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ON THE BEHAVIOR OF EXTENSIONS OF VECTOR BUNDLES UNDER THE FROBENIUS MAP

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Introduction.

Let k be an algebraically closed field of characteristic $p > 0$, and let X be a curve defined over k. The aim of this paper is to study the behavior of the Frobenius map $F^*: H^1(X, E) \to H^1(X, F^*E)$ for a vector bundle E.

Our main result is the following.

THEOREM 15. *Let X be a curve of genus $g > 0$. Let $n(X)$ be the integer defined by*

$$n(X) = \max\left\{\sum_{x\in X}\left[\frac{v_x(\mathrm{d}f)}{p}\right];\ f \text{ runs over all rational functions on } X \text{ with } \mathrm{d}f \neq 0\right\}.$$

Then

(i) *for any line bundle L such that $\deg L > n(X)$, the Frobenius map $F^*: H^1(X, \check{L}) \to H^1(X, F^*\check{L})$ is injective.*

(ii) *if $n(X) > 0$, then there exists a line bundle M of degree $n(X)$ such that the Frobenius map $F^*: H^1(X, \check{M}) \to H^1(X, F^*\check{M})$ is not injective. (where $\check{L}$ is the dual line bundle of L)*

This main result leads us to a counter example to a question posed by R. Hartshorne:

QUESTION. Assume the Hasse-Witt matrix of X is non-singular. Is the Frobenius map $F^*: H^1(X, \check{L}) \to H^1(X, F^*\check{L})$ injective for any ample line bundle L?

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Notations.

Throughout this paper, we mean by a variety (resp. curve) an irreducible complete non-singular variety (resp. curve) defined over an algebraically closed field of characteristic $p > 0$. We denote by $\mathcal{O}_X$ the structure sheaf of X, by $K = K(X)$ the field of rational functions on X and by Ω_X^i the sheaf of germs of regular differential i-forms.

We use the words vector bundle and locally free sheaf interchangeably. For any vector bundle E of rank n on a curve, there exists a series of subbundles of E

$$0 = E_0 \subset E_1 \subset E_2 \subset, \cdots, \subset E_n = E$$

where $L_i = E_i/E_{i-1}$ is a line bundle (cf. Atiyah [1])
$(L_1, L_2, \cdots, L_n)$ will be called a splitting of E. A line subbundle L of E will be called a maximal line subbundle of E, if L satisfies the following condition: for any line subbundle M of E, $\deg L \geqq \deg M$.

A splitting $(L_1, L_2, \cdots, L_n)$ will be called a maximal splitting of E, if it satisfies the following conditions:

(i) L_1 is a maximal line subbundle of E,

(ii) $(L_2, L_3, \cdots, L_n)$ is a maximal splitting of E/L_1.

We denote by $\check{E}$ the dual vector bundle of E and denote by $h^i(E)$ the dimension of the k-vector space $H^i(X, E)$.

1. Let X be a variety of dim n. Let $F: X \to X$ be the Frobenius morphism. (cf. [4]). The natural derivation $\mathrm{d}: \mathcal{O}_X \to \Omega_X^1$ gives rise to a k-linear map $\mathrm{d}: \Omega_X^i \to \Omega_X^{i+1}$ for each i, which induce *a* $\mathcal{O}_X$-homomorphism $F_*\mathrm{d}: F_*\Omega_X^i \to F_*\Omega_X^{i+1}$ for each i. We denote by $\mathscr{Z}_X^i$ (resp. $\mathscr{B}_X^{i+1}$) the kernel (resp. image) of $F_*\mathrm{d}: F_*\Omega_X^i \to F_*\Omega_X^{i+1}$. Let x be a point of X and let $u_1, u_2, \cdots, u_n$ be local parameters of X at x. Then we have the following Propositions, due to Cartier (cf. [10]).

PROPOSITION 1. *$\mathscr{Z}_{XX}^i = \mathscr{B}_{X,x}^i \oplus (\oplus\, \mathcal{O}_{X,x}^P (u_{j_1}, u_{j_2}, \cdots, u_{j_i})^{p-1} \mathrm{d}\, u_{j_1} \wedge \mathrm{d}\, u_{j_2} \wedge \cdots \wedge \mathrm{d}\, u_{j_i})$ where $\mathcal{O}_{X,x}^P = \{f^p ; f \in \mathcal{O}_{X,x}\}$, $\mathscr{Z}_{X,x}^i$ is an $\mathcal{O}_{X,x}$-module through the p-th power map.*

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PROPOSITION 2. *There are $\mathcal{O}_X$-homomorphisms $C: \mathcal{Z}^i_X \to \Omega^i_X$, called the Cartier operator, with the following properties.*

(i) $C(\omega_1 + \omega_2) = C(\omega_1) + C(\omega_2)$

(ii) $C(f^p\omega) = fC(\omega)$

(iii) $C(\omega) = 0$ *if* $\omega \in \mathcal{B}^i_{X,x}$

(iv) $C((f_1 f_2 \cdots f_i)^{p-1} \mathrm{d} f_1 \wedge \mathrm{d} f_2 \wedge \cdots \wedge \mathrm{d} f_i = \mathrm{d} f_1 \wedge \mathrm{d} f_2 \wedge \cdots \wedge \mathrm{d} f_i$

where $\omega_1, \omega_2, \omega \in \mathcal{Z}^i_{X,x}$ *and* $f, f_1, f_2, \cdots, f_i \in \mathcal{O}_{X,x}$.

PROPOSITION 3. *The following sequence of $\mathcal{O}_X$-Modules are exact.*

(i) $0 \longrightarrow \mathcal{Z}^i_X \longrightarrow F_*\Omega^i_X \xrightarrow{F_*\mathrm{d}} \mathcal{B}^{i+1}_X \longrightarrow 0$

(ii) $0 \longrightarrow \mathcal{O}_X \xrightarrow{F'} F_*\mathcal{O}_X \xrightarrow{F_*\mathrm{d}} \mathcal{B}^1_X \longrightarrow 0$

(iii) $0 \longrightarrow \mathcal{B}^i_X \longrightarrow \mathcal{Z}^i_X \xrightarrow{C} \Omega^i_X \longrightarrow 0$

Since the Frobenius morphism F is affine, the canonical p-linear map $\alpha: H^i(X, F_*\mathcal{F}) \to H^i(X, \mathcal{F})$ is bijective, for any coherent sheaf $\mathcal{F}$ on X and for any integer i, (cf. [3] III. 1. 3. 3.). Since $\mathcal{Z}^n_X = F_*\Omega^n_X$, $\dim H^n(X, \mathcal{Z}^n_X) = \dim H^n(X, \Omega^n_X) = 1$ and the Cartier operator $C^*: H^n(X, \mathcal{Z}^n_X) \to H^n(X, \Omega^n_X)$ is surjective, so we have that C^* is bijective. Let E be a vector bundle on X. Then there exists a natural map $\psi: E \otimes \check{E} \otimes \Omega^n_X \to \Omega^n_X$ and the cup product

$$U: H^i(X, E) \times H^{n-i}(X, \check{E} \otimes \Omega^n_X) \longrightarrow H^n(X, E \otimes \check{E} \otimes \Omega^n_X) .$$

The composition map

$$H^i(X, E) \times H^{n-i}(X, \check{E} \otimes \Omega^n_X) \longrightarrow H^n(X, E \otimes \check{E} \otimes \Omega^n_X) \longrightarrow H^n(X, \Omega^n_X) \approx k .$$

gives the Serre duality between $H^i(X, E)$ and $H^{n-i}(X, E \otimes \Omega^n_X)$.

The following is well known (e.g. for curves Serre [9]).

PROPOSITION 4. *Let E be a vector bundle on X. Then the following two k-linear maps are dual to each other.*

(i) $F'^*(i, E): H^i(X, E) \longrightarrow H^i(X, E \otimes F_*\mathcal{O}_X)$

(ii) $C^*(n-i, \check{E}): H^{n-i}(X, \check{E} \otimes \mathcal{Z}^n_X) \longrightarrow H^{n-i}(X, \check{E} \otimes \Omega^n_X)$.

In particular, we have $\dim \operatorname{Image} F'^*(i, E) = \dim \operatorname{Image} C^*(n-i, \check{E})$.

For the sake of completeness we include a proof:

$$
\begin{array}{ccccc}
H^i(X,E)\times H^{n-i}(X,\check{E}\otimes\Omega^n_X) & \xrightarrow{U} & H^n(X,E\otimes\check{E}\otimes\Omega^n_X) & \xrightarrow{\psi^*} & H^n(X,\Omega^n_X)\approx k \\
\uparrow{\scriptstyle id\times C^*(n-i,\check{E})} & & \uparrow{\scriptstyle C^*(n,E\otimes\check{E})} & & \uparrow{\scriptstyle C^*(n,\mathcal{O}_X)} \\
H^i(X,E)\times H^{n-i}(X,\check{E}\otimes\mathscr{Z}^n_X) & \xrightarrow{U} & H^n(X,E\otimes\check{E}\otimes\mathscr{Z}^n_X) & \xrightarrow{\psi^*} & H^n(X,\mathscr{Z}^n_X) \\
\downarrow{\scriptstyle F'^*(i,E)\times id} & & \wr\downarrow\alpha & & \wr\downarrow\alpha \\
H^i(X,E\otimes F_*\mathcal{O})\times H^{n-i}(X,\check{E}\otimes\mathscr{Z}^n_X) & & & & \\
\wr\downarrow\alpha \qquad \wr\downarrow\alpha & & & & \\
H^i(X,F^*E)\times H^{n-i}(X,F^*\check{E}\otimes\Omega^n_X) & \xrightarrow{U} & H^n(X,F^*E\otimes F^*\check{E}\otimes\Omega^n_X & \xrightarrow{\psi^*} & H^n(X,\Omega^n_X)
\end{array}
$$

Giving the duality between $H^i(X,E\otimes F_*\mathcal{O}_X)$ and $H^{n-i}(X,\check{E}\otimes\mathscr{Z}^n_X)$ by the composition map $C^*(n,\mathcal{O}_X)\circ\alpha^{-1}\circ\psi^*\circ U\circ(\alpha\times a)$, we have the duality between $F'^*(i,E)$ and $C^*(n-i,E)$.

2. Let E be a vector bundle on X. We denote by $F^*(i,E)$, the composition map $\alpha\circ F'^*(i,E): H^i(X,E)\to H^i(X,F^*E)$.

THEOREM 5. *Let X be a curve and let E be a vector bundle on X. Then*

(i) $\dim \operatorname{Cokernel} F^*(1,E) = h^0(\check{E}\otimes\mathscr{B}^1_X)$

(ii) $\dim \operatorname{Kernel} F^*(1,\check{E}) = h_0(\check{E}\otimes\mathscr{B}^1_X) - (h^0(F^*\check{E}) - h^0(\check{E}))$
$\leqq h^0(\check{E}\otimes\mathscr{B}^1_X)$

Proof. By virtue of Proposition 3, we have the following exact sequences,

$$
\begin{array}{l}
0\longrightarrow \check{E}\otimes\mathscr{B}^1_X\longrightarrow\check{E}\otimes\mathscr{Z}^1_X\xrightarrow{C}\check{E}\otimes\Omega^1_X\longrightarrow 0\\
0\longrightarrow \check{E}\longrightarrow\check{E}\otimes F_*\mathcal{O}_X\longrightarrow\check{E}\otimes\mathscr{B}^1_X\longrightarrow 0
\end{array}
$$

and hence following cohomology exact sequences

$$
\begin{array}{l}
0\longrightarrow H^0(X,\check{E}\otimes\mathscr{B}^1_X)\longrightarrow H^0(X,\check{E}\otimes\mathscr{Z}^1_X)\xrightarrow{C^*(0,\check{E})} H^0(X,\check{E}\otimes\Omega^1_X)\\
0\longrightarrow H^0(X,\check{E})\longrightarrow H^0(X,\check{E}\otimes F_*\mathcal{O}_X)\longrightarrow H^0(X,\check{E}\otimes\mathscr{B}^1_X)\\
\longrightarrow H^1(X,\check{E})\xrightarrow{F'^*(1,\check{E})} H^1(X,\check{E}\otimes F_*\mathcal{O}_X)
\end{array}
$$

Hence we have

$$
\begin{aligned}
\dim \operatorname{Cokernel} F^*(1,E) &= \dim \operatorname{Cokernel} F'^*(1,E)\\
&= h^1(E\otimes F_*\mathcal{O}_X) - \dim \operatorname{Image} F'^*(1,E)\\
&= h^1(F^*E) - \dim \operatorname{Image} C^*(0,\check{E}) \qquad \text{(by virtue of Proposition 4)}\\
&= h^1(F^*E) - (h^0(\check{E}\otimes F_*\Omega^1_X) - h^0(\check{E}\otimes\mathscr{B}^1_X))\\
&= h^0(\check{E}\otimes\mathscr{B}^1_X)
\end{aligned}
$$

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And we have

$$\begin{aligned}\dim \text{Kernel}\, F^*(1,\check{E}) &= \dim \text{Kernel}\, F'^*(1,\check{E}) \\ &= h^0(\check{E}\otimes \mathscr{B}_X^1) - h^0(\check{E}\otimes F_*\mathcal{O}_X) + h^0(\check{E}) \\ &= h^0(\check{E}\otimes \mathscr{B}_X^1) - (h^0(F^*\check{E}) - h^0(\check{E})) \\ &\leqq h^0(\check{E}\otimes \mathscr{B}_X^1)\end{aligned}$$

COROLLARY 6. *Let X be a curve and let E be a vector bundle. Assume that the Frobenius map $F^*(1,E)$ is surjective, then $F^*(1,\check{E})$ is injective and $h^0(F^*\check{E}) = h^0(\check{E})$.*

As a corollary of this Theorem 5, we have the following Theorem of Oda:

THEOREM 7. (*T. Oda*). *Let X be an elliptic curve and let E be an indecomposable vector bundle of rank r and of degree d. Then we have the following results.*

(i) *When the Hasse-Witt matrix of X is not zero (i.e., $F^*(1,\mathcal{O}_X)$ is injective), the Frobenius map $F^*(1,E)$ is injective.*

(ii) *When the Hasse-Witt matrix of X is zero (i.e., $F^*(1,\mathcal{O}_X)$ is the zero map), the Frobenius map $F^*(1,E)$ is not injective (and in fact the zero map) if and only if $r < p$, $d = 0$ and E has a non-zero section (i.e., in Atiyah's notation $E = F_r$ with $r < p$).*

COROLLARY 8. (*Corollary of the proof of Theorem* 7) (*cf.* [1] *p.* 451) *Let X be an elliptic curve.*

(i) *When the Hasse-Witt matrix of X is not zero, then*
$\mathscr{B}_X^1 \approx L_1 \oplus L_2 \oplus \cdots \oplus L_{p-1}$ *where*
$\{\mathcal{O}_X, L_1, L_2, \cdots, L_{p-1}\} = \{L;$ *line bundles with* $L^{\otimes p} \approx \mathcal{O}_X\}$

(ii) *When the Hasse-Witt matrix of X is zero, then $\mathscr{B}_X^1 \approx F_{p-1}$.*

(iii) $F^*F_*\mathcal{O}_X \approx \overset{p}{\oplus}\, \mathcal{O}_X$

Proof. Let E be an indecomposable vector bundle of rank r and of degree d. We use the following results of Atiyah (cf. [1]).

$h^0(E) = d$ and $h^1(E) = 0$ when d is positive

$h^0(E) = 0$ and $h^1(E) = -d$ when d is negative.

$h^0(E) = h^1(E) = 0$ when $d = 0$ and $E \not\approx F_r$.

$h^0(E) = h^1(E) = 1$ when $E \approx F_r$.

When $d=0$, there is a line bundle of degree 0 with $E \approx L \otimes F_r$. It is easy to see that $\mathscr{B}_X^1$ is a vector bundle of rank $p-1$. Let $\mathscr{B}_X^1 \approx E_1 \oplus E_2 \oplus \cdots \oplus E_s$ be the decomposition of $\mathscr{B}_X^1$ into indecomposable factors. Let r_i be the rank of E_i and let d_i be the degree of E_i. Then we have $\sum d_i = \deg \mathscr{B}_X^1 = \chi(\mathscr{B}_X^1) = \chi(F_*\mathcal{O}_X) - \chi(\mathcal{O}_X) = 0$. Let L be a non trivial line bundle of degree 0, then $h^0(L \otimes \mathscr{B}_X^1) \neq 0$ (in fact equal to 1) if and only if $L^{\otimes p} \approx \mathcal{O}_X$ by virtue of following exact sequence.

$$0 = H^0(X, L) \longrightarrow H^0(X, L \otimes F_*\mathcal{O}_X) \longrightarrow H^0(X, L \otimes \mathscr{B}_X^1) \longrightarrow H^1(X, L) = 0 .$$

This shows that $d_i \leqq 0$ for all i and so $d_i = 0$ for all i. Let L_i be the line bundle with $E_i \approx L_i \otimes F_{r_i}$, then $L_i^{\otimes p} \approx \mathcal{O}_X$. By virtue of Lemma 13, we have the following results. When $h^0(\mathscr{B}_X^1) = 1$, then $s=1$, $r_1 = p-1$ and $L_1 \approx \mathcal{O}_X$. And when $h^0(\mathscr{B}_X^1) = 0$, then $s = p-1$, $r_i = 1$ and $\{\mathcal{O}_X, L_1, L_2, \cdots, L_{p-1}\} = \{L;$ line bundles with $L^{\otimes p} \approx \mathcal{O}_X\}$. Let E be an indecomposable vector bundle of rank r and of degree d. If $d>0$, then $h^1(E) = 0$. If $d<0$, then $h^0(E \otimes L) = 0$ for all line bundle L of degree 0, and so $h^0(E \otimes \mathscr{B}_X^1) = 0$. Thus the Frobenius map $F^*(1, E)$ is injective when $d \neq 0$. When $d=0$ and $E \not\approx F_r$ then $h^1(E) = 0$ and the Frobenius map $F^*(1, E)$ is injective. When $E = F_r$ and the Hasse-Witt matrix of X is not zero, then $h^0(E \otimes \mathscr{B}_X^1) = 0$ and the Frobenius map $F^*(1, E)$ is injective. When $E = F_r$ and the Hasse-Witt matrix of X is zero, then we have the following results by induction on r. $h^0(F^*F_r) = \min\{p, r\}$, $F^*F_r \approx \overset{r}{\oplus} \mathcal{O}_X$ for all r with $r \leqq p$ and $F^*(1, F_r)$ is the zero map if and only if $r \leqq p-1$.

$r=1$. It is obvious.

$p \geqq r > 1$. We have the following exact sequence

$$0 \longrightarrow F_{r-1} \longrightarrow F_r \longrightarrow \mathcal{O}_X \longrightarrow 0 .$$

Hence we have $F^*F_r \approx \overset{r}{\oplus} \mathcal{O}_X$ and $h^0(F^*F_r) = r$ by the induction assumption. But we have $h^0(F_r \otimes \mathscr{B}_X^1) = h^0(F_r \otimes F_{p-1}) = \min\{r, p-1\}$ (for all r, cf. [1] Lemma 17). Hence we have $h^0(F_r \otimes \mathscr{B}_X^1) - h^0(F^*F_r) + h^0(F_r) = 1$, if $r<p$. This shows that when $r<p$ the Frobenius map $F^*(1, F_r)$ is the zero map.

$p \leqq r$. F_r has F_p as a subbundle and so $h^0(F^*F_r) \geqq p$ by the induction assumption. Hence we have
$0 \leqq h(F_r \otimes \mathscr{B}_X^1) - h^0(F^*F_r) + h^0(F_r) \leqq 0$. This shows that $h_0(F^*F_r) = p$ and the Frobenius map $F^*(1, F_r)$ is injective.

3. Let X be a curve. For any divisor D on X, we denote by $\mathcal{O}(D)$ the line bundle associated with D.

DEFINITION 9. *For any function $f \in K$, we denote by $n(f)$ the integer (or infinity) $\sum_{x \in X}[v_x(\mathrm{d}\,f)/p]$ where $[\ \ ]$ is the Gauss symbol, and v_x is the valuation associated with x.*

LEMMA 10. *Let g be the genus of the curve X. Then*
(i) *$n(f) = \infty$ if and only if $f \in K^p$,*
(ii) *$n(f) \leqq [2(g-1)/p]$, if $n(f) < \infty$.*

Proof. $\mathrm{d}\,f = 0$ if and only if $f \in K^p$, and if $\mathrm{d}\,f = 0$, then $n(f) = \infty$. If $\mathrm{d}\,f \neq 0$, the divisor $D = \sum_{x \in X} v_x(\mathrm{d}\,f)x$ is a canonical divisor, and so the degree of D is $2(g-1)$. Therefore we have

$$n(f) \leqq \left[\frac{2(g-1)}{p}\right].$$

DEFINITION 11. *We define $n(X)$ by the following formula*

$$n(X) = \max\{n(f)\,;\, f \in K \text{ and } f \notin K^p\}.$$

Note that $n(X) \leqq [2(g-1)/p]$, by virtue of Lemma 10.

LEMMA 12. *Let D be a divisor on X, Then we have*

$$H^0(X, \mathcal{O}(-D) \otimes \mathscr{B}_X^1) \cong \{\mathrm{d}\,f\,;\, f \in K \text{ and } (\mathrm{d}\,f) > pD\}.$$

Proof. By virtue of Proposition 3, we have the following exact sequence.

$$0 \longrightarrow \mathcal{O}(-D) \otimes \mathscr{B}_X^1 \longrightarrow \mathcal{O}(-D) \otimes \mathscr{Z}_X^1 \xrightarrow{C} \mathcal{O}(-D) \otimes \Omega_X^1 \longrightarrow 0$$

Hence, we have the following cohomology exact sequence.

$$0 \longrightarrow H^0(X, \mathcal{O}(-D) \otimes \mathscr{B}_X^1) \longrightarrow H^0(X, \mathcal{O}(-D) \otimes \mathscr{Z}_X^1) \xrightarrow{C*} H^0(X, \mathcal{O}(-D) \otimes \Omega_X^1) \cong H^0(X, \mathcal{O}(-pD) \otimes \Omega_X^1)$$

Since, $H^0(X, \mathcal{O}(-pD) \otimes \Omega_X^1) = \{\omega \in \Omega^1(K/k)\,;\,(\omega) > pD\}$. The assersion is obvious by virtue of Proposition 3.

Remark: *By Lemma 12, it is easy to see that $n(X)$ coincides with the degree of a maximal line subbundle of $\mathscr{B}_X^1$.*

LEMMA 13. *Let G be the group of linear equiualence classes of* *divisors on X, and let G_p be the subgroup of elements $\bar{D} \in G$ such that $p\bar{D} = 0$. Then G_p is a finite group of order p^σ, where σ is the rank of the Hasse-Witt matrix of X.*

Proof. See Serre [9] Proposition 10 § 2.

PROPOSITION 14. *Let X be a curve of genus $g > 0$. Then $\boldsymbol{n}(X) \geqq 0$.*

Proof. When the Hasse-Witt matrix of X is not zero. $G_p \neq 0$, by virtue of Lemma 13. So there exists a non-zero element $\bar{D} \in G$ such that $p\bar{D} = 0$. Therefore, there exists a rational function f such that $f \notin K^p$ and $(f) = pD$. Hence $(\mathrm{d}f) > pD$. Thus $\boldsymbol{n}(X) \geqq \deg D = 0$.

When the Hasse-Witt matrix of X is zero, i.e., $F^*(1, \mathcal{O}_X)$ is the zero map. We have

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, F^*\mathcal{O}_X) \longrightarrow H^0(X, \mathscr{B}^1_X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

and hence we have $H^0(X, \mathscr{B}^1_X) \approx H^1(X, \mathcal{O}_X) \neq 0$. Therefore $\boldsymbol{n}(X) \geqq 0$, by virtue of Remark.

THEOREM 15. *Let X be a curve of genus $g > 0$. Then*

(i) *for any line bundle L such that $\deg L > \boldsymbol{n}(X)$, the Frobenius map $F^*(1, \check{L}) : H^1(X, \check{L}) \to H^1(X, F^*\check{L})$ is injective.*

(ii) *if $\boldsymbol{n}(X) > 0$, then there exists a line bundle M of degree $\boldsymbol{n}(X)$ such that the Frobenius map $F^*(1, \check{M})$ is not injective.*

Proof. Let $\deg L > \boldsymbol{n}(X)$. Then $H^0(X, \check{L} \otimes \mathscr{B}^1_X) = 0$ by virtue of Remark. Therefore the Frobenius map $F^*(1, \check{L})$ is injective by virtue Theorem 5.

(ii) $\boldsymbol{n}(X) > 0$. There exists a line bundle M of degree $\boldsymbol{n}(X) > 0$, with $H^0(X, \check{M} \otimes \mathscr{B}^1_X) \neq 0$. Since $h^0(F^*(\check{M})) = 0$, the Frobenius map $F^*(1, \check{M})$ is not injective by virtue of Theorem 5.

The following Proposition gives the relation between the number $\boldsymbol{n}(X)$ and the rank of the Hasse-Witt matrix.

PROPOSITION 16. *Let X be a curve of genus $g > 0$, and let $\boldsymbol{h}(X)$ be the rank of the Hasse-Witt matrix of x. Then we have*

$$g - \boldsymbol{h}(X) \leqq (p - 1)(\boldsymbol{n}(X) + 1)$$

Proof. Let D be an effective divisor of degree $d > 0$, such that the

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Frobenius map $F^*(1, \mathcal{O}(-D))$ is injective. Then we have the following exact commutative diagram.

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & H^1(X, \mathcal{O}(-D)) & \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & 0 \\ & & & & \downarrow {\scriptstyle F^*(1, \mathcal{O}(-D))} & & \Big| {\scriptstyle F^*(1, \mathcal{O})} & & \\ 0 & \longrightarrow & \text{Kernel}\,\varphi & \longrightarrow & H^1(X, \mathcal{O}(-pD)) & \stackrel{\varphi}{\longrightarrow} & H^1(X, \mathcal{O}_X) & \longrightarrow & 0 \end{array}$$

And we have

$$\dim \text{Image}\, \varphi \circ F^*(1, \mathcal{O}(-D)) \geqq h^1(\mathcal{O}(-D)) - \dim \text{Kernel}\, \varphi = g + d - pd\ .$$

Hence we have $\boldsymbol{h}(X) \geqq g + d - pd$, i.e., $g - \boldsymbol{h}(X) \leqq (p-1)d$. Since, for any effective divisor D of degree $\boldsymbol{n}(X) + 1$, the Frobenius map $F^*(1, \mathcal{O}(-D))$ is injective, we have

$$g - \boldsymbol{h}(X) \leqq (p-1)(\boldsymbol{n}(X) + 1)\ .$$

4. In this section we shall extend Theorem 15 from line bundles to indecomposable vector bundles of arbitrary rank.

PROPOSITION 17. *Let X be a curve of genus $g > 0$. Then for any r, there exists an indecomposable vector bundle which has a splitting*

$$(\Omega_X^{1\otimes(r-1)}, \Omega_X^{1\otimes(r-2)}, \cdots, \Omega_X^1, \mathcal{O}_X)\ .$$

In order to prove Proposition 17, we need the following Lemmas.

LEMMA 18. *Let E and E' be vector bundle on X, and let $(L_1, L_2, \cdots, L_r)$ be a splitting of E, and suppose that $\varphi: E \to E'$ is a generically surjective morphism. Then there exists a splitting $(L'_1, L'_2, \cdots, L'_s)$ of E' which satisfies the following condition; There exists a sequence $1 \leqq i_1 < i_2 <, \cdots, < i_s$ such that* $\text{Hom}\,(L_{i_j}, L'_j) \neq 0$ *for all j, in particular* $\deg L_{i_j} \leqq \deg L'_j$.

Proof of Lemma 18. It is easy.

LEMMA 19. *Let X be a curve and let E' be an indecomposable vector bundle which has a splitting $(L_1, L_2, \cdots, L_r)$. Let L be a line bundle such that $\deg L < \deg L_j$ for all j. If an exact sequence $0 \longrightarrow E' \stackrel{\alpha}{\longrightarrow} E \stackrel{\varphi}{\longrightarrow} L \longrightarrow 0$ does not split, then E is indecomposable.*

Proof of Lemma 19. Tensoring the sequence with $\check{L}$ we may assume that $L = \mathcal{O}_X$ and $\deg L_j > 0$ for all j. Suppose E is decomposable. Let $E = E_1 \oplus E_2$ and let ψ_i be the injection $E_i \to E$ $(i = 1, 2)$. We may assume that $\varphi \circ \psi_1 \neq 0$. By virtue of Lemma 18, there exists a splitting $(L_1', L_2', \cdots, L_{r_1}')$ of E_1 such that $\deg L_i' \geqq 0$ for all i. Therefore $\varphi \circ \psi$ is surjective. And we have the following exact commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E'' & \longrightarrow & E_1 & \xrightarrow{\varphi\circ\psi_1} & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_1'} & & \downarrow{\scriptstyle \psi_1} & & \| & & \\
0 & \longrightarrow & E' & \xrightarrow{\alpha} & E & \xrightarrow{\varphi} & \mathcal{O}_X & \longrightarrow & 0 \\
 & & {\scriptstyle \psi_2'}\uparrow\downarrow{\scriptstyle \eta'} & & {\scriptstyle \psi_2}\uparrow\downarrow{\scriptstyle \eta} & & & & \\
 & & E''' & \xrightarrow{\alpha'} & E_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where E'' is the kernel of $\varphi \circ \psi_1$, ψ_1' is the injection induced by ψ_1, E''' is the cokernel of ψ_1' and α' is the homomorphism induced by α. By virtue of Snake Lemma, the map α' is an isomorphism. Since $\deg L_i > 0$ for all j, the composition map $\varphi \circ \psi_2 \circ \alpha' \circ \eta' = 0$, by virtue of Lemma 18, since η is a surjection, $\varphi \circ \psi_2 \circ \alpha' = 0$. Hence, the exists a map $\psi_2' : E''' \to E'$ such that $\alpha \circ \psi_1' = \psi_2 \circ \alpha'$. It is easy to show that $\eta' \circ \psi_2' =$ identity. Therefore $E' = E'' \oplus E''' \cdot E'' = 0$, since E' is indecomposable and $E''' \approx E_2 \neq 0$. Hence $E_1 = \mathcal{O}_X$. This shows that the exact sequence $0 \longrightarrow E' \xrightarrow{\alpha} E \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow 0$ splits. This is a contradiction. Therefore E is indecomposable.

Proof of Proposition 17. When $g = 1$. $\Omega_X^1 \approx \mathcal{O}_X$ and F_r has a splitting $(\mathcal{O}_X, \mathcal{O}_X, \cdots, \mathcal{O}_X)$ (cf. [1]).

When $g > 1$. We prove this by induction on r.

$r = 1$. It is obvious.

$r > 1$. By induction assumption, there exists an indecomposable vector F_{r-1} which has a splitting $(\Omega_X^{1\otimes(r-2)}, \Omega_X^{1\otimes(r-3)}, \cdots, \Omega_X^1, \mathcal{O}_X)$. Since $H^1(X, F_{r-1} \otimes \Omega_X^1) = H^0(X, \check{F}_{r-1}) \neq 0$, the exists a non-split exact sequence $0 \to F_{r-1} \otimes \Omega_X^1 \to E \to \mathcal{O}_X \to 0$.

Applying Lemma 19 to this exact sequence, we see that E is indecomposable. It is easy to show that E has a splitting $(\Omega_X^{1\otimes(r-1)}, \Omega_X^{1\otimes(r-2)}, \cdots, \mathcal{O}_X)$.

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PROPOSITION 20. *Let X be a curve of genus $g \geqq 2$. Let E be an indecomposable vector bundle of rank r on X and let $(L_1, L_2, \cdots, L_r)$ be a maximal splitting of E. If* $d_0 = \min\{\deg L_1, \deg L_2, \cdots, \deg L_r\}$, *then*

$$\deg E \leqq r(r-1)(g-1) + rd_0 .$$

In order to prove Proposition 20, we need the following Lemmas.

LEMMA 21. *Let X be a curve of genus g. Let E (resp. E') be a vector bundle of rank r (resp. r') on X and let (resp. $(M_1, M_2, \cdots, M_s)$) be a splitting of E (resp. E'). Suppose that* $\deg L_i > \deg M_j + 2(g-1)$, *for all i, j, then $H^1(X, \check{E} \otimes E') = 0$.*

Proof of Lemma 21. $(L_i \otimes \check{M}_j)$ i, j is a splitting of $E \otimes \check{E}'$. Since $\deg \Omega_X^1 \otimes M_j \otimes \check{L}_i < 0$, we have $H^1(X, L_i \otimes \check{M}_j) = H^0(X, \Omega_X^1 \otimes M_j \otimes \check{L}_i) = 0$. Therefore we have $H^1(X, E \otimes \check{E}') = 0$.

LEMMA 22. *Let X be a curve of genus g. Let E be an indecomposable vector bundle of rank r on X and let $(L_1, L_2, \cdots, L_r)$ be a splitting of E. Then for any m with $1 \leqq m \leqq r$, we have*

$$\begin{aligned} &\min\{\deg L_1, \deg L_2, \cdots, \deg L_{m-1}\} \\ &\qquad \leqq \max\{\deg L_m, \deg L_{m+1}, \cdots, \deg L_r\} + 2(g-1) . \end{aligned}$$

Proof of Lemma 22. It is obvious by virtue of Lemma 21.

LEMMA 23 (M. Nagata). *Let X be a curve of genus g. Let E be a vector bundle of rank 2 and let (L_1, L_2) be a maximal splitting of E. Then*

$$\deg L_2 \leqq \deg L_1 + g .$$

Proof of Lemma 23. See M. Nagata [7] or M. Maruyama [6] Theorem 3. 13.

LEMMA 24. *Let X be a curve of genus g. Let E be a vector bundle of rank r on X and let $(L_1, L_2, \cdots, L_r)$ be a maximal splitting of E. Then*

$$\deg L_r \leqq \deg L_1 + (r-1)g .$$

Proof of Lemma 24. It is obvious by virtue of Lemma 23.

Proof of Proposition 20. We shall define a sequence of integers,

$1 = i_n < i_{n-1} <, \cdots, < i_2 < i_1 < i_0 = r + 1$, which satisfies the following condition.

$$\deg L_{i_m} = \min\{\deg L_1, \deg L_2, \cdots, \deg L_{i_{m-1}-1}\} \qquad (m > 0) .$$

We define a one-to-one onto map

$$\varphi : \{1, 2, \cdots, r\} \longrightarrow \{0, 1, \cdots, r-1\} ,$$

such that $\varphi(j) = r + j - i_m - i_{m-1} + 1$ where $i_m \leqq j < i_{m-1}$. We shall prove that

$$\deg L_j \leqq d_0 + 2\varphi(j)(g-1)$$

by induction on m such that $i_m \leqq j < i_{m-1}$.

For $m = 1$. Since $(L_{i_1}, L_{i_1+1}, \cdots, L_j)$ is a maximal splitting of a vector bundle, we have $\deg L_j \leqq d_0 + (j - i_1)g$, by virtue of Lemma 24. Since $\varphi(j) = j - i_1$ and $g \leqq 2(g-1)$, we have

$$\deg L_j \leqq d_0 + 2\varphi(j)(g-1) .$$

For $m > 1$. Since $(L_{i_m}, L_{i_m} + 1, \cdots, L_j)$ is a maximal splitting of a vector bundle, we have $\deg L_j \leqq \deg L_{i_m} + (j - i_m)g \leqq \deg L_{i_m} + 2(j - i_m)(g-1)$. Since $\varphi(i_{m-2} - 1) \geqq \varphi(q)$ for all $i_{m-1} \leqq q \leqq r$, we have

$$\deg L_q \leqq d_0 + 2\varphi(q)(g-1) \leqq d_0 + 2\varphi(i_{m-2} - 1)(g-1) ,$$

for all $i_{m-1} \leqq q \leqq r$, by induction assumption. For any $1 \leqq q' < i_{m-1}$, $\deg L_{q'} \geqq \deg L_{i_m}$. Hence by virtue of Lemma 22, we have

$$\deg L_{i_m} \leqq d_0 + 2\varphi(i_{m-2} - 1)(g-1) + 2(g-1) .$$

Hence we have

$$\begin{aligned}\deg L_j &\leqq d_0 + 2(r - i_{m-1} + 1)(g-1) + 2(j - i_m)(g-1) \\ &= d_0 = 2\varphi(j)(g-1) .\end{aligned}$$

Therefore, we have

$$\deg E = \sum_{j=1}^{r} \deg L_j \leqq rd_0 + \sum_{j=1}^{r} 2\varphi(j)(g-1) = rd_0 + r(r-1)(g-1) .$$

THEOREM 25. *Let X be a curve of genus $g > 1$. Then*

(i) *for any indecomposable vector bundle of rank r such that $\deg E > r(r-1) + (g-1) + rn(X)$, the Frobenius map $F^*(1, \check{E})$ is injective.*

(ii) *if* $\boldsymbol{n}(X) > 0$, *then for any* $r > 0$, *there exists an indecomposable vector bundle* E' *of rank* r *with* $\deg E' = r(r-1)(g-1) + r\boldsymbol{n}(X)$ *such that the Frobenius map* $F^*(1, \check{E}')$ *is not injective*

Proof. (i) Let $(L_1, L_2, \cdots, L_r)$ be a maximal splitting of E. Then $\deg L_j > \boldsymbol{n}(X)$, by virtue of Proposition 20. Hence the Frobenius map $F^*(1, L_j)$ is surjective for all j, and the Frobenius map $F^*(1, E)$ is surjective. Therefore, the Frobenius map $F^*(1, \check{E})$ is injective by virtue of Corollary 8.

(ii) When $\boldsymbol{n}(X) > 0$, there exists a line bundle M of degree $\boldsymbol{n}(X)$, such that the Frobenius map $F^*(1, \check{M})$ is not injective. There exists an indecomposable vector bundle F_r which has a splitting $(\Omega_X^{1\otimes(r-1)}, \Omega_X^{1\otimes(r-2)}, \cdots, \Omega_X^1, \mathcal{O}_X)$. Put $E' = F_r \otimes M$. Then E' is an indecomposable vector bundle of rank r, and of degree $r(r-1)(g-1) + r\boldsymbol{n}(X)$, which has M as a quotient line bundle. And $H^0(X, \check{E}') = H^0(X, \check{E}'^{(p)}) = 0$. Therefore, the Frobenius map $F^*(1, \check{E}')$ is not injective, by virtue of Corollary 6.

5. In this section we shall give an example of a curve with positive $\boldsymbol{n}(X)$ although the Hasse-Witt matrix of X is non-singular. We also give other examples of a curve X with positive $\boldsymbol{n}(X)$.

EXAMPLE 1. Let k be an algebraically closed field of characteristic 3. Let $X \subset \boldsymbol{P}_k^2$ be the curve defined by the homogeneous equation

$$X_0^3 X_1 + X_1^3 X_2 + X_2^3 X_0 = 0 .$$

One verifies easily that X is non-singular. Being a plane curve of degree 4, it has genus 3. (This example was given in [5]). The Hasse-Witt matrix of X is identically zero. (cf. [5]).

PROPOSITION 26. *If* X *is the curve in Example* 1, *then* $\boldsymbol{n}(X) = 1$.

Proof. By Definition 11, $\boldsymbol{n}(X) \leqq 1$. Let $f = (X_0 - X_2/X_1) \in K = K(X)$. We have $(f)_\infty = (0,0,1) + 3(1,0,0)$. This shows that $f \notin K^3$. It is easy to show that $v_x(d f) \geqq -3$, if $x = (0,0,1)$ or $x = (1,0,0)$, and $v_x(d f) \geqq 3$, if $x = (1-\alpha, -1, 1)$ $i = 1,2,3$ where α_i are the distinct roots of the equation $\alpha^3 = \alpha + 1$, and $v_x(d f) \geqq 0$, if $x \neq (1,0,0)$. This shows that $\boldsymbol{n}(f) \geqq 1$, and $\boldsymbol{n}(X) = 1$.

EXAMPLE 2. Let k be an algebraically closed field of characteristic 3. Let $X \subset \boldsymbol{P}_k^2$ be the curve defined by the homogeneous equation

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$$X_0^4 - X_1^3X_2 - X_1X_2^3 = 0 .$$

One verifies easily that X is non-singular. Being a plane curve of degree 4, it has genus 3. (This example was given in [2]). The Hasse-Witt matrix of X is identically zero. (cf. [2]).

PROPOSITION 27. *If X is the curve in Example 2, theh $\boldsymbol{n}(X) = 1$.*

Proof. We prove this in the same way as in Proposition 26. We have $\boldsymbol{n}(X) \leqq 1$. Put $f = (X_2/X_1) \in K$, then $\boldsymbol{n}(f) = 1$. Therefore we have $\boldsymbol{n}(X) = 1$.

EXAMPLE 3. Let k be an algebraically closed field of characteristic $p \geqq 3$. Let $X \subset \boldsymbol{P}_k^2$ be the curve defined by the homogeneous equation

$$X_0^{p+1} = X_1X_2(X_0^{p-1} + X_1^{p-1} - X_2^{p-1}) .$$

One verifies easily that X is non-singular. Being a plane curve of degree $p+1$, it has genus $(1/2)p(p-1)$.

PROPOSITION 28. *If X is the curve in Example 3, then $\boldsymbol{n}(X) = p - 2 > 0$.*

Proof. We have $\boldsymbol{n}(X) \leqq p - 2$. Put $f = (X_0/X_1) \in K$, then we have $\boldsymbol{n}(X) = p - 2$.

PROPOSITION 29. *If X is the curve in Example 3, then the Hasse-Witt matrix is non-singular, i.e., the Frobenius endomorphism of $H^1(X, \mathcal{O}_X)$ is injective.*

Proof. $U_i = \{(X_0, X_1, X_2) ; X_i \neq 0\}$ $i = 1, 2$ are affine open subsets of $\boldsymbol{P}_k^2$. Then $X \subset U_1 \cup U_2$. Let $f = X_0^{p+1} - X_1X_2(X_0^{p-1} + X_1^{p-1} - X_2^{p-1}) \in k[X_0, X_1, X_2]$. Now let $\alpha \in H^1(X, \mathcal{O}_X)$. Since $\{X \cap U_1, X \cap U_2\}$ is an affine open covering of X, we can realize α as a function $\bar{h}$ on $X \cap U_1 \cap U_2$. This function extends to a function h on $U_1 \cap U_2$, i.e., to an element of the ring $k[X_0/X_1, X_2/X_1, X_1/X_2]$. The set of coboundaries is

$$\left\{h_1 - h_2 ; h_1 \in k\left[\frac{X_0}{X_1}, \frac{X_2}{X_1}\right], h_2 \in k\left[\frac{X_0}{X_2}, \frac{X_1}{X_2}\right]\right\} .$$

h is a linear combination of monomials $X_0^i/X_1^jX_2^{i-j}$. Now if $i \geqq p + 1$, we can write

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$$\frac{X_0^i}{X_1^j X_2^{i-j}} \equiv \frac{X_0^{i-2}}{X_1^{j-1} X_2^{i-j-1}} + \frac{X_0^{i-p-1}}{X_1^{j-p} X_2^{i-j-1}} + \frac{X_0^{i-p-1}}{X_1^{j-1} X_2^{i-j-p}} \qquad (\mathrm{mod}\ f)\ .$$

If $i \leqq j$ or $j \leqq 0$, then $X_0^i/X_1^j X_2^{i-j}$ is a coboundary. Let φ be the natural map $k[X_0/X_1, X_2X_1, X_1X_2] \to H^1(K, \mathcal{O}_X)$. Then we can choose $\varphi(X_0^2/X_1X_2)$, $\varphi(X_0^3/X_1X_2^2)$, $\varphi(X_0^3/X_1^2X_2)$, $\cdots$, $\varphi(X_0^p/X_1X_2^{p-1})$, $\varphi(X_0^p/X_1^2X_2^{p-2})$, $\cdots$, $\varphi(X_0^p/X_1^{p-1}X_2)$ as a basis of $H^1(X, \mathcal{O}_X)$. Let $\alpha_{\varepsilon ij} = (X_0^{p-2j+\varepsilon-1}/X_1^{i-j}X_2^{p-i-j+\varepsilon-1})$, for all i, j and $\varepsilon = 0$ or 1. To complete the proof we need the following Lemma.

LEMMA 30. *Under the same notation as above,*

(i) *let* $V_{\varepsilon i}$ *be a vector subspace of* $H^1(X, \mathcal{O}_X)$ *which is spanned by* $\alpha_{\varepsilon i0}, \alpha_{\varepsilon i1}, \cdots, \alpha_{\varepsilon ij(\varepsilon i)}$ *where* $j(\varepsilon i) = \min\{i-1, p-i+\varepsilon-2\}$, *for all* i *such that* $p+\varepsilon-2 \geqq i \geqq 1$. *Then* $V_{\varepsilon i}$ *is stable under the Frobenius endomorphism.*

(ii) $F^*(1, \mathcal{O}_X | V_{\varepsilon i}$ *is an injection.*

(iii) $\bigoplus_{\varepsilon, i} V_{\varepsilon i} = H^1(X, \mathcal{O}_X)$.

Proof. Let $1/2(p-1) \geqq j \geqq 1$, then we have

$$\begin{aligned}
X_0^{p(p-2j+\varepsilon+1)} &- X_0^{p(p-2j+\varepsilon-1)}(X_1X_2)^p \\
&= X_0^{p(p-2j+\varepsilon-1)}(X_0^2 - X_1X_2)^p \\
&\equiv X_0^{p-2j+\varepsilon-1}(X_0^2 - X_1X_2)^{2j-\varepsilon+1}X_0^{p-1}(X_0^2 - X_1X_2)^{p-2j+\varepsilon-1} \\
&\quad X_0^{p-2j+\varepsilon-1}(X_0^2 - X_1X_2)^{2j-\varepsilon+1}(X_1X_2)^{p-2j+\varepsilon-1}(X_1^{p-1} - X_2^{p-1})^{p-2j+\varepsilon-1} \\
&\hspace{22em} (\mathrm{mod}\ f)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{j}(-1)^m\binom{2j-\varepsilon+1}{m}X_0^{p-2j+2m+\varepsilon-1}(X_1X_2)^{p-m}(X_1^{p-1} - X_2^{p-1})^{p-2j+\varepsilon-1} \\
(1)\qquad &\quad + \sum_{m=1}^{j-\varepsilon+1}(-1)^{j+m}\binom{2j-\varepsilon+1}{j+\mathrm{m}}X_0^{p+2m+\varepsilon-1}(X_1X_2)^{p-m-j}(X_1^{p-1} - X_2^{p-1})^{p-2j+\varepsilon-1} \\
&\equiv \sum_{m=0}^{j}(-1)^m\binom{2j-\varepsilon+1}{m}X_0^{p-2j+2m+\varepsilon-1}(X_1X_2)^{p-m}(X_1^{p-1} - X_2^{p-1})^{p-2j+\varepsilon-1} \\
&\quad + \sum_{m=1}^{j-\varepsilon+1}\left(\sum_{n=m}^{j-\varepsilon+1}(-1)^{j+n}\binom{2j-\varepsilon+1}{j+n}\right)X_0^{2m+\varepsilon-2}(X_1X_2)^{p-m-j+1} \\
&\hspace{18em} \times (X_1^{p-1} - X_2^{p-1})^{p-2j+\varepsilon} \\
&\quad + \left(\sum_{n=1}^{j-\varepsilon+1}(-1)^{j+n}\binom{2j-\varepsilon+1}{j+n}\right)X_0^{p+\varepsilon-1}(X_1X_2)^{p-j}(X_1^{p} - X_2^{p-1})^{p-2j+\varepsilon-1} \\
&\hspace{22em} (\mathrm{mod}\ f)\ .
\end{aligned}$$

In the sequel, let $p > i$ and $0 \leqq j-1 \leqq j(\varepsilon i)$. Then we have

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$$
\begin{aligned}
&\varphi\left(\frac{X_0^{p-2j+2m+\varepsilon-1}(X_1X_2)^{p-m}(X_1^{p-1}-X_2^{p-1})^{p-2j+\varepsilon-1}}{(X_1^{i-j+1}X_2^{p-i-j+\varepsilon})^p}\right)\\
&\qquad=(-1)^{i-j}\binom{p-2j+\varepsilon-1}{i-j}\varphi\left(\frac{X_0^{p-2j+2m+\varepsilon-1}}{X_1^{i-j+m}X_2^{p-i-j+m+\varepsilon-1}}\right)\\
&\qquad=(-1)^{i-j}\binom{p-2j+\varepsilon-1}{i-j}\alpha_{ij-m}\,.
\end{aligned}
\tag{2}
$$

$$
\varphi\left(\frac{X^{2m+\varepsilon-2}(X_1X_2)^{p-m-j+1}(X_1^{p-1}-X_2^{p-1})^{p-2j+\varepsilon}}{(X_0^{i-j+1}X_2^{p-i-j+\varepsilon})}\right)=0\,.
\tag{3}
$$

By virtue of formulas (1), (2) and (3), we have

$$
\begin{aligned}
&F^*(1,\mathcal{O}_X)(\alpha_{\varepsilon ij-1})-F^*(1,\mathcal{O}_X)(\alpha_{\varepsilon ij})\\
&\quad=\sum_{m=0}^{j-1}(-1)^m\binom{2j-\varepsilon+1}{m}(-1)^{i-j}\binom{p-2j+\varepsilon-1}{i-j}\alpha_{\varepsilon ij-m}\\
&\quad\quad+a_{\varepsilon ij-1}(-1)^{i-j}\binom{p-2j+\varepsilon-1}{i-j}\alpha_{\varepsilon i0}\,,
\end{aligned}
$$

where $a_{\varepsilon ij-1}=\sum_{m=0}^{j-\varepsilon+1}(-1)^{j+m}\binom{2j-\varepsilon+1}{j+m}$

Since $j(\varepsilon i)+1\leqq(1/2)(p-1)$ and $\alpha_{\varepsilon ij(\varepsilon i)+1}=0$, formula (4) shows that (i) is true.

(ii) Since $a_{\varepsilon ij-1}+\sum_{m-0}^{j-1}(-1)^m\binom{2j-\varepsilon+1}{m}=0$, it is easy to verify that $F^*(1,\mathcal{O}_X)\,|\,V_i$ is injective

(iii) It is obvious.

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