	$2_+^{1+12}.2^4.(A_6 \times S_3)$
$C(72)$	$1 < 2^{3+12} < 2_+^{1+12}.2^4 < 2^{11}.2^4.2^4$	$2^{11}.2^4.2^4.(S_3 \times S_3)$
$C(73)$	$1 < 2^8 < 2_+^{1+12}.2$	$2_+^{1+12}.2.U_4(2)$
$C(74)$	$1 < 2^8$	$2^8{:}O_8^-(2)$
$C(75)$	$1 < 2^8 < 2^8{:}2^{3+6}$	$2^8{:}2^{3+6}.(L_3(2) \times 3)$
$C(76)$	$1 < 2^8 < 2^8{:}2_+^{1+8} < 2^8{:}2^6.2^4$	$2^8{:}2^6.2^4.A_5$
$C(77)$	$1 < 2^8 < 2^8{:}2_+^{1+8}$	$2^8{:}2_+^{1+8}.(S_3 \times A_5)$
$C(78)$	$1 < 2^8 < 2^8{:}2_+^{1+8} < 2^8{:}2^{3+6}.2^2$	$2^8{:}2^{3+6}.2^2.(3 \times S_3)$
$C(79)$	$1 < 2^8 < 2^8{:}2^{3+6} < 2^8.2^6.2.2^4 < 2^8.2^{3+6}.D_8$	$2^8.2^{3+6}.D_8.3$
$C(80)$	$1 < 2^8 < 2^8{:}2^{3+6} < 2^8.2^6.2.2^4$	$2^8.2^6.2.2^4.(3 \times S_3)$
$C(81)$	$1 < 2^{6+8} < 2^{11}.2^4$	$2^{11}.2^4.A_8$
$C(82)$	$1 < 2^{6+8}$	$2^{6+8}.(S_3 \times A_8)$
$C(83)$	$1 < 2^{6+8} < 2^3.2^3.2^6.2^5$	$2^3.2^3.2^6.2^5.(L_3(2) \times S_3)$
$C(84)$	$1 < 2^{6+8} < 2^3.2^3.2^6.2^5 < 2^{11}.2^3.2^4$	$2^{11}.2^3.2^4.L_3(2)$
$C(85)$	$1 < 2^{6+8} < 2_+^{1+12}.2.2^4$	$2_+^{1+12}.2.2^4.(S_3 \times S_3 \times S_3)$
$C(86)$	$1 < 2^{6+8} < 2_+^{1+12}.2.2^4 < 2^{1+12}.2.2^4.2$	$2^{1+12}.2.2^4.2.(S_3 \times S_3)$
$C(87)$	$1 < 2^{6+8} < 2_+^{1+12}.2.2^4 < 2^{6+8}.2^4.2 < 2^{11}.2^2.2^3.2^4$	$2^{11}.2^2.2^3.2^4.S_3$
$C(88)$	$1 < 2^{6+8} < 2_+^{1+12}.2.2^4 < 2^{6+8}.2^4.2$	$2^{6+8}.2^4.2.(S_3 \times S_3)$

Table 13: Some radical 2-chains of Fi_{24}'

Let $R \in \mathcal{R}_0(V_2, 2) \setminus \{(2 \times 2_+^{1+8}).2.2^4\}$ and $\sigma(R) : 1 < 2^2 < 2^{6+8} < Q = (2 \times 2_+^{1+8}).2.2^4 < R$, so that $\sigma(R)' : 1 < 2^2 < 2^{6+8} < R$. We may suppose (5.4) holds with 3^{1+10} replaced by $(2 \times 2_+^{1+8}).2.2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(V_2, 2) \setminus \{(2 \times 2_+^{1+8}){:}2^4\}$.

If $V_3 = 2^{11}.2^4.A_5.S_3 \leqslant K_3$, then we may take

$$\mathcal{R}_0(V_3, 2) = \{2^{11}.2^4, 2^{6+8}.2^2, (2 \times 2_+^{1+8})2^4.2^2, (2 \times 2_+^{1+8}){:}2^4.2^2.2\} \subseteq \mathcal{R}_0(K_3, 2).$$

In addition, $N_{K_3}(R) = N_{V_3}(R)$, $N_{K_3.2}(R) = N_{N_{K_3.2}(V_3)}(R)$ for each $R \in \mathcal{R}_0(V_3, 2)$.

Let $\sigma : 1 < 2^2 < 2^{6+8} < Q = 2^{11}.2^4 < 2^{6+8}.2^2$, so that $\sigma' : 1 < 2^2 < 2^{6+8} < 2^{6+8}.2^2$. A similar proof to above shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $2^{11}.2^4$.

Let $C' : 1 < 2^2 < 2^{6+8} < 2^{11}.2^4 < (2 \times 2_+^{1+8}){:}2^4.2^2 < (2 \times 2_+^{1+8}){:}2^4.2^2.2$ and $g(C') : 1 < 2^2 < 2^{6+8} < 2^{11}.2^4 < (2 \times 2_+^{1+8}){:}2^4.2^2.2$. Then $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C'))$ and we may suppose $C \neq_G C', g(C')$.

It follows that if $P_1 = 2^2$, then $C \in \{C(i) : 43 \leqslant i \leqslant 50\}$.

Case (2.2) Let

$$\Omega = \{2_+^{1+12}.2, 2_+^{1+12}.2^4, 2_+^{1+12}.2.2^4, 2_+^{1+12}.2^4.2, 2_+^{1+12}.2^2.2^4, 2^{6+8}.2^4.2, 2^{3+12}.(D_8 \times 2^2)\}$$

be a subset of $\mathcal{R}_0(M_4, 2)$, $R \in \Omega$ and let $\sigma(R) : 1 < Q = 2_+^{1+12} < R$, so that $\sigma(R)' : 1 < R$. A similar proof to Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by 2_+^{1+12} and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by Ω. Thus $P_1 \neq_G R$ and if $P_1 = 2_+^{1+12}$, then $P_2 \neq_G R$.

If $W = 2^{11}.2^6.3S_6 \leqslant M_4$, then we may take $\mathcal{R}_0(W, 2) =$

$$\{2^{11}.2^6, 2_+^{1+12}.2.2^4.2, 2^{11}.2^4.2^4, 2^{6+8}.2^4.2^2, 2^{11}.2^2.2^3.2^4, 2_+^{1+12}.2^4.2^2.2, S\} \subseteq \mathcal{R}_0(M_4, 2),$$

so that $\mathcal{R}_0(M_4, 2) = \Omega \cup \mathcal{R}_0(W, 2)$. In addition, $N_W(R) = N_{M_4}(R)$ and $N_{N_{M_4.2}(W)}(R) = N_{M_4.2}(R)$ for each $R \in \mathcal{R}_0(W, 2)$. Let $R \in \mathcal{R}_0(W, 2) \setminus \{2^{11}.2^6\}$ and $\sigma(R) : 1 < 2_+^{1+12} < Q = 2^{11}.2^6 < R$, so that $\sigma(R)' : 1 < 2_+^{1+12} < R$. We may suppose (5.4) holds with 3^{1+10} replaced by $2^{11}.2^6$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(W, 2) \setminus \{2^{11}.2^6\}$. Thus if $P_1 = 2_+^{1+12}$, then $C \in_G \{C(51), C(52)\}$.

Case (2.3) Let $\sigma : 1 < Q = (2^2)^* < D_8$ with $D_8 \in \mathcal{R}_0(M_5, 2)$, so that $\sigma : 1 < D_8$. Since $N(D_8) = N_{M_5}(D_8)$ and $N_E(D_8) = N_{M_5.2}(D_8) = N(D_8).2$, it follows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $(2^2)^*$. In particular, $P_1 \neq_G D_8$ and if $P_1 = (2^2)^*$, then $P_2 \neq_G D_8$.

If $X_1 = 2^3.2^8.3(S_3 \times S_3 \times S_3).S_3 \leqslant M_5$, we may take $\mathcal{R}_0(X_1, 2) =$

$$\{2^3.2^8, 2^8.2^4, 2^3.2^8.2, 2^3.2^8.2^2, 2^8.2^2.2^3, 2^3.2^8.2^3, 2^3.2^8.2^2.2, 2^8.2^4.2^3\} \subseteq \mathcal{R}_0(M_5, 2)$$

and in addition, $N_{M_5}(R) = N_{X_1}(R)$ and $N_{M_5.2}(R) = N_{X_1.2}(R)$ for all $R \in \mathcal{R}_0(X_1, 2)$.

Let $R \in \mathcal{R}_0(X_1, 2) \setminus \{2^3.2^8\}$ and $\sigma(R) : 1 < (2^2)^* < Q = 2^3.2^8 < R$, so that $\sigma(R)' : 1 < (2^2)^* < R$. We may suppose (5.4) holds with 3^{1+10} replaced by $2^3.2^8$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(X_1, 2) \setminus \{2^3.2^8\}$. Thus if $P_1 = (2^2)^*$, then $P_2 \notin_G \mathcal{R}_0(X_1, 2) \setminus \{2^3.2^8\}$ and if moreover, $P_2 = 2^3.2^8$, then $C = C(53)$.

Let $X_2 = (A_4 \times 2^6{:}A_8).2 \leqslant M_5$,

$$\mathcal{X}_1 = \{2^8, 2^8.2, 2^8.2^4, 2^8.2^2.2^3, 2^8.2^3.2^2.2, 2^8.2^4.2^3\} \subseteq \mathcal{R}_0(M_5, 2)$$

and $\mathcal{X}_1^* = \{2^8.2^3, 2^4.2^3.2^6, 2^3.2^8.2^2, 2^3.2^8.2^3\}$. We may take $\mathcal{R}_0(X_2,2) = \mathcal{X}_1 \cup \mathcal{X}_1^*$ such that $N_{M_5}(R) = N_{X_2}(R)$ for $R \in \mathcal{X}_1$ and

$$N_{X_2}(R) = \begin{cases} 2^8.2^3.(L_3(2)\times 3) & \text{if } R = 2^8.2^3, \\ 2^4.2^3.2^6.(S_3\times S_3) & \text{if } R = 2^4.2^3.2^6, \\ 2^3.2^8.2^2.(S_3\times 3) & \text{if } R = 2^3.2^8.2^2, \\ 2^3.2^8.2^3.S_3 & \text{if } R = 2^3.2^8.2^3. \end{cases}$$

Let $R \in \{2^8.2, 2^8.2^3.2^2.2\} \subseteq \mathcal{X}_1$ and $\sigma(R) : 1 < (2^2)^* < Q = 2^8 < R$, so that $\sigma(R)' : 1 < (2^2)^* < R$. We may suppose (5.4) holds with 3^{1+10} replaced by 2^8 and $\mathcal{R}_0(M_3,3) \setminus \{3^{1+10}\}$ by $\{2^8.2, 2^8.2^3.2^2.2\}$.

If $X_3 = 2^8.2^3.(L_3(2)\times 3) \leqslant X_2$, then we may take

$$\mathcal{R}_0(X_3,2) = \{2^8.2^3, 2^3.2^8.2^2, 2^4.2^3.2^6, 2^4.2^3.2^6.2\} \quad \text{and}$$

$N_{X_3}(2^3.2^8.2^2) = N_{X_2}(2^3.2^8.2^2)$, $N_{X_3}(2^4.2^3.2^6) = 2^4.2^3.2^6.(S_3\times 3) \neq N_{X_2}(2^4.2^3.2^6)$, $N_{X_3}(2^4.2^3.2^6.2) = 2^4.2^3.2^6.6$ and $N_{X_2.2}(R) = N_{X_3}(R).2$ for all $R \in \mathcal{R}_0(X_3,2)$.

Let $\sigma : 1 < (2^2)^* < 2^8 < Q = 2^8.2^3 < 2^3.2^8.2^2$, so that $\sigma' : 1 < (2^2)^* < 2^8 < 2^3.2^8.2^2$. We may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $2^8.2^3$. Let $C' : 1 < (2^2)^* < 2^8 < 2^8.2^3 < 2^4.2^3.2^6 < 2^4.2^3.2^6.2$ and $g(C') : 1 < (2^2)^* < 2^8 < 2^8.2^3 < 2^4.2^3.2^6.2$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C')) = N(C').2$ and we may suppose $C \notin_G \{C', g(C')\}$.

If $X_4 = 2^8.2^4.(S_3\times S_3).S_3 \leqslant X_2$, then we may take

$$\mathcal{R}_0(X_4,2) = \{2^8.2^4, 2^3.2^8.2^2, 2^8.2^2.2^3, 2^3.2^8.2^3, 2^8.2^4.2^3\} \subseteq \mathcal{R}_0(X_2,2)$$

and in addition, $N_{X_2}(R) = N_{X_4}(R)$ and $N_{X_2.2}(R) = N_{X_4.2}(R)$ for all $R \in \mathcal{R}_0(X_4,2)$.

Let $R \in \mathcal{R}_0(X_4,2)\setminus\{2^8.2^4, 2^3.2^8.2^2\}$ and let $\sigma(R) : 1 < (2^2)^* < 2^8 < Q = 2^8.2^4 < R$, so that $\sigma(R)' : 1 < (2^2)^* < 2^8 < R$. We may suppose (5.4) holds with 3^{1+10} replaced by $2^8.2^4$ and $\mathcal{R}_0(M_3,3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(X_4,2)\setminus\{2^8.2^4, 2^3.2^8.2^2\}$.

If $X_5 = 2^4.2^3.2^6.(S_3\times S_3) \leqslant X_2$, then we may take

$$\mathcal{R}_0(X_5,2) = \{2^4.2^3.2^6, 2^8.2^3.2^2.2, 2^3.2^8.2^3, 2^8.2^4.2^3\} \subseteq \mathcal{R}_0(X_2,2) \tag{5.6}$$

and in addition, $N_{X_2}(R) = N_{X_5}(R)$ and $N_{X_2.2}(R) = N_{X_5.2}(R)$ for all $R \in \mathcal{R}_0(X_5,2)$.

If $X_6 = 2^4.2^3.2^6.(S_3\times S_3).3 \leqslant M_5$, then we may take

$$\mathcal{R}_0(X_6,2) = \{2^4.2^3.2^6, 2^8.2^3.2^2.2, 2^3.2^8.2^3, 2^8.2^4.2^3\} \subseteq \mathcal{R}_0(M_5,2)$$

and in addition, $N_{M_5}(R) = N_{X_6}(R)$ and $N_{M_5.2}(R) = N_{X_6.2}(R)$ for all $R \in \mathcal{R}_0(X_6,2)$. In particular, $N_{X_6}(R) =_G N_{X_5}(R)$ for $R \in \{2^8.2^3.2^2.2, 2^8.2^4.2^3\}$ and

$$N_{X_6}(2^3.2^8.2^3) = 2^3.2^8.2^3.3^2.2 \neq_G N_{X_5}(2^3.2^8.2^3) = 2^3.2^8.2^3.S_3.$$

Let $R \in \mathcal{R}_0(X_6,2)\setminus\{2^4.2^3.2^6, 2^3.2^8.2^3\}$ and let $\sigma(R) : 1 < (2^2)^* < Q = 2^8 < 2^4.2^3.2^6 < R$, so that $\sigma(R)' : 1 < (2^2)^* < 2^4.2^3.2^6 < R$. We may suppose (5.4) holds with 3^{1+10} replaced by 2^8 and $\mathcal{R}_0(M_3,3)\setminus\{3^{1+10}\}$ by $\mathcal{R}_0(X_6,2)\setminus\{2^4.2^3.2^6, 2^3.2^8.2^3\}$.

Let $C' : 1 < (2^2)^* < 2^8 < 2^4.2^3.2^6 < 2^3.2^8.2^3 < 2^8.2^4.2^3$ and $g(C') : 1 < (2^2)^* < 2^4.2^3.2^6 < 2^3.2^8.2^3 < 2^8.2^4.2^3$. Then $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C'))$ and we may suppose $C \neq_G C', g(C')$.

If $X_7 = 2^8.2^3.(L_3(2)\times S_3) \leqslant M_5$, then we may take $\mathcal{R}_0(X_7,2) =$

$$\{2^8.2^3, 2^4.2^4.2^4, 2^3.2^8.2^2, 2^4.2^3.2^6, 2^3.2^8.2^3, 2^3.2^8.2^2.2, 2^8.2^3.2^2.2, 2^8.2^4.2^3\}$$

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and $N_{X_7}(R) = N_{M_5}(R)$ for $R \in \mathcal{R}_0(X_7, 2) \backslash \{2^4.2^3.2^6, 2^3.2^8.2^3\}$, $N_{X_7}(R) =_G N_{X_2}(R)$ for $R \in \{2^4.2^3.2^6, 2^3.2^8.2^3\}$ and $N_{X_7.2}(R) = N_{X_7}(R).2$ for all $R \in \mathcal{R}_0(X_7, 2)$.

Let $\sigma : 1 < (2^2)^* < Q = 2^8.2^3 < 2^4.2^4.2^4$, so that $\sigma' : 1 < (2^2)^* < 2^4.2^4.2^4$. We may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $2^8.2^3$.

We may suppose $X_5 = 2^4.2^3.2^6.(S_3 \times S_3) \leqslant X_7$, so that $\mathcal{R}_0(X_5, 2) \subseteq \mathcal{R}_0(X_7, 2)$ and $N_{X_5}(R) = N_{X_7}(R)$. Let $R \in \mathcal{R}_0(X_5, 2) \backslash \{2^4.2^3.2^6\}$ and let $\sigma(R) : 1 < (2^2)^* < 2^8.2^3 < Q = 2^4.2^3.2^6 < R$, so that $\sigma(R)' : 1 < (2^2)^* < 2^8.2^3 < R$. We may suppose (5.4) holds with 3^{1+10} replaced by $2^4.2^3.2^6$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(X_5, 2) \backslash \{2^4.2^3.2^6\}$.

If $X_8 = 2^3.2^8.2^2.(S_3 \times S_3) \leqslant X_7$, then we may take

$$\mathcal{R}_0(X_8, 2) = \{2^3.2^8.2^2, 2^3.2^8.2^3, 2^3.2^8.2^2.2, 2^8.2^4.2^3\} \subseteq \mathcal{R}_0(X_7, 2)$$

and $N_{X_8}(R) = N_{X_7}(R)$ and $N_{X_8.2}(R) = N_{X_7.2}(R)$ for all $R \in \mathcal{R}_0(X_8, 2)$.

Let $\sigma : 1 < (2^2)^* < 2^8.2^3 < Q = 2^3.2^8.2^2 < 2^3.2^8.2^2.2$, so that $\sigma' : 1 < (2^2)^* < 2^8.2^3 < 2^3.2^8.2^2.2$. We may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $2^3.2^8.2^2$.

Let $C' : 1 < (2^2)^* < 2^8.2^3 < 2^3.2^8.2^2 < 2^3.2^8.2^3 < 2^8.2^4.2^3$ and $g(C') : 1 < (2^2)^* < 2^8.2^3 < 2^3.2^8.2^2 < 2^8.2^4.2^3$. Then $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C'))$ and we may suppose $C \neq_G C', g(C')$.

It follows that if $P_1 = (2^2)^*$, then $C \in_G \{C(j) : 53 \leqslant j \leqslant 68\}$.

Case (2.4) Let $\mathcal{Y}_1 = \{2^{3+12}.2^2, 2^3.2^3.2^6.2^5, 2^{6+8}.2^3.2^2\} \subseteq \mathcal{R}_0(M_6, 2)$, $R \in \mathcal{Y}_1$ and $\sigma(R) : 1 < Q = 2^{3+12} < R$, so that $\sigma(R)' : 1 < R$. We may suppose (5.4) holds with 3^{1+10} replaced by 2^{3+12} and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{Y}_1$. Thus we may suppose $P_1 \notin_G \mathcal{Y}_1$ and if $P_1 = 2^{3+12}$, then $P_2 \notin \mathcal{Y}_1$.

If $Y_1 = 2^3.2^6.2^8.(S_4 \times L_3(2)) \leqslant M_6$, then we may take $\mathcal{R}_0(Y_1, 2) =$

$$\{2^3.2^6.2^8, 2^{11}.2^3.2^4, 2^{11}.2^4.2^4, 2^{11}.2^2.2^6, 2^{11}.2^2.2^3.2^4, 2^{1+12}.2^4.2^2.2, 2^{6+8}.2^3.2^3, S\}$$

as a subset of $\mathcal{R}_0(M_6, 2)$, so $N_{M_6}(R) = N_{Y_1}(R)$ and $N_{M_6.2}(R) = N_{N_{M_6.2}(Y_1)}(R)$ for $R \in \mathcal{R}_0(Y_1, 2)$.

Let $R \in \mathcal{R}_0(Y_1, 2) \setminus \{2^3.2^6.2^8\}$, and $\sigma(R) : 1 < 2^{3+12} < Q = 2^3.2^6.2^8 < R$, so that $\sigma(R)' : 1 < 2^{3+12} < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^3.2^6.2^8$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(Y_1, 2) \setminus \{2^3.2^6.2^8\}$.

If $Y_2 = 2^{1+12}.2^4.(S_4 \times A_6) \leqslant M_6$, then we may take $\mathcal{R}_0(Y_2, 2) =$

$$\{2^{1+12}.2^4, 2^{1+12}.2^4.2, 2^{6+8}.2^4.2, 2^{11}.2^4.2^4, \\ 2^{11}.2^2.2^3.2^4, 2^{1+12}.2^4.2^2.2, 2^{3+12}.(D_8 \times 2^2), S\}$$

as a subset of $\mathcal{R}_0(M_6, 2)$, so $N_{M_6}(R) = N_{Y_2}(R)$ and $N_{M_6.2}(R) = N_{N_{M_6.2}(Y_2)}(R)$ for $R \in \mathcal{R}_0(Y_2, 2)$.

Let $\mathcal{Y}_2 := \{2^{1+12}.2^4.2, 2^{6+8}.2^4.2, 2^{3+12}.(D_8 \times 2^2)\} \subseteq \mathcal{R}_0(Y_2, 2)$, $R \in \mathcal{Y}_2$, and $\sigma(R) : 1 < 2^{3+12} < Q = 2^{1+12}.2^4 < R$, so that $\sigma(R)' : 1 < 2^{3+12} < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^{1+12}.2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{Y}_2$.

If $Y_3 = 2^{11}.2^4.2^4.(S_3 \times S_3) \leqslant Y_2$, then we may take

$$\mathcal{R}_0(Y_3, 2) = \{2^{11}.2^4.2^4, 2^{11}.2^2.2^3.2^4, 2^{1+12}.2^4.2^2, S\} \subseteq \mathcal{R}_0(M_6, 2),$$

so that $N_{M_6}(R) = N_{Y_2}(R) = N_{Y_3}(R)$ and $N_{M_6.2}(R) = N_{N_{M_6.2}(Y_3)}(R)$ for $R \in \mathcal{R}_0(Y_3, 2)$.

Let $R \in \mathcal{R}_0(Y_3, 2) \setminus \{2^{11}.2^4.2^4\}$, and $\sigma(R) : 1 < 2^{3+12} < 2^{1+12}.2^4 < Q = 2^{11}.2^4.2^4 < R$, so that $\sigma(R)' : 1 < 2^{3+12} < 2^{1+12}.2^4 < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^{11}.2^4.2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(Y_3, 2) \setminus \{2^{11}.2^4.2^4\}$.

It follows that if $P_1 = 2^{3+12}$, then then $C \in_G \{C(i) : 69 \leqslant i \leqslant 73\}$.

Case (2.5) Let $\sigma : 1 < Q = 2^{6+8} < 2^{6+8}.2^3$, so that $\sigma' : 1 < 2^{6+8}.2^3$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by 2^{6+8}. Thus we may suppose $P_1 \neq_G 2^{6+8}.2^3$ and if $P_1 = 2^{6+8}$, then $P_2 \neq 2^{6+8}.2^3$.

If $Z_1 = 2^{11}.2^4.A_8 \leqslant M_7$, then we may take $\mathcal{R}_0(Z_1, 2) =$

$$\{2^{11}.2^4, 2^{11}.2^{1+6}, 2^{11}.2^3.2^4, 2^{1+12}.2.2^4.2, 2^{6+8}.2^4.2^2, 2^{6+8}.2^3.2^3, 2^{11}.2^2.2^3.2^4, S\}$$

as a subset of $\mathcal{R}_0(M_7, 2)$, so that $N_{M_7}(R) = N_{Z_1}(R)$ and $N_{M_7.2}(R) = N_{N_{M_7.2}(Z_1)}(R)$.

Let $R \in \mathcal{R}_0(Z_1, 2) \setminus \{2^{11}.2^4\}$, and $\sigma(R) : 1 < 2^{6+8} < Q = 2^{11}.2^4 < R$, so that $\sigma(R)' : 1 < 2^{6+8} < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^{11}.2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(Z_1, 2) \setminus \{2^{11}.2^4\}$.

If $Z_2 = 2^3.2^3.2^6.2^5.(L_3(2) \times S_3) \leqslant M_7$, then we may take

$$\mathcal{R}_0(Z_2, 2) = \{2^3.2^3.2^6.2^5, 2^{11}.2^3.2^4, 2^{6+8}.2^4.2, 2^{6+8}.2^3.2^2,\\ 2^{11}.2^2.2^3.2^4, 2^{3+12}(D_8 \times 2^2), 2^{6+8}.2^3.2^3, S\} \subseteq \mathcal{R}_0(M_7, 2)$$

so that $N_{M_7}(R) = N_{Z_2}(R)$ and $N_{M_7.2}(R) = N_{N_{M_7.2}(Z_2)}(R)$.

Let $\mathscr{Z}_1 := \{2^{6+8}.2^4.2, 2^{6+8}.2^3.2^2, 2^{3+12}.(D_8 \times 2^2)\} \subseteq \mathcal{R}_0(Z_2, 2)$, $R \in \mathscr{Z}_1$, and $\sigma(R) : 1 < 2^{6+8} < Q = 2^3.2^3.2^6.2^5 < R$, so that $\sigma(R)' : 1 < 2^{6+8} < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^3.2^3.2^6.2^5$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathscr{Z}_1$.

If $Z_3 = 2^{11}.2^3.2^4.L_3(2) \leqslant Z_2$, then we may take

$$\mathcal{R}_0(Z_3, 2) = \{2^{11}.2^3.2^4, 2^{11}.2^2.2^3.2^4, 2^{6+8}.2^3.2^3, S\} \subseteq \mathcal{R}_0(Z_2, 2)$$

so that $N_{M_7}(R) = N_{Z_2}(R) = N_{Z_3}(R)$ and $N_{M_7.2}(R) = N_{N_{M_7.2(Z_3)}}(R)$.

Let $R \in \mathcal{R}_0(Z_3, 2) \setminus \{2^{11}.2^3.2^4\}$, and $\sigma(R) : 1 < 2^{6+8} < 2^3.2^6.2^5 < Q = 2^{11}.2^3.2^4 < R$, so that $\sigma(R)' : 1 < 2^{6+8} < 2^3.2^6.2^5 < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^{11}.2^3.2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(Z_3, 2) \setminus \{2^{11}.2^3.2^4\}$.

If $Z_4 = 2^{1+12}.2.2^4.(S_3 \times S_3 \times S_3) \leqslant M_7$, then we may take

$$\mathcal{R}_0(Z_4, 2) = \{2^{1+12}.2.2^4, 2^{6+8}.2^4.2, 2^{1+12}.2.2^4.2, 2^{1+12}.2^2.2^4, 2^{6+8}.2^4.2^2,\\ 2^{3+12}(D_8 \times 2^2), 2^{11}.2^2.2^3.2^4, S\} \subseteq \mathcal{R}_0(M_7, 2)$$

so that $N_{M_7}(R) = N_{Z_4}(R)$ and $N_{M_7.2}(R) = N_{N_{M_7.2(Z_4)}}(R)$.

Let $\sigma : 1 < 2^{6+8} < Q = 2^{1+12}.2.2^4 < 2^{1+12}.2^2.2^4$, so that $\sigma' : 1 < 2^{6+8} < 2^{1+12}.2^2.2^4$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $2^{1+12}.2.2^4$.

If $Z_5 = 2^{1+12}.2.2^4.2.(S_3 \times S_3) \leqslant Z_4$, then we may take

$$\mathcal{R}_0(Z_5, 2) = \{2^{1+12}.2.2^4.2, 2^{6+8}.2^4.2^2, 2^{11}.2^2.2^3.2^4, S\} \subseteq \mathcal{R}_0(Z_4, 2)$$

so that $N_{M_7}(R) = N_{Z_5}(R) = N_{Z_4}(R)$ and $N_{M_7.2}(R) = N_{N_{M_7.2(Z_5)}}(R)$.

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Let $R \in \mathcal{R}_0(Z_5, 2) \setminus \{2^{1+12}.2.2^4.2\}$, and $\sigma(R) : 1 < 2^{6+8} < 2^{1+12}.2.2^4 < Q = 2^{1+12}.2.2^4.2 < R$, so that $\sigma(R)' : 1 < 2^{6+8} < 2^{1+12}.2.2^4 < R$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $2^{1+12}.2.2^4.2$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(Z_5, 2) \setminus \{2^{1+12}.2.2^4.2\}$.

If $Z_6 = 2^{6+8}.2^4.2.(S_3 \times S_3) \leqslant Z_4$, then we may suppose

$$\mathcal{R}_0(Z_6, 2) = \{2^{6+8}.2^4.2, 2^{11}.2^2.2^3.2^4, 2^{3+12}(D_8 \times 2^2), S\} \subseteq \mathcal{R}_0(Z_4, 2)$$

so that $N_{M_7}(R) = N_{Z_4}(R) = N_{Z_6}(R)$ and $N_{M_7.2}(R) = N_{N_{M_7.2}(Z_6)}(R)$.

Let $\sigma : 1 < 2^{6+8} < 2^{1+12}.2.2^4 < Q = 2^{6+8}.2^4.2 < 2^{3+12}(D_8 \times 2^2)$, so that $\sigma' : 1 < 2^{6+8} < 2^{1+12}.2.2^4 < 2^{3+12}(D_8 \times 2^2)$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $2^{6+8}.2^4.2$.

Let $C' : 1 < 2^{6+8} < 2^{1+12}.2.2^4 < 2^{6+8}.2^4.2 < 2^{11}.2^2.2^3.2^4 < S$ and $g(C') : 1 < 2^{6+8} < 2^{1+12}.2.2^4 < 2^{6+8}.2^4.2 < S$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C')) = N(C').2$ and we may suppose $C \notin_G \{C', g(C')\}$.

It follows that if $P_1 = 2^{6+8}$, then

$$C \in_G \{C(j) : 81 \leqslant j \leqslant 88\}.$$

Case (2.6) If $P_1 = 2^8$, then $N(P_1) = M_8 = 2^8.O_8^-(2)$. Applying the Borel-Tits theorem to $O_8^-(2)$, it follows that $C \in_G \{C(j) : 73 \leqslant j \leqslant 80\}$.

Case (2.7) Let $\sigma : 1 < 2 < Q = (2^2)^* < D_8$, so that $\sigma' : 1 < 2 < D_8$, where $D_8 \in \mathcal{R}_0(T_1, 2)$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $(2^2)^*$. Similarly, if $\sigma : 1 < Q = 2 < 2^2 < D_8$ with $\sigma' : 1 < 2^2 < D_8$, then (5.5) holds with $3^2.3^4.3^8$ replaced by 2. Thus if $P_1 = 2$, then we may suppose $P_2 \neq_G D_8$ and if moreover, $P_2 \in \{2^2, (2^2)^*\}$, then we may suppose $P_3 \neq_G D_8$.

Let $R \in \mathcal{R}_0(T_2, 2) \setminus \{2^{11}\}$, and $\sigma(R) : 1 < 2 < Q = 2^{11} < R$, so that $\sigma(R)' : 1 < 2 < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by 2^{11} and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(T_2, 2) \setminus \{2^{11}\}$.

Let $\mathcal{T}_1 := \{2^8.2^5, 2^2.2^{10}.2.2^4, 2^{6+8}.D_8.2\} \subseteq \mathcal{R}_0(T_4, 2)$, $R \in \mathcal{T}_1$, and $\sigma(R) : 1 < 2 < Q = 2^2.2^{10} < R$, so that $\sigma(R)' : 1 < 2 < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^2.2^{10}$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{T}_1$.

If $T_7 = 2^{11}.2^4.2.S_5 \leqslant T_4$, then we may suppose

$$\mathcal{R}_0(T_7, 2) = \{2^{11}.2^4.2, 2^{11}.2^2.2^3.2, 2^{11}.2^2.2^4.2, 2^{6+8}.2^3.2^2\} \subseteq \mathcal{R}_0(T_4, 2)$$

so that $N_{M_1}(R) = N_{T_7}(R) = N_{T_4}(R)$ and $N_{M_1.2}(R) = N_{N_{M_1.2}(T_7)}(R)$.

Let $R \in \mathcal{R}_0(T_7, 2) \setminus \{2^{11}.2^4.2\}$, and $\sigma(R) : 1 < 2 < 2^2.2^{10} < Q = 2^{11}.2^4.2 < R$, so that $\sigma(R)' : 1 < 2 < 2^2.2^{10} < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^{11}.2^4.2$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(T_7, 2) \setminus \{2^{11}.2^4.2\}$.

Let $\sigma : 1 < 2 < Q = 2^{6+8} < 2^{6+8}.2^3$, so that $\sigma' : 1 < 2 < 2^{6+8}.2^3$, where $2^{6+8}.2^3 \in \mathcal{R}_0(T_5, 2)$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by 2^{6+8}.

If $T_8 = 2^{11}.2^4.S_6 \leqslant T_5$, then we may suppose

$$\mathcal{R}_0(T_8, 2) = \{2^{11}.2^4, 2^{11}.2^2.2^4.2, 2^{6+8}.2^3.2, 2^{6+8}.2^3.2^2\} \subseteq \mathcal{R}_0(T_5, 2)$$

so that $N_{M_1}(R) = N_{T_8}(R) = N_{T_5}(R)$ and $N_{M_1.2}(R) = N_{N_{M_1.2}(T_8)}(R)$.

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Let $R \in \mathcal{R}_0(T_8, 2) \setminus \{2^{11}.2^4\}$ and $\sigma(R) : 1 < 2 < 2^{6+8} < Q = 2^{11}.2^4 < R$, so that $\sigma(R)' : 1 < 2 < 2^{6+8} < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^{11}.2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(T_8, 2) \setminus \{2^{11}.2^4\}$.

If $T_9 = 2^2.2^{10}.2.2^4.(S_3 \times S_3) \leqslant T_5$, then we may suppose

$$\mathcal{R}_0(T_9, 2) = \{2^2.2^{10}.2.2^4, 2^{11}.2^2.2^4.2, 2^{6+8}.D_8.2, 2^{6+8}.2^3.2^2\} \subseteq \mathcal{R}_0(T_5, 2)$$

so that $N_{M_1}(R) = N_{T_9}(R) = N_{T_5}(R)$ and $N_{M_1.2}(R) = N_{N_{M_1.2}(T_9)}(R)$.

Let $\sigma : 1 < 2 < 2^{6+8} < Q = 2^2.2^{10}.2.2^4 < 2^{6+8}.D_8.2$, so that $\sigma' : 1 < 2 < 2^{6+8} < 2^{6+8}.D_8.2$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $2^2.2^{10}.2.2^4$.

Let $C' : 1 < 2 < 2^{6+8} < 2^2.2^{10}.2.2^4 < 2^{11}.2^2.2^4.2 < 2^{6+8}.2^3.2^2$ and $g(C') : 1 < 2 < 2^{6+8} < 2^2.2^{10}.2.2^4 < 2^{6+8}.2^3.2^2$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C')) = N(C').2$ and we may suppose $C \notin_G \{C', g(C')\}$. Thus if $P_1 = 2$ and $P_2 = 2^{6+8}$, then $C \in_G \{C(j) : 38 \leqslant j \leqslant 41\}$.

If $P_1 = 2$ and $P_2 = 2^8 \in \mathcal{R}_0(T_3, 2)$, then $N_{M_1}(P_2) = T_3 = 2^8.S_6(2)$. Applying the Borel-Tits Theorem to $S_6(2)$, it follows that $C \in_G \{C(j) : 27 \leqslant j \leqslant 34\}$.

Let $T_{10} = 2^8.S_8 \leqslant T_6$, $\mathcal{T}_4 = \{2^8.2, 2^8.2^4, 2^8.2^4.2, 2^4.2^4.2^4.2^2, 2^3.2^8.D_8.2\}$ and $\mathcal{T}_4^* = \{2^8.2^3, 2^8.2^2.2^3\}$. Then we may suppose

$$\mathcal{R}_0(T_{10}, 2) = \{2^8\} \cup \mathcal{T}_4 \cup \mathcal{T}_4^*$$

and in addition, $N_{T_{10}}(R) = N_{T_6}(R)$, $N_{T_6.2}(R) = N_{N_{T_6.2}(T_{10})}(R)$ for $R \in \mathcal{T}_4$, and $N_{T_{10}}(2^8.2^3) = 2^8.2^3.L_3(2)$, $N_{T_{10}}(2^8.2^2.2^3) = 2^8.2^2.2^3.S_3$.

Let $R \in \mathcal{T}_4$, and $\sigma(R) : 1 < 2 < (2^2)^* < Q = 2^8 < R$, so that $\sigma(R)' : 1 < 2 < (2^2)^* < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by 2^8 and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{T}_4$.

If $T_{11} = 2^8.2^3.L_3(2) \leqslant T_{10} \leqslant T_6$, then we may suppose

$$\mathcal{R}_0(T_{11}, 2) = \{2^8.2^3, 2^8.2^2.2^3, 2^8.2^3.2^2, 2^8.2^3.D_8\} \subseteq \mathcal{R}_0(T_{10}, 2)$$

so that $N_{T_{10}}(R) = N_{T_{11}}(R)$ and $N_{T_{10}.2}(R) = N_{N_{T_{10}.2}(T_{11})}(R)$.

Let $\sigma : 1 < 2 < (2^2)^* < 2^8 < Q = 2^8.2^3 < 2^8.2^2.2^3$, so that $\sigma' : 1 < 2 < (2^2)^* < 2^8 < 2^8.2^2.2^3$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by $2^8.2^3$.

Let $C' : 1 < 2 < (2^2)^* < 2^8 < 2^8.2^3 < 2^8.2^3.2^2 < 2^8.2^3.D_8$ and $g(C') : 1 < 2 < (2^2)^* < 2^8 < 2^8.2^3 < 2^8.2^3.D_8$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C')) = N(C').2$ and we may suppose $C \notin_G \{C', g(C')\}$. Thus if $P_1 = 2, P_2 = (2^2)^*$ and $P_3 = 2^8$, then $C \in_G \{C(12), C(13), C(14)\}$.

Let $T_{12} = 2^3.2^8.3.(S_3 \times S_3 \times S_3).2 \leqslant T_6$, $\mathcal{T}_5 = \{2^3.2^8.2, 2^4.2^3.2^6.2, 2^3.2^8.D_8\}$ and $\mathcal{T}_5^* = \{2^8.2^4, 2^8.2^4.2, 2^3.2^8.D_8.2\}$. Then we may suppose

$$\mathcal{R}_0(T_{12}, 2) = \{2^3.2^8\} \cup \mathcal{T}_5 \cup \mathcal{T}_5^*$$

and in addition, $N_{T_{12}}(R) = N_{T_6}(R)$, $N_{T_6.2}(R) = N_{N_{T_6.2}(T_{12})}(R)$ for $R \in \mathcal{R}_0(T_{12}, 2)$.

Let $R \in \mathcal{T}_5$, and $\sigma(R) : 1 < 2 < (2^2)^* < Q = 2^3.2^8 < R$, so that $\sigma(R)' : 1 < 2 < (2^2)^* < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^3.2^8$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{T}_5$.

If $T_{13} = 2^8.2^4.(S_3 \times S_3).2 \leqslant T_{12} \leqslant T_6$, then we may suppose

$$\mathcal{R}_0(T_{13}, 2) = \{2^8.2^4, 2^8.2^4.2, 2^3.2^2.2^4.2^4, 2^3.2^8.D_8.2\}$$

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so that $N_{T_{13}}(R) = N_{T_{12}}(R)$ and $N_{T_{12}.2}(R) = N_{N_{T_{12}.2}(T_{13})}(R)$ for $R \in \mathcal{R}_0(T_{13}, 2) \setminus \{2^3.2^2.2^4.2^4,\}$ and $N_{T_{12}}(2^3.2^2.2^4.2^4) = 2^3.2^2.2^4.2^4.S_3$.

Let $R \in \mathcal{R}_0(T_{13}, 2) \setminus \{2^8.2^4, 2^3.2^2.2^4.2^4\}$, and $\sigma(R) : 1 < 2 < (2^2)^* < 2^3.2^8 < Q = 2^8.2^4 < R$, so that $\sigma(R)' : 1 < 2 < (2^2)^* < 2^3.2^8 < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^8.2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(T_{13}, 2) \setminus \{2^8.2^4, 2^3.2^2.2^4.2^4,\}$. Thus if $P_1 = 2$, $P_2 = (2^2)^*$ and $P_3 = 2^3.2^8$, then $C \in_G \{C(15), C(16), C(17), C(18)\}$.

Let $P_1 = 2$, $P_2 = (2^2)^*$, $P_3 = 2^4.2^4.2^4 \in \mathcal{R}_0(T_6, 2)$. Then $N_{T_6}(P_3) = 2^4.2^4.2^4.L_3(2)$ and we may take

$$\mathcal{R}_0(2^4.2^4.2^4.L_3(2), 2) = \{2^4.2^4.2^4, 2^4.2^4.2^4.2^2, 2^3.2^8.D_8, 2^3.2^8.D_8.2\} \subseteq \mathcal{R}_0(T_6, 2).$$

Applying the Borel-Tits Theorem to $L_3(2)$, it follows that $C \in_G \{C(j) : 19 \leqslant j \leqslant 22\}$.

Let $P_1 = 2$, $P_2 = (2^2)^*$ and $P_3 = 2^4.2^3.2^6 \in \mathcal{R}_0(T_6, 2)$. Then

$$N_{T_6}(P_3) = 2^4.2^3.2^6.(L_2(2) \times L_2(2))$$

and we may take $\mathcal{R}_0(2^4.2^3.2^6.(L_2(2) \times L_2(2)), 2) =$

$$\{2^4.2^3.2^6, 2^4.2^3.2^6.2, 2^4.2^4.2^4.2^2, 2^3.2^8.D_8.2\} \subseteq \mathcal{R}_0(T_6, 2).$$

Applying the Borel-Tits Theorem to $L_2(2) \times L_2(2)$, it follows that $C \in_G \{C(j) : 23 \leqslant j \leqslant 26\}$.

It follows that if $P_1 = 2$ and $P_2 = (2^2)^*$, then

$$C \in_G \{C(j) : 11 \leqslant j \leqslant 26\}.$$

Finally, suppose $P_1 = 2$ and $P_2 = 2^2$, so that $N_{M_1}(P_2) = T_1$.

If $T_{14} = 2^{11}.L_3(4){:}2 \leqslant T_1$, then we may take

$$\begin{aligned} \mathcal{R}_0(T_{14}, 2) \quad &= \{2^{11}, 2^{11}.2, (2^2 \times 2_+^{1+8}).2^4, 2^{11}.2^4, (2^2 \times 2_+^{1+8}).2^4.2, \\ &2^{6+8}.2^2, (2^2 \times 2_+^{1+8}).2^4.2^2, (2^2 \times 2_+^{1+8}).2^4.D_8\} \subseteq \mathcal{R}_0(T_1, 2), \end{aligned}$$

so that $N_{T_1}(R) = N_{T_{14}}(R)$ and $N_{T_1.2}(R) = N_{N_{T_1.2}(T_{14})}(R)$ for $R \in \mathcal{R}_0(T_{14}, 2)$.

Let $R \in \mathcal{R}_0(T_{14}, 2) \setminus \{2^{11}\}$, and $\sigma(R) : 1 < 2 < 2^2 < Q = 2^{11} < R$, so that $\sigma(R)' : 1 < 2 < 2^2 < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by 2^{11} and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(T_{14}, 2) \setminus \{2^{11}\}$.

If $T_{15} = (2^2 \times 2^{1+8}){:}U_4(2){:}2 \leqslant T_1$, then we may take

$$\begin{aligned} \mathcal{R}_0(T_{15}, 2) = \{&2^2 \times 2^{1+8}, (2^2 \times 2^{1+8}).2, (2^2 \times 2^{1+8}){:}2^4, (2^2 \times 2^{1+8}).2.2^4, \\ &(2^2 \times 2^{1+8}).2^4.2, (2^2 \times 2^{1+8}).2^4.2^2, \\ &(2^2 \times 2^{1+8}).2.2^4.2, (2^2 \times 2_+^{1+8}).2^4.D_8\} \subseteq \mathcal{R}_0(T_1, 2), \end{aligned}$$

so that $N_{T_1}(R) = N_{T_{15}}(R)$ and $N_{T_1.2}(R) = N_{N_{T_1.2}(T_{15})}(R)$ for $R \in \mathcal{R}_0(T_{15}, 2)$.

Let $\mathcal{T}_6 = \{(2^2 \times 2^{1+8}).2, (2^2 \times 2^{1+8}).2.2^4, (2^2 \times 2^{1+8}).2.2^4.2\} \subseteq \mathcal{R}_0(T_{15}, 2)$, $R \in \mathcal{T}_6$, and $\sigma(R) : 1 < 2 < 2^2 < Q = 2^2 \times 2^{1+8} < R$, so that $\sigma(R)' : 1 < 2 < 2^2 < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^2 \times 2^{1+8}$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{T}_6$.

If $T_{16} = (2^2 \times 2^{1+8}).2^4.S_5 \leqslant T_{15} \leqslant T_1$, then we may take $\mathcal{R}_0(T_{16}, 2) =$

$$\{(2^2 \times 2^{1+8}){:}2^4, (2^2 \times 2^{1+8}).2^4.2, (2^2 \times 2^{1+8}).2^4.2^2, (2^2 \times 2_+^{1+8}).2^4.D_8\} \subseteq \mathcal{R}_0(T_{15}, 2)$$

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so that $N_{T_1}(R) = N_{T_{16}}(R)$ and $N_{T_1.2}(R) = N_{N_{T_1.2}(T_{16})}(R)$.

Let $R \in \mathcal{R}_0(T_{16}, 2) \setminus \{(2^2 \times 2^{1+8}){:}2^4\}$, and $\sigma(R) : 1 < 2 < 2^2 < 2^2 \times 2^{1+8} < Q = (2^2 \times 2^{1+8}){:}2^4 < R$, so that $\sigma(R)' : 1 < 2 < 2^2 < 2^2 \times 2^{1+8} < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $(2^2 \times 2^{1+8}).2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(T_{16}, 2) \setminus \{(2^2 \times 2^{1+8}){:}2^4\}$.

If $T_{17} = 2^{6+8}.(S_3 \times S_5) \leqslant T_1$, then we may take

$$\mathcal{R}_0(T_{17}, 2) = \{2^{6+8}, 2^{11}.2^4, 2_+^{6+8}.2, (2^2 \times 2^{1+8}).2.2^4, 2_+^{6+8}.2^2, (2^2 \times 2^{1+8}).2.2^4.2, (2^2 \times 2^{1+8}).2^4.2^2, (2^2 \times 2_+^{1+8}).2^4.D_8\} \subseteq \mathcal{R}_0(T_1, 2),$$

so that $N_{T_1}(R) = N_{T_{17}}(R)$ and $N_{T_1.2}(R) = N_{N_{T_1.2}(T_{17})}(R)$.

Let $\sigma : 1 < 2 < 2^2 < Q = 2^{6+8} < 2^{6+8}.2$, so that $\sigma' : 1 < 2 < 2^2 < 2^{6+8}.2$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.3^8$ replaced by 2^{6+8}.

If $T_{18} = 2^{11}.2^4.S_5 \leqslant T_{17} \leqslant T_1$, then we may take

$$\mathcal{R}_0(T_{18}, 2) = \{2^{11}.2^4, 2_+^{6+8}.2^2, (2^2 \times 2^{1+8}).2^4.2^2, (2^2 \times 2_+^{1+8}).2^4.D_8\} \subseteq \mathcal{R}_0(T_{17}, 2)$$

so that $N_{T_1}(R) = N_{T_{18}}(R)$ and $N_{T_1.2}(R) = N_{N_{T_1.2}(T_{18})}(R)$.

Let $R \in \mathcal{R}_0(T_{18}, 2) \setminus \{2^{11}.2^4\}$, and $\sigma(R) : 1 < 2 < 2^2 < 2^{6+8} < Q = 2^{11}.2^4 < R$, so that $\sigma(R)' : 1 < 2 < 2^2 < 2^{6+8} < R$. A similar proof to that of Case (1) shows that we may suppose (5.4) holds with 3^{1+10} replaced by $2^{11}.2^4$ and $\mathcal{R}_0(M_3, 3) \setminus \{3^{1+10}\}$ by $\mathcal{R}_0(T_{18}, 2) \setminus \{2^{11}.2^4\}$.

If $T_{19} = (2^2 \times 2^{1+8}).2.2^4.(S_3 \times S_3) \leqslant T_{17} \leqslant T_1$, then we may take $\mathcal{R}_0(T_{19}, 2) =$

$$\{(2^2 \times 2^{1+8}).2.2^4, (2^2 \times 2^{1+8}).2.2^4.2, (2^2 \times 2^{1+8}).2^4.2^2, (2^2 \times 2_+^{1+8}).2^4.D_8\} \subseteq \mathcal{R}_0(T_{17}, 2)$$

so that $N_{T_1}(R) = N_{T_{19}}(R)$ and $N_{T_1.2}(R) = N_{N_{T_1.2}(T_{19})}(R)$.

Let $\sigma : 1 < 2 < 2^2 < 2^{6+8} < Q = (2^2 \times 2^{1+8}).2.2^4 < (2^2 \times 2^{1+8}).2.2^4.2$, so that $\sigma' : 1 < 2 < 2^2 < 2^{6+8} < (2^2 \times 2^{1+8}).2.2^4.2$. A similar proof to that of Case (1) shows that we may suppose (5.5) holds with $3^2.3^4.2^8$ replaced by $(2^2 \times 2^{1+8}).2.2^4$.

Let $C' : 1 < 2 < 2^2 < 2^{6+8} < (2^2 \times 2^{1+8}).2.2^4 < (2^2 \times 2^{1+8}).2^4.2^2 < (2^2 \times 2_+^{1+8}).2^4.D_8$ and $g(C') : 1 < 2 < 2^2 < 2^{6+8} < (2^2 \times 2^{1+8}).2.2^4 < (2^2 \times 2_+^{1+8}).2^4.D_8$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C')) = N(C').2$ and we may suppose $C \notin_G \{C', g(C')\}$.

It follows that if $P_1 = 2$ and $P_2 = 2^2$, then we may suppose

$$C \in_G \{C(i) : 3 \leqslant i \leqslant 10\}.$$

This completes the classification of the radical p-chains. The normalizers of the chains are determined by MAGMA. □

REMARK 5.2. *Let $\hat{G}$ be a covering group of $G = \mathrm{Fi}_{24}'$, ρ a faithful linear character of $Z(\hat{G})$ and $\hat{B}$ a block of $\hat{G}$ covering the block $B(\rho)$ containing ρ. If $D(\hat{B}) \neq O_p(Z(\hat{G}))$ and $p = 2, 3$, then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, u, \rho, [r]) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, u, \rho, [r])$$

for all integers $d, u \geqslant 0$.

The proof of the Remark is the same as that of Lemma 5.1, since $N(C') = N(g(C'))$ implies $N_{\hat{G}}(C') = N_{\hat{G}}(g(C'))$.

6. *The proof of Uno's invariant conjecture for* Fi_{24}'

Let $N(C)$ be the normalizer of a radical p-chain. If $N(C)$ is a maximal subgroup of Fi_{24}', then the character table of $N(C)$ may be found in the library of character tables distributed with GAP [13]. If this is not the case, we construct a "useful" description of $N(C)$ and attempt to compute directly its character table using Magma. Many of these normalisers posed difficulties for the standard character-table algorithm of [18]; this motivated the development by Unger [19] of a new and more powerful algorithm to compute character tables.

If $N(C)$ is soluble, we construct a power-conjugate presentation for $N(C)$ and use this presentation to obtain the character table.

If $N(C)$ is insoluble, we construct a faithful representation for $N(C)$ and use this as input to the character table construction function. We employ two strategies to obtain faithful representations of $N(C)$.

1. Construct the action of $N(C)$ on the cosets of soluble subgroups of $N(C)$.
2. Construct the actions of $N(C)$ on the cosets of its stabilizers acting on the underlying set of Fi_{24}'.

The tables in Appendix A list the degrees of the irreducible characters used in the proof of Theorem 6.1.

Theorem 6.1. *Let B be a p-block of $G = \mathrm{Fi}_{24}'$ with a positive defect. Then B satisfies Uno's invariant conjecture.*

Proof. We may suppose B has a non-cyclic defect group.

(1) Suppose $p = 7$, so that by Lemma 4.2 (a), $B = B_0$. By Tables A–235 — A–238 (Appendix A),

$$\mathrm{k}(N(C(2)), B_0, d, u, [r]) = \mathrm{k}(N(C(3)), B_0, d, u, [r]) = \begin{cases} 21 & \text{if } (d,u,r) = (2,2,4), \\ 14 & \text{if } (d,u,r) = (2,1,1), \\ 0 & \text{otherwise.} \end{cases}$$

If $i = 4, 5, 6, 7$, then by Tables A–239 — A–242 (Appendix A),

$$\mathrm{k}(N(C(i)), B_0, d, u, [r]) = \begin{cases} 16 & \text{if } (d,u,r) = (3,1,2), \\ 3 & \text{if } (d,u,r) = (3,1,1), \\ 2 & \text{if } (d,u,r) = (2,1,2), \\ 0 & \text{otherwise.} \end{cases}$$

Thus Theorem 6.1 follows by

$$\mathrm{k}(N(C(1)), B_0, d, u, [r]) = \mathrm{k}(N(C(8)), B_0, d, u, [r]) = \begin{cases} 6 & \text{if } (d,u,r) = (3,2,1), \\ 6 & \text{if } (d,u,r) = (3,1,1), \\ 4 & \text{if } (d,u,r) = (3,2,2), \\ 9 & \text{if } (d,u,r) = (3,1,3), \\ 2 & \text{if } (d,u,r) = (2,2,1), \\ 4 & \text{if } (d,u,r) = (2,1,1), \\ 0 & \text{otherwise.} \end{cases}$$

(2) Suppose $p = 5$, so that by Lemma 4.2 (b), $B = B_0, B_1$, or B_2 and $D(B) \simeq 5^2$. Apply Lemma 2.1 to both $N_{\mathrm{Fi}_{24}'}(5)$ and $N_{\mathrm{Fi}_{24}}(5)$, where $5 \in \mathcal{R}_0(\mathrm{Fi}_{24}', 5)$. Thus

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Theorem 6.1 follows by

$$\mathrm{k}(\mathrm{Fi}_{24}', B_i, d, u, [r]) = \mathrm{k}(N(5^2), B_i, d, u, [r]) = \begin{cases} 10 & \text{if } (d,u,r) = (2,2,1), \\ 10 & \text{if } (d,u,r) = (2,2,2), \\ 0 & \text{otherwise,} \end{cases}$$

where $i = 0, 1$ and by

$$\mathrm{k}(\mathrm{Fi}_{24}', B_2, d, u, [r]) = \mathrm{k}(N(5^2), B_2, d, u, [r]) = \begin{cases} 4 & \text{if } (d,u,r) = (2,2,1), \\ 4 & \text{if } (d,u,r) = (2,2,2), \\ 4 & \text{if } (d,u,r) = (2,1,1), \\ 4 & \text{if } (d,u,r) = (2,1,2), \\ 0 & \text{otherwise.} \end{cases}$$

Defect d	8	8	7	7	6	6	otherwise
Value u	2	1	2	1	2	1	otherwise
k(3, d, u)	60	30	24	12	4	2	0
k(4, d, u)	60	30	60	30	4	2	0
k(31, d, u)	44	22	52	26	4	2	0
k(32, d, u)	44	22	16	8	4	2	0

Table 14: Values of $\mathrm{k}(N(C(i)), B_0, d, u)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 8$

(3) Suppose $p = 3$, so that by Lemma 4.2 (c), $B \in \{B_0, B_1\}$ with $D(B_1) \simeq 3^2$.

If $B = B_1$, then $N(3^2) = N(C(32)) = (3^2{:}2 \times G_2(3)).2$, so Theorem 6.1 follows by Lemma 2.1 and

$$\mathrm{k}(G, B_1, d, u) = \mathrm{k}(N(C(32)), B_1, d, u) = \begin{cases} 4 & \text{if } d = 2 \text{ and } u = 2, \\ 2 & \text{if } d = 2 \text{ and } u = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $B = B_0$ and let $\mathrm{k}(j, d, u) = \mathrm{k}(N(C(j)), B_0, d, u)$. First, we consider the radical 3-chains $C(j)$ with $\mathrm{d}(N(C(j))) = 8$, and so $j = 3, 4, 31, 32$. The values $\mathrm{k}(j, d, u)$ are given in Table 14.

It follows that

$$\sum_{i=3,31} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = \sum_{i=4,32} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u).$$

Next we consider the radical 3-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 13$ or 14, so that $j \in \{2, 5-19\}$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 15.

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=13,14} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0.$$

Finally, we consider the radical 3-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 16$, so that $j \in \{1, 20-30, 33-36\}$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 16.

Defect d	14	14	13	13	12	12	11	11	10	10	9	9	8	8	7	7
Value u	2	1	2	1	2	2	2	1	2	1	2	1	2	1	2	1
k(2, d, u)	36	18	48	6	12	0	51	12	10	2	3	0	0	0	13	2
k(5, d, u)	0	0	75	24	25	8	66	6	46	2	7	2	0	0	9	0
k(6, d, u)	0	0	45	90	15	30	45	90	30	60	4	8	0	0	0	0
k(7 d, u)	0	0	45	90	15	30	81	162	114	228	4	8	0	0	0	0
k(8, d, u)	0	0	75	24	94	56	66	6	46	2	72	36	9	18	9	0
k(9, d, u)	0	0	45	90	78	156	45	90	30	60	54	108	9	18	0	0
k(10, d, u)	0	0	45	90	78	156	81	162	114	228	54	108	0	0	0	0
k(11, d, u)	0	0	75	24	94	56	93	60	162	72	72	36	0	0	0	0
k(12, d, u)	0	0	75	24	25	8	93	60	162	72	7	2	0	0	0	0
k(13, d, u)	36	18	48	6	60	6	51	12	30	12	38	4	12	6	13	2
k(14, d, u)	0	0	75	24	25	8	93	60	162	72	7	2	0	0	0	0
k(15, d, u)	0	0	75	24	25	8	45	36	50	16	7	2	0	0	0	0
k(16, d, u)	0	0	75	24	94	56	45	36	50	16	72	36	12	6	0	0
k(17, d, u)	0	0	75	24	94	56	93	60	162	72	72	36	0	0	0	0
k(18, d, u)	36	18	48	6	60	6	126	36	108	18	38	4	0	0	0	0
k(19, d, u)	36	18	48	6	12	0	126	36	88	8	3	0	0	0	0	0

Table 15: Values of $\mathrm{k}(i, d, u)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 13, 14$

Defect d	16	16	15	15	14	14	13	13	12	12	11	11	10	10	9	7	6
Value u	2	1	2	1	2	1	2	2	2	1	2	1	2	1	2	2	2
k(1, d, u)	40	2	12	6	9	4	3	6	9	0	1	0	2	2	0	3	2
k(20, d, u)	24	12	31	2	18	16	13	14	30	0	11	4	3	2	0	3	0
k(21, d, u)	24	12	43	8	59	50	13	14	34	8	38	4	24	0	6	3	0
k(22, d, u)	24	12	43	8	69	198	25	56	41	46	49	0	39	0	6	0	0
k(23 d, u)	24	12	31	2	12	66	25	56	38	10	22	0	3	2	0	0	0
k(24, d, u)	24	12	35	16	21	84	55	110	77	34	79	18	3	2	0	0	0
k(25, d, u)	24	12	35	16	27	34	43	68	69	24	53	4	3	2	0	0	0
k(26, d, u)	24	12	47	22	68	68	43	68	73	32	80	4	24	0	0	0	0
k(27, d, u)	24	12	47	22	78	216	55	110	80	70	106	18	39	0	0	0	0
k(28, d, u)	40	2	24	12	28	12	3	6	12	0	28	0	9	0	6	3	2
k(29, d, u)	40	2	36	18	37	30	42	24	55	8	49	6	9	0	0	0	0
k(30 d, u)	40	2	24	12	18	22	42	24	52	8	22	6	2	2	0	0	0
k(33, d, u)	40	2	47	10	30	48	66	36	92	10	83	6	4	0	0	0	0
k(34, d, u)	40	2	35	4	21	30	27	18	49	2	23	0	4	0	0	0	0
k(35, d, u)	40	2	47	10	87	90	27	18	56	16	50	0	39	0	6	0	0
k(36, d, u)	40	2	59	16	96	108	66	36	99	24	110	6	39	0	0	0	0

Table 16: Values of $\mathrm{k}(i, d, u)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 16$

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=16} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0$$

and the theorem follows when $p = 3$.

(4) Suppose $p = 2$, so that by Lemma 4.2 (c), $B \in \{B_0, B_1, B_2\}$ with $D(B_1) \simeq 2^2$ and $D(B_2) \simeq D_8$. If $B \neq B_0$, then by [**20**], Dade's invariant conjecture holds for B. Suppose $B = B_0$.

First we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 14$, so that $j \in \{13, 14, 16, 17, 62, 63, 67, 68\}$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 17.

Defect d	14	14	13	13	12	12	11	11	10	10	9	9
Value u	2	1	2	1	2	1	2	1	2	1	2	1
k(13, d, u) = k(62, d, u)	32	32	16	16	32	32	10	10	0	0	0	0
k(14, d, u) = k(63, d, u)	32	32	16	16	56	56	46	46	0	0	0	0
k(16, d, u) = k(67, d, u)	32	32	56	56	32	32	10	10	16	16	8	8
k(17, d, u) = k(68, d, u)	32	32	56	56	56	56	46	46	16	16	0	0

Table 17: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 14$

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=14} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0.$$

Next we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 15$, so that $j \in \{11, 12, 15, 18-26, 53-61, 64-66\}$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 18.

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Defect d	15	14	14	13	13	12	12	11	11	10	10	9	9	8	7	7
Value u	2	2	1	2	1	2	1	2	1	2	1	2	1	2	2	1
k(11, d, u) = k(54, d, u)	32	32	8	40	4	28	8	7	4	0	0	8	0	6	3	2
k(12, d, u) = k(61, d, u)	32	32	8	56	4	24	0	3	0	0	0	8	0	2	0	0
k(15, d, u) = k(66, d, u)	32	48	8	76	20	24	0	19	0	8	4	16	8	2	0	0
k(18, d, u) = k(53, d, u)	32	48	8	44	4	28	8	23	4	16	4	12	2	6	3	2
k(19, d, u) = k(64, d, u)	32	32	8	56	4	72	16	23	12	0	0	0	0	0	0	0
k(20, d, u) = k(55, d, u)	32	32	8	40	4	28	8	5	2	0	0	0	0	0	0	0
k(21, d, u) = k(56, d, u)	32	48	8	60	20	28	8	21	2	16	4	4	2	0	0	0
k(22, d, u) = k(57, d, u)	32	48	8	76	20	72	16	39	12	8	4	0	0	0	0	0
k(23, d, u) = k(60, d, u)	32	48	8	60	4	64	8	37	10	8	4	0	0	0	0	0
k(24, d, u) = k(59, d, u)	32	32	8	56	4	64	8	21	10	0	0	0	0	0	0	0
k(25, d, u) = k(58, d, u)	32	32	8	56	4	72	16	23	12	0	0	0	0	0	0	0
k(26, d, u) = k(65, d, u)	32	48	8	76	20	72	16	39	12	8	4	0	0	0	0	0

Table 18: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 15$

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=15} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0.$$

If $\mathrm{d}(N(C(j))) = 17$, then $27 \leqslant j \leqslant 34$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 19.

Defect d	17	16	15	14	14	13	13	12	12	11	11	8
Value u	2	2	2	2	1	2	1	2	1	2	1	2
k(27, d, u)	64	32	8	32	8	8	4	4	0	2	2	4
k(28, d, u)	64	112	24	48	8	16	4	36	0	8	0	4
k(29, d, u)	64	112	88	64	24	28	20	36	0	8	0	0
k(30, d, u)	64	32	24	48	24	12	4	4	0	2	2	0
k(31, d, u)	64	144	120	96	8	64	4	52	8	8	0	0
k(32, d, u)	64	64	104	48	8	56	4	20	8	2	2	0
k(33, d, u)	64	64	120	64	24	60	4	20	8	2	2	0
k(34, d, u)	64	144	184	112	24	76	20	52	8	8	0	0

Table 19: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 17$

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=17} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0.$$

If $\mathrm{d}(N(C(j))) = 18$, then $j \in \{3-10, 43-50\}$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 20.

Defect d	18	17	16	15	14	13	12	11	10	8	7
Value u	2	2	2	2	2	2	2	2	2	2	2
k(3, d, u)	16	12	10	24	20	20	24	9	1	4	1
k(4, d, u)	16	12	10	20	19	7	8	5	0	0	0
k(5, d, u)	16	12	10	28	37	31	12	5	0	0	0
k(6, d, u)	16	12	10	24	40	36	32	9	5	4	1
k(7, d, u)	16	12	26	26	35	35	8	2	0	0	0
k(8, d, u)	16	12	26	30	36	40	24	6	1	0	0
k(9, d, u)	16	12	26	30	56	64	32	14	5	0	0
k(10, d, u)	16	12	26	34	53	59	12	10	0	0	0
k(43, d, u)	16	20	10	24	11	2	8	3	0	0	0
k(44, d, u)	16	20	18	28	16	17	14	9	1	4	1
k(45, d, u)	16	20	18	28	36	33	24	9	5	4	1
k(46, d, u)	16	20	10	32	29	26	10	3	0	0	0
k(47, d, u)	16	20	34	38	32	29	18	8	1	0	0
k(48, d, u)	16	20	34	38	52	45	28	16	5	0	0
k(49, d, u)	16	20	26	42	45	38	14	10	0	0	0
k(50, d, u)	16	20	26	34	27	14	12	2	0	0	0

Table 20: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 18$

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It follows that

$$\sum_{\mathrm{d}(N(C(i)))=18} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0.$$

If $\mathrm{d}(N(C(j))) = 19$, then $j \in \{2, 35 - 41\}$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 21.

Defect d	19	18	17	16	15	14	14	13	13	12	12	11	10	8
Value u	2	2	2	2	2	2	1	2	1	2	1	2	2	2
k(2, d, u)	32	24	4	8	8	25	4	10	8	8	4	0	2	4
k(35, d, u)	32	24	4	8	6	5	4	6	4	2	4	0	0	0
k(36, d, u)	32	24	20	48	24	31	4	14	4	10	4	0	0	0
k(37, d, u)	32	24	20	40	22	43	4	26	0	18	4	4	2	4
k(38, d, u)	32	24	36	20	38	13	4	14	4	0	0	4	0	0
k(39, d, u)	32	24	36	20	40	33	4	18	8	6	0	4	2	0
k(40, d, u)	32	24	52	52	54	67	4	34	0	16	0	8	2	0
k(41, d, u)	32	24	52	60	56	55	4	22	4	8	0	4	0	0

Table 21: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 19$

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=19} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0.$$

If $\mathrm{d}(N(C(j))) = 20$, then $73 \leqslant j \leqslant 80$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 22.

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=20} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0.$$

If $\mathrm{d}(N(C(j))) = 21$, then $j \in \{1, 42, 51, 52, 69-72, 81-88\}$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 23.

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=21} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0$$

and the theorem follows when $p = 2$. □

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Defect d	20	20	19	19	18	18	17	17	16	16	15	15	14	14	13	13	12	12	11	11	8
Value u	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
k(73, d, u)	16	16	16	16	36	0	30	10	12	0	4	2	18	8	6	8	4	0	2	4	2
k(74, d, u)	16	16	8	8	20	0	22	2	4	0	2	0	10	0	2	0	0	0	0	0	2
k(75, d, u)	16	16	8	8	20	16	30	14	20	16	10	8	10	16	2	4	0	0	0	0	0
k(76, d, u)	16	16	24	16	44	8	34	14	40	20	32	22	26	8	14	0	4	0	0	0	0
k(77, d, u)	16	16	16	8	28	8	26	6	32	20	30	20	18	0	10	0	0	0	0	0	0
k(78, d, u)	16	16	16	8	28	24	34	18	48	36	38	28	18	16	10	20	0	0	0	0	0
k(79, d, u)	16	16	24	16	44	24	66	50	56	36	56	30	26	24	14	20	4	8	0	0	0
k(80, d, u)	16	16	16	16	36	16	62	46	28	16	28	10	18	24	6	12	4	8	2	4	0

Table 22: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 20$

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Defect d	21	20	19	18	18	17	17	16	16	15	15	14	14	13	13	12	12	11	10	8	7
Value u	2	2	2	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	2	2	2
k(1, d, u)	32	8	4	8	8	3	4	2	0	2	4	10	4	3	0	0	0	0	1	2	2
k(42, d, u)	32	8	4	4	8	3	4	0	0	2	2	0	4	1	0	0	0	0	0	0	0
k(51, d, u)	32	24	12	20	8	7	4	4	8	5	6	4	8	7	4	4	2	1	0	0	0
k(52, d, u)	32	24	12	8	8	3	4	6	0	5	4	14	8	9	8	10	2	5	1	2	2
k(69, d, u)	32	24	20	16	8	5	4	10	8	8	12	2	4	1	0	0	0	0	0	0	0
k(70, d, u)	32	24	4	12	8	5	4	8	8	8	6	8	4	3	0	0	0	0	0	0	0
k(71, d, u)	32	40	12	12	8	5	4	12	8	11	6	12	8	9	4	4	2	3	0	0	0
k(72, d, u)	32	40	28	32	8	17	4	14	16	11	16	6	8	7	8	2	2	1	0	0	0
k(81, d, u)	32	24	12	20	8	11	4	0	0	19	2	4	0	0	0	0	0	2	0	0	0
k(82, d, u)	32	24	12	24	8	11	4	2	0	19	4	14	0	2	0	0	0	2	1	0	0
k(83, d, u)	32	40	12	28	8	13	4	8	8	25	6	12	0	2	0	0	0	0	0	0	0
k(84, d, u)	32	40	28	32	8	13	4	10	8	25	12	6	0	0	0	0	0	0	0	0	0
k(85, d, u)	32	40	48	24	8	27	20	18	0	24	4	18	4	16	8	14	0	6	1	0	0
k(86, d, u)	32	40	48	36	8	31	20	16	8	24	6	8	4	14	4	8	0	2	0	0	0
k(87, d, u)	32	56	64	48	8	41	20	26	16	30	16	10	4	14	8	6	0	0	0	0	0
k(88, d, u)	32	56	48	28	8	29	20	24	8	30	6	16	4	16	4	8	0	2	0	0	0

Table 23: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 21$

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7. *The proof of Uno's projective conjecture for* $3.\mathrm{Fi}_{24}'$

Let C be a radical p-chain of Fi_{24}' and $N_{3.\mathrm{Fi}_{24}'}(C) = 3.N_{\mathrm{Fi}_{24}'}(C)$. Let $Z = Z(3.\mathrm{Fi}_{24}')$, $\mathrm{Irr}(Z) = \{1, \zeta, \zeta'\}$ and $B(\zeta)$ the block of Z containing ζ.

The tables in Appendix B list the degrees of the irreducible characters used in the proof of Theorem 7.1.

Theorem 7.1. *Let B be a p-block of $G = 3.\mathrm{Fi}_{24}'$ with $D(B) \neq O_p(G)$. Then B satisfies Uno's projective conjecture.*

Proof. (1). Suppose $p = 7$. Then G has exactly one block $B = \hat{B}$ covering the block $B(\zeta)$. Let C be a radical 7-chain of Fi_{24}'. If $|C| = 0$, then $N_G(C) = G$, $N_{\mathrm{Fi}_{24}'}(C) = \mathrm{Fi}_{24}'$ and so $\mathrm{k}(N_G(C), B, d, \zeta, [r])$ is given by Lemma 4.2, and $\mathrm{k}(N_{\mathrm{Fi}_{24}'}(C), B_0, d, u, [r])$ is given in the proof (1) of Theorem 6.1. Thus

$$\mathrm{k}(N_G(C), B, d, \zeta, [r]) = \sum_{u=1}^{2} \mathrm{k}(N_{\mathrm{Fi}_{24}'}(C), B_0, d, u, [5r]). \tag{7.1}$$

If $|C| \neq 0$, then $N_{3.\mathrm{Fi}_{24}'}(C) = 3 \times N_{\mathrm{Fi}_{24}'}(C)$, so that (7.1) still holds. Thus Theorem 7.1 follows by the proof (1) of Theorem 6.1 when $p = 7$.

(2). Let $p = 5$ and $B = \hat{B}_1$ or $\hat{B}$ given by Lemma 4.2. If $C(j) = C(2)$ or $C(3)$ given by Table 9, then

$$N_{3.\mathrm{Fi}_{24}'}(C(j)) = 3 \times N_{\mathrm{Fi}_{24}'}(C(j)). \tag{7.2}$$

If $C = C(4)$, then $N_G(C) = N_G(5^2)$ and

$$\mathrm{k}(3.\mathrm{Fi}_{24}', B, d, \zeta, [r]) = \mathrm{k}(N_{3.\mathrm{Fi}_{24}'}(5^2), B, d, \zeta, [r]) = \begin{cases} 10 & \text{if } (d, r) = (2, 1), \\ 10 & \text{if } (d, r) = (2, 2), \\ 0 & \text{otherwise.} \end{cases}$$

The theorem holds.

(3). Suppose $p = 3$, so that Uno's projective conjecture is Dade's projective conjecture. By Lemma 4.2, $B \in \{\hat{B}_1, \hat{B}\}$ with $D(\hat{B}_1) \simeq 3^{1+2}$.

If $B = \hat{B}_1$, then $N_G(3^2) = N(C(32)) = 3.(3^2{:}2 \times G_2(3)).2$, so Theorem 7.1 follows by Lemma 2.1 and

$$\mathrm{k}(G, \hat{B}_1, d, \zeta) = \mathrm{k}(N(C(32)), \hat{B}_1, d, \zeta) = \begin{cases} 4 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $B = \hat{B}$ and let $\mathrm{k}(j, d) = \mathrm{k}(N(C(j)), \hat{B}, d, \zeta)$. First, we consider the radical 3-chains $C(j)$ with $\mathrm{d}(N(C(j))) = 9$, and so $j = 3, 4, 31, 32$. Then the values $\mathrm{k}(j, d)$ are given in Table 24.

Defect d	8	7	6	otherwise
k(3, d)	30	12	2	0
k(4, d)	30	30	2	0
k(31, d)	36	30	4	0
k(32, d)	36	12	4	0

Table 24: Values of $\mathrm{k}(N(C(i)), \hat{B}, d, \zeta)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 9$

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It follows that

$$\sum_{i=3,31}(-1)^{|C(i)|}\mathrm{k}(N(C(i)),\hat{B},d,\zeta)=\sum_{i=4,32}(-1)^{|C(i)|}\mathrm{k}(N(C(i)),\hat{B},d,\zeta).$$

Next we consider the radical 3-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j)))=14$ or 15, so that $j\in\{2,5-19\}$. If $\mathrm{d}(N(C(j)))=14$, then $j\in\{5-12,14-17\}$ and (7.2) holds. Thus if $d\geqslant 1$, then

$$\mathrm{k}(N_{3.\mathrm{Fi}_{24}'}(C(j)),\hat{B},d,\zeta)=\sum_{u=1}^{2}\mathrm{k}(N_{\mathrm{Fi}_{24}'}(C(j)),B_0,d-1,u)$$

and so $\mathrm{k}(N_{3.\mathrm{Fi}_{24}'}(C(j)),\hat{B},d,\zeta)$ is given by Table 15. If $\mathrm{d}(N(C(j)))=15$, then $j\in\{2,13,18,19\}$ and the non-zero values $\mathrm{k}(i,d)$ are given in Table 25.

Defect d	14	13	12	11	10	9	8	7
k(2, d)	54	18	27	18	5	0	9	2
k(13, d)	54	72	27	18	54	18	9	2
k(18, d)	54	72	72	120	54	0	0	0
k(19, d)	54	18	72	120	5	0	0	0

Table 25: Values of $\mathrm{k}(i,d)$ when $p=3$ and $\mathrm{d}(N(C(i)))=15$

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=14,15}(-1)^{|C(i)|}\mathrm{k}(N(C(i)),\hat{B},d,\zeta)=0.$$

Finally, we consider the radical 3-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j)))=17$, so that $j\in\{1,20-30,33-36\}$.

The non-zero values $\mathrm{k}(i,d)$ are given in Table 26.

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=17}(-1)^{|C(i)|}\mathrm{k}(N(C(i)),\hat{B},d,\zeta)=0$$

and the theorem follows when $p=3$.

(4). Suppose $p=2$, so that by Lemma 4.2, $B\in\{\hat{B}_1,\hat{B}\}$ with $D(\hat{B}_1)\simeq D_8$. If $B=\hat{B}_1$, then Theorem 7.1 follows by

$$\mathrm{k}(G,\hat{B}_1,d,\zeta)=\mathrm{k}(N(C(54)),\hat{B}_1,d,\zeta)=\begin{cases}4 & \text{if } d=3,\\ 1 & \text{if } d=2,\\ 0 & \text{otherwise.}\end{cases}$$

Suppose $B=\hat{B}$ and let $\Omega=\{1,52,53,54,59,60,70,71\}$. If $1\leqslant j\leqslant 88$ with $j\notin\Omega$, then (7.2) holds and so $\mathrm{k}(N_{3.\mathrm{Fi}_{24}'}(C(j)),\hat{B},d,\zeta)$ is given by (7.1). It follows by Tables 17-22 that

$$\sum_{\mathrm{d}(N(C(i)))=w}(-1)^{|C(i)|}\mathrm{k}(N(C(i)),\hat{B},d,\zeta)=0,$$

where $w\in\{14,17-20\}$.

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Defect d	14	13	12	11	10	8	7
k(1, d)	42	9	6	7	0	3	2
k(20, d)	60	12	24	8	0	3	0
k(21, d)	60	36	51	24	6	3	0
k(22, d)	72	48	75	39	6	0	0
k(23, d)	72	12	48	11	0	0	0
k(24, d)	72	30	141	11	0	0	0
k(25, d)	60	30	84	8	0	0	0
k(26, d)	60	54	111	24	0	0	0
k(27, d)	72	66	168	39	0	0	0
k(28, d)	42	24	33	9	6	3	2
k(29, d)	42	51	78	9	0	0	0
k(30, d)	42	36	51	7	0	0	0
k(33, d)	81	39	126	10	0	0	0
k(34, d)	81	12	42	10	0	0	0
k(35, d)	81	45	69	39	6	0	0
k(36, d)	81	72	153	39	0	0	0

Table 26: Values of $\mathrm{k}(i, d)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 17$

Suppose $\mathrm{d}(N(C(j))) = 15$, so that $j \in \{11, 12, 15, 18-26, 53-61, 64-66\}$. If moreover, $j \notin \Omega$, then $\mathrm{k}(N_{3.\mathrm{Fi}_{24}'}(C(j)), \hat{B}, d, \zeta)$ is given by (7.1) and Table 18. In addition, if $j \in \{53, 54, 59, 60\} \subseteq \Omega$, then the non-zero values $\mathrm{k}(j, d)$ are given in Table 27.

Defect d	15	14	13	12	11	10	9	8	7
k(53, d)	32	40	44	20	19	12	14	6	1
k(54, d)	32	24	36	20	3	0	8	6	1
k(59, d)	32	24	52	56	11	0	0	0	0
k(60, d)	32	40	60	56	27	4	0	0	0

Table 27: Values of $\mathrm{k}(i, d)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 15$

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=15} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), \hat{B}, d, \zeta) = 0.$$

Suppose $\mathrm{d}(N(C(j))) = 21$, so that $j \in \{1, 42, 51, 52, 69-72, 81-88\}$. If moreover, $j \notin \Omega$, then $\mathrm{k}(N_{3.\mathrm{Fi}_{24}'}(C(j)), \hat{B}, d, \zeta)$ is given by (7.1) and Table 23. In addition, if $j \in \{1, 52, 70, 71\} \subseteq \Omega$, then the non-zero values $\mathrm{k}(j, d)$ are given in Table 28.

It follows that

$$\sum_{\mathrm{d}(N(C(i)))=21} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), \hat{B}, d, \zeta) = 0$$

and the theorem is now proved. □

Defect d	21	20	19	18	17	16	15	14	13	12	11	10	8
k(1, d)	32	8	4	8	1	2	4	6	1	0	0	1	2
k(52, d)	32	24	12	8	5	6	5	14	15	8	3	1	2
k(70, d)	32	24	4	12	3	16	12	4	1	0	0	0	0
k(71, d)	32	40	12	12	7	20	13	12	11	2	1	0	0

Table 28: Values of $k(i, d)$ when $p = 2$ and $d(N(C(i))) = 21$

Appendices

Two appendices are provided as add-ons to this paper.

- Appendix A gives the degrees of irreducible characters of chain normalizers of Fi'_{24} and Fi_{24}; it may be found at http://www.lms.ac.uk/jcm/11/lms2007-003/appendix-a/app-a.pdf.
- Appendix B gives the degrees of irreducible characters of chain normalizers of $3.Fi'_{24}$; it may be found at http://www.lms.ac.uk/jcm/11/lms2007-003/appendix-b/app-b.pdf.

These characters were used in the proofs of Theorems 6.1 and 7.1, respectively.

References

1. J. L. Alperin, 'Weights for finite groups', *The Arcata Conference on Representations of Finite Groups*, Proc. Sympos. Pure Math. 47 (1987) 369-379. 100
2. J. An and C. W. Eaton, 'The p-local rank of a block', *J. Group Theory* 3 (2000) 369-380. 101
3. J. An and C. W. Eaton, 'Modular representation theory of blocks with trivial intersection defect groups', *Algebra Represent. Theory* 8 (2005) 427-448. 104
4. J. An and E. A. O'Brien, 'A local strategy to decide the Alperin and Dade conjectures', *J. Algebra* 206 (1998) 183-207. 100, 103, 104
5. J. An and E. A. O'Brien, 'The Alperin and Dade conjectures for the simple Fischer group Fi_{23}', *Internat. J. Algebra Comput.* 9 (1999) 621-670. 100, 103, 104
6. J. An and R. A. Wilson, 'The Alperin and Uno conjectures for the Baby Monster $\mathbb{B}$, p odd', *LMS J. Comput. Math.* 7 (2004) 120-166. 102
7. W. Bosma, J. Cannon and C. Playoust, 'The Magma algebra system I: The user language', *J. Symbolic Comput.* 24 (1997) 235–265. 104
8. N. Burgoyne and C. Williamson, 'On a theorem of Borel and Tits for finite Chevalley groups', *Arch. Math.* 27 (1976) 489-491. 118
9. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups* (Clarendon Press, Oxford, 1985). 103, 104, 106, 108, 113, 114
10. F. Celler, C. R. Leedham-Green, S. H. Murray, A. C. Niemeyer and E. A. O'Brien, 'Generating random elements of a finite group', *Comm. Algebra* 23 (1995) 4931–4948. 104

https://doi.org/10.1112/S1461157000000541 Published online by Cambridge University Press

11. E. C. Dade, 'Counting characters in blocks. I', *Invent. Math.* 109 (1992) 187-210. 100, 104, 114

12. E. C. Dade, 'Counting characters in blocks. II.9.', Representation theory of finite groups (Columbus, OH, 1995), Ohio State Univ. Math. Res. Inst. Publ. 6 (de Gruyter, Berlin, 1997) 45-59. 100, 101

13. The GAP Group, *GAP – Groups, Algorithms, and Programming, Version* 4.4.9, 2006, http://www.gap-system.org. 131

14. G. Hiss and K. Lux, *Brauer Trees of Sporadic Groups* (Oxford Science Publications, 1989). 112, 114

15. I. M. Isaacs and G. Navarro, 'New refinements of the McKay conjecture for arbitrary finite groups', *Ann of Math.* (2) 156 (2002) 333–344. 100, 102

16. H. Nagao and Y. Tsushima, *Representations of Finite Groups* (Academic Press, Boston, MA, 1989). 114

17. J. B. Olsson and K. Uno, 'Dade's conjecture for general linear groups in the defining characteristic', *Proc. London Math. Soc.* (3) 72 (1996) 359-384. 118

18. G. J. A. Schneider, 'Dixon's character table algorithm revisited', *J. Symbolic Comput.* 9 (1990) 601–606. 104, 131

19. W. R. Unger, 'Computing the character table of a finite group', *J. Symbolic Comput.* 41 (2006) 847–862. 104, 131

20. K. Uno, 'Dade's conjecture for tame blocks', *Osaka J. Math.* 31 (1994) 747-772. 114, 115, 135

21. K. Uno, 'Conjectures on character degrees for the simple Thompson group', *Osaka J. Math.* 41 (2004) no. 1, 11-36. 100, 102

22. R. A. Wilson, 'The local subgroups of the Fischer groups', *J. London Math. Soc.* (2) 36 (1987) 77-94. 103, 105

23. R. A. Wilson, 'Standard generators for sporadic simple groups', *J. Algebra* 184 (1996) 505–515. 104

24. R. A. Wilson et al., Atlas of Finite Group Representations, http://brauer.maths.qmul.ac.uk/Atlas 104

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