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MAXIMAL FUNCTIONS AND TRANSFERENCE FOR GROUPS OF OPERATORS

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Abstract Let Δ be the Laplace operator on $\mathbb{R}^d$ and $1 < \delta < 2$. Using transference methods we show that, for $\max\{q, q/(q-1)\} < 4d/(2d+1-\delta)$, the maximal function $\sup_{t>0} |e^{it\Delta} f|$ for the Schrödinger group is in L^q, for $f \in L^q$ with $\Delta^{\delta/2} f \in L^q$. We obtain a similar result for the Airy group $\exp it\Delta^{3/2}$. An abstract version of these results is obtained for bounded C_0-groups e^{itL} on subspaces of L^p spaces. Certain results extend to maximal functions defined for functions with values in UMD Banach spaces.

Keywords: transference; operator groups; Schrödinger equation

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1. Introduction

Let Ω be a complete and separable metric space, and μ a σ-finite Radon measure on Ω. Let E be a closed linear subspace of the Lebesgue space $L^p(\Omega; \mu)$, where $1 < p < \infty$. Suppose that L is a closed and densely defined linear operator in E; for convenience we will take its null space to be zero. We are concerned with the abstract Cauchy problem

$$iw_t = Lw, \qquad w(x, 0) = f(x), \tag{1.1}$$

which has a unique solution for initial data $f \in E$, in the sense of Hille and Phillips [**13**, p. 622], whenever $(-iL)$ is the generator of a C_0-semigroup of bounded linear operators on E. It is of interest to determine when the solution satisfies $w(x,t) \to f(x)$ μ-almost everywhere as $t \to 0$. This involves imposing extra conditions on f in order to control the maximal function $\sup_{t>0} |w(x,t)|$, as in the following theorem.

Theorem 1.1. *Let* $\Delta = -\sum_{j=1}^d \partial_{x_j}^2$ *be the Laplace operator in* $L^p(\mathbb{R}^d)$, *and let* $2 < \alpha < 3$ *and* $\max\{p, p/(p-1)\} < 12d/(6d+3-2\alpha)$.

(i) *Then there is a uniformly bounded family of linear operators on* $L^p(\mathbb{R}^d)$

$$m(t^{1/3}\sqrt{\Delta}) = \frac{e^{it(\sqrt{\Delta})^3} - I}{(t^{1/3}\sqrt{\Delta})^\alpha} \qquad (t > 0), \tag{1.2}$$

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for which the maximal function

$$f \mapsto \sup_{t>0} |m(t^{1/3}\sqrt{\Delta})f| \qquad (f \in L^p) \tag{1.3}$$

defines a strongly bounded sublinear operator $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$.

(ii) *Further, the solution* w *of (1.1) with* $L = \Delta^{3/2}$ *satisfies* $w(x,t) \to f(x)$ *almost everywhere as* $t \to 0+$, *when* $f \in L^p$ *is in the domain of* $\Delta^{\alpha/2}$.

The method of proof is based upon a device of Cowling [**10**, Theorem 6]. For suitable m, we can take the Fourier transform of $h(x) = m(\mathrm{e}^x)$ and write

$$m(tL)f = \int_{-\infty}^{\infty} \hat{h}(\xi) t^{\mathrm{i}\xi} L^{\mathrm{i}\xi} f \, \mathrm{d}\xi / 2\pi \qquad (f \in E,\ t > 0) \tag{1.4}$$

for the *multiplier operator* $m(tL)$. We will usually take $m(tL)$ to be defined by this formula, whenever the right-hand side is a convergent Bochner–Lebesgue integral. The operators to which we apply our results have a rich functional calculus which is essentially unique, so this mode of definition may be adopted without ambiguity.

In § 3 I prove Theorem 1.1, and the result on the Schrödinger group stated in the abstract, by showing that the right-hand side of (1.4) is suitably convergent. The key step, which suggests the subsequent generalizations, is to use the Marcinkiewicz multiplier theorem to control the group $L^{\mathrm{i}\xi}$ of imaginary powers of $L = \sqrt{\Delta}$. In cases of interest involving L^p ($1 < p < \infty$, $p \neq 2$), the operator group $L^{\mathrm{i}\xi}$ is locally but not uniformly bounded.

In § 4 we extend the method for application to the generators ($\mathrm{i}L$) of bounded C_0-groups $\mathrm{e}^{\mathrm{i}tL}$ of operators on subspaces E of $L^p(\Omega; \mu)$. In order to achieve bounds on (1.4), we use the functional calculus for spectral integration, as developed in [**1–3**]. In § 2 we describe the class of q-Marcinkiewicz multipliers of [**2**,**9**,**11**,**20**] which include the classical Marcinkiewicz multipliers. We then transfer bounds on the mulipliers into bounds on the operators $m(L)$.

In § 5 we use similar techniques to deal with almost everywhere convergence of solutions of an abstract wave equation. See [**8**] for an introduction to cosine families and hyperbolic equations. The technique works for functions with values in a suitable Banach space, precisely a UMD Banach space. Throughout the paper we use transference techniques for C_0-groups of operators, as in [**3**,**4**], rather than analytic semigroups as in [**10**, Theorem 1].

Notation. The Bochner–Lebesgue space of strongly measurable functions $f : \mathbb{R} \to E$ with $\|f(t)\|_E^p$ integrable is denoted $L^p(\mathbb{R}; E)$ [**13**, p. 79]. When $W : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is a bounded linear operator, it has a natural extension $W \otimes I : L^p(\mathbb{R}) \otimes E \to L^p(\mathbb{R}) \otimes E$ which, under favourable circumstances, extends to define a bounded linear operator on $L^p(\mathbb{R}; E)$; we then write W for this extension. We denote by (ε_n) the Rademacher random variables, and by $\mathbb{E}_\varepsilon$ the expectation with respect to their usual probability measure.

2. q-variation and multipliers

Definition. Let $\Delta_k = [2^k, 2^{k+1})$ and $\tilde{\Delta}_k = (-2^{k+1}, -2^k]$ for $k \in \mathbb{Z}$ be the intervals defining the standard dyadic decomposition of $\mathbb{R}$. For any bounded interval I, $1 \leqslant q < \infty$ and complex-valued function H, the *q-variation* of H over I is defined to be the supremum over all finite partitions:

$$\operatorname{var}_q(H; I) = \sup\left\{ \left(\sum_{j=1}^{N-1} |H(t_{j+1}) - H(t_j)|^q \right)^{1/q} \mid t_j \in I;\ t_1 < t_2 < \cdots < t_N \right\}. \tag{2.1}$$

When $q = 1$ this gives the usual notion of variation, as in the classical Marcinkiewicz multiplier theorem [**16**, Theorem 6]. The Marcinkiewicz *q-multipliers* of $\mathbb{R}$ are the functions H for which the norm

$$\|H\|_{\mathbb{M}^q(\mathbb{R})} = \sup_{t \in \mathbb{R}} |H(t)| + \sup_{k \in \mathbb{Z}} \operatorname{var}_q(H; \Delta_k) + \sup_{k \in \mathbb{Z}} \operatorname{var}_q(H; \tilde{\Delta}_k) \tag{2.2}$$

is finite. The space $\mathbb{M}^q(\mathbb{R})$ thus formed is a Banach algebra under pointwise multiplication of functions. In [**9**] it is shown that *q-multipliers* do indeed give bounded Fourier multipliers on suitable L^p spaces. Further, $\mathbb{M}^1(\mathbb{R})$ includes all functions of bounded variation on $\mathbb{R}$.

Let L be a closed and densely defined operator in a Banach space E with zero null space, and suppose that the imaginary powers L^{iu} ($u \in \mathbb{R}$) form a C_0-group of operators on E. (In the cases of interest, the powers may be defined by functional calculus, see (4.2) below.) As in [**4**], we define the *modular function* of this group to be

$$\tau_E(u) = \sup\{\|L^{is}\|_{E \to E} \mid |s| \leqslant |u|\} \qquad (u \in \mathbb{R}). \tag{2.3}$$

By an application of the uniform boundedness theorem, given in [**13**, p. 306], the function $\tau_E(u)$ is at most of exponential growth in $|u|$.

Proposition 2.1. *Let $m(\lambda)$ ($\lambda > 0$) be a function with $h(x) = m(e^x)$ integrable, and let $\hat{h}(\xi)$ be the Fourier transform. Let E be a closed linear subspace of $L^p(\Omega; \mu)$ for some $1 < p < \infty$, and let q with $1 \leqslant q < \infty$ have $|1/2 - 1/p| < 1/q$.*

(i) *Then $m(L)$, defined as in (1.4), has operator norm*

$$\|m(L)\|_{E \to E} \leqslant C_p \sum_{k=-\infty}^{\infty} \tau_E(2^k)^3 (\|h * \psi_k\|_{\mathbb{M}^q(\mathbb{R})} + \|h * \bar{\psi}_k\|_{\mathbb{M}^q(\mathbb{R})}), \tag{2.4}$$

where

$$\psi_k(x) = \frac{\sin 2^{k-1}x \exp(i3 \cdot 2^{k-1}x)}{\pi x} \qquad (x \in \mathbb{R});$$

so that $m(L)$ is a bounded linear operator, whenever this series converges.

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(ii) *Suppose further that $L^{i\xi}$ is a bounded C_0-group of operators, so that the modular function is bounded by τ. Then $\|m(L)\|_{E\to E} \leqslant C(p,q)\tau^2\|h\|_{M^q(\mathbb{R})}$.*

Proof. (i) Let us suppose that the series in (2.4) converges. By the Fourier inversion theorem we can write

$$m(\lambda) = \sum_{k=-\infty}^{\infty} \int_{\Delta_k} + \int_{\tilde{\Delta}_k} \lambda^{i\xi}\hat{h}(\xi)\,d\xi/2\pi \qquad (\lambda > 0), \tag{2.5}$$

so that, at least formally, the multiplier operator is

$$m(L)f = \sum_{k=-\infty}^{\infty} \int_{\Delta_k} + \int_{\tilde{\Delta}_k} \hat{h}(\xi)L^{i\xi}f\,d\xi/2\pi \qquad (f \in E). \tag{2.6}$$

Our task is to show that the right-hand side is norm convergent; this we do by considering a typical summand. The operator

$$H_k : f \mapsto \int_{\Delta_k} \hat{h}(\xi)L^{i\xi}f\,d\xi/2\pi \qquad (f \in E) \tag{2.7}$$

is certainly well defined, since $\hat{h}$ is integrable over Δ_k and $\xi \to L^{i\xi}f$ is norm continuous. By the transference theorem for C_0-groups [**4**, Theorem 2.1], the operator norm of H_k is $\leqslant 2^{1/p}\tau_E^3(2^{k+1})\|\Lambda(\hat{h}_k)\|$, where $\|\Lambda(\hat{h}_k)\|$ is the operator norm of the convolution operator

$$\Lambda(\hat{h}_k) : g \mapsto \int_{\Delta_k} \hat{h}(\nu)g(\nu+\xi)\,d\nu \qquad (g \in L^p(\mathbb{R};E)) \tag{2.8}$$

on the Bochner–Lebesgue space $L^p(\mathbb{R};E)$. By Fubini's Theorem, this is just the norm of $\Lambda(\hat{h}_k)$ acting on the scalar-valued function space $L^p(\mathbb{R})$; and by the multiplier theorem 1 of [**9**], this is $\leqslant C_{p,q}\|h * \psi_k\|_{M^q(\mathbb{R})}$. Summing up these estimates over k we obtain the stated result.

(ii) This is a special case of [**2**, Theorem 1.2]. □

Proposition 2.2. *Let E and L be as in Proposition 2.1(i). Suppose further that $\tau_E(\xi)\hat{h}(\xi)$ is integrable. Then the Bochner integrals*

$$m(tL)f = \int_{-\infty}^{\infty} \hat{h}(\xi)t^{i\xi}L^{i\xi}f\,d\xi/2\pi \qquad (f \in E, t > 0) \tag{2.9}$$

define a uniformly bounded family of linear operators $m(tL)$ on E for which the maximal function

$$f \mapsto \sup_{t>0} |m(tL)f| \qquad (f \in E) \tag{2.10}$$

defines a strongly bounded sublinear operator $E \to L^p(\Omega;\mu)$.

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Proof. Since $\xi \mapsto L^{i\xi}f \in E$ is norm-continuous and each image is bounded by $\|L^{i\xi}\|_{E\to E}\|f\|_E$, the expression (2.9) is a well-defined Bochner–Lebesgue integral. By [**13**, Theorem 3.8.2] the integral defines a bounded linear operator $m(tL)$, and the following computation will additionally achieve a uniform bound on the operator norms.

To estimate the maximal function we take the supremum norm in the t-variable, and then the $L^p(\Omega;\mu)$ norm, to obtain

$$\|\sup_{t>0}|m(tL)f|\|_{L^p(\Omega;\mu)} \leqslant \int_{-\infty}^{\infty} |\hat{h}(\xi)|\|L^{i\xi}f\|_E \, d\xi/2\pi \qquad (f \in E); \tag{2.11}$$

and by considering the triangle inequality for integrals we see this is

$$\leqslant \int_{-\infty}^{\infty} |\hat{h}(\xi)|\tau_E(\xi)\, d\xi/2\pi \times \|f\|_E \qquad (f \in E). \tag{2.12}$$

By the assumptions on $\hat{h}$ and the modular function, the latest integral is finite. □

3. Maximal functions for Schrödinger and KdV groups

Let Δ be the operator $-\sum_{j=1}^{d} \partial_{x_j}^2$ in $L^2(\mathbb{R}^d, dx)$. This defines an essentially self-adjoint and positive operator in $C_c^{\infty}(\mathbb{R}^d)$; and we can use the spectral theorem to form functions of Δ, in particular, we form the square root $\sqrt{\Delta} \geqslant 0$. One can extend suitably bounded operators, defined initially on $L^2 \cap L^p$, to L^p for $1 < p < \infty$ [**8**, p. 412].

Lemma 3.1. *Let $0 < \beta < d$ and $\max\{p, p/(p-1)\} < 2d/(d-\beta)$. Then there is $C(p,\beta) < \infty$ such that the modular function τ_p of the group of imaginary powers Δ^{iu} on $L^p(\mathbb{R}^d)$ satisfies*

$$\|\Delta^{iu}\|_{L^p\to L^p} \leqslant \tau_p(u) \leqslant C(p,\beta)(1+|u|^{\beta}) \qquad (u \in \mathbb{R}). \tag{3.1}$$

Proof. We may suppose that $2 < p < \infty$, for the other cases follow by duality. We set $\theta = 1 - \beta/d$, so that there exists q with $p < q < \infty$ for which $1/p = \theta/2 + (1-\theta)/q$; this exploits the assumptions on β and p.

The operator Δ^{iu} corresponds to the Fourier multiplier $(\sum_{j=1}^{d} \xi_j^2)^{iu/2}$. By the Marcinkiewicz multiplier theorem [**16**, Theorem 6′], Δ^{iu} defines a C_0-group of operators on $L^q(\mathbb{R}^d)$ with

$$\|\Delta^{iu}\|_{L^q(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \leqslant C(q,d)(1+|u|)^d \qquad (u \in \mathbb{R}); \tag{3.2}$$

and by the Plancherel formula

$$\|\Delta^{iu}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = 1 \qquad (u \in \mathbb{R}). \tag{3.3}$$

We interpolate between the estimates (3.3) and (3.2) using the Riesz–Thorin Theorem [**21**, p. 95] to achieve the desired bound (3.1). □

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In the course of the proofs of Theorems 3.3 and 1.1, we require to estimate various oscillatory integrals. The basic technique may be summarized in the following variant of van der Corput's Lemma from [**18**, p. 334].

Lemma 3.2. *Suppose that the phase function ϕ is real-valued and smooth on (a,b) and that $|\phi^{(k)}(\lambda)| \geqslant 1$ for all $\lambda \in (a,b)$, where either*

(i) *$k \geqslant 2$; or*

(ii) *$k = 1$, and $\phi'(\lambda)$ is monotonic.*

Then there is c_k, independent of ϕ and s, with

$$\left|\int_a^b e^{is\phi(\lambda)}\psi(\lambda)\,d\lambda\right| \leqslant c_k s^{-1/k}\left(|\psi(b)| + \int_a^b |\psi'(\lambda)|\,d\lambda\right) \qquad (s>0).$$

The following result was stated in the abstract. The group of operators $e^{it\Delta}$ involved is used to solve the Cauchy problem for the Schrödinger equation in free Euclidean space.

Theorem 3.3. *For $1<\delta<2$ and $4d/(2d+\delta-1) < q < 4d/(2d+1-\delta)$, the maximal function*

$$f \mapsto \sup_{t>0}\left|\frac{e^{it\Delta}-I}{(t\Delta)^{\delta/2}}f\right| \qquad (f \in L^2 \cap L^q) \tag{3.4}$$

defines a strongly bounded sublinear operator $L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$.

Proof. We begin by making a Fourier transform representation of $h(x) = m(e^x)$, where m is the multiplier

$$m(\lambda) = \frac{e^{i\lambda^2}-1}{\lambda^\delta} \qquad (\lambda > 0). \tag{3.5}$$

The maximal function involves a scaling of m. We calculate

$$\hat{h}(\xi) = \int_{-\infty}^{\infty} h(x)e^{-ix\xi}\,dx = \int_0^\infty \frac{e^{i\lambda^2}-1}{\lambda^{1+\delta}}\lambda^{-i\xi}\,d\xi \qquad (\xi \in \mathbb{R}), \tag{3.6}$$

where we have introduced the variable $\lambda = e^x$; the latest integral is absolutely convergent when $0<\delta<2$. Consequently, $\hat{h}(\xi)$ is bounded near to $\xi = 0$; by considering large ξ, we shall show $\hat{h}(\xi)\tau_p(\xi) \in L^1_\xi$. Integration by parts gives us, since the integrated term vanishes, the expression

$$\hat{h}(\xi) = \frac{2i}{\delta + i\xi}\int_0^\infty \lambda^{1-\delta-i\xi}e^{i\lambda^2}\,d\lambda, \tag{3.7}$$

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which invites the method of stationary phase. The phase function $\phi(\lambda) = \lambda^2 - \xi \log \lambda$ has derivative $\phi'(\lambda) = 2\lambda - \xi/\lambda$, which vanishes only when $\xi > 0$ and $\lambda = \sqrt{\xi/2}$; there $\phi''(\sqrt{\xi/2}) = 4$. Consequently, we have an asymptotic estimate [**12**, p. 51]

$$|\hat{h}(\xi)| \sim \left| \frac{\sqrt{\pi}(\mathrm{i}-1)}{\delta + \mathrm{i}\xi} \mathrm{e}^{\mathrm{i}\xi/2} (\xi/2)^{(1-\delta-\mathrm{i}\xi)/2} \right| \qquad (\xi \to \infty), \tag{3.8}$$

since Fresnel's integral takes the value

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\alpha\nu^2}\, \mathrm{d}\nu = \sqrt{\frac{\pi}{2\alpha}}(\mathrm{i}+1) \qquad (\alpha > 0). \tag{3.9}$$

When $\xi < 0$ there is no point of stationary phase; so we split the integral (3.7) and write

$$\begin{aligned} &\int_1^{\infty} \lambda^{1-\delta} \mathrm{e}^{\mathrm{i}\phi(\lambda)}\, \mathrm{d}\lambda \\ &\quad = \left[\frac{\lambda^{1-\delta}\mathrm{e}^{\mathrm{i}\phi(\lambda)}}{\mathrm{i}\phi'(\lambda)} \right]_1^{\infty} - \int_1^{\infty} \frac{(2-\delta)\lambda^{1-\delta}}{\mathrm{i}(2\lambda^2 - \xi)} \mathrm{e}^{\mathrm{i}\phi(\lambda)}\, \mathrm{d}\lambda + \int_1^{\infty} \frac{4\lambda^{3-\delta}}{\mathrm{i}(2\lambda^2-\xi)^2} \mathrm{e}^{\mathrm{i}\phi(\lambda)}\, \mathrm{d}\lambda. \end{aligned} \tag{3.10}$$

The integrated term is $O(1/|\xi|)$ as $\xi \to -\infty$; whereas, for $1 < \delta < 2$, the other integrals in (3.10) contribute at most

$$\int_1^{\infty} \frac{1}{(2\lambda^2 - \xi)^{1/2}} \frac{\lambda}{(2\lambda^2 - \xi)^{1/2}} \frac{\mathrm{d}\lambda}{\lambda^{\delta}} \leqslant \frac{C}{|\xi|^{1/2}} \int_1^{\infty} \frac{\mathrm{d}\lambda}{\lambda^{\delta}} \leqslant \frac{C(\delta)}{|\xi|^{1/2}}, \tag{3.11}$$

and

$$\int_1^{\infty} \frac{1}{(2\lambda^2 - \xi)^{1/2}} \frac{4\lambda^3}{(2\lambda^2 - \xi)^{3/2}} \frac{\mathrm{d}\lambda}{\lambda^{\delta}} \leqslant \frac{C}{|\xi|^{1/2}} \int_1^{\infty} \frac{\mathrm{d}\lambda}{\lambda^{\delta}} \leqslant \frac{C(\delta)}{|\xi|^{1/2}}, \tag{3.12}$$

respectively. By repeated integration by parts, one can show that the contribution to (3.6) arising from $\int_0^1$ decays fast enough as $\xi \to -\infty$; see (3.15) below for a more delicate case. (Alternatively, one can transform (3.7) into a Gamma function integral and use Stirling's formula [**4**, p. 248].)

By Lemma 3.1, $\tau_p(u) = O(|u|^{\beta})$ as $|u| \to \infty$, where $\beta < (\delta - 1)/2$; and hence on combining the preceding estimates we see that $\tau_p(\xi)\hat{h}(\xi)$ is integrable. The result follows from Proposition 2.2. □

We can establish the result stated in the introduction in a similar fashion. This involves a variant of the Airy group $\mathrm{e}^{-t\partial_x^3}$ which solves the linearized KdV equation ($u_t + u_{xxx} = 0$) in one space dimension. We have $-\mathrm{i}\partial/\partial x = H\sqrt{\Delta}$, where the Hilbert transform H is defined as in (5.7) below, and all the operators commute. The Riesz projection $R_+ : L^p \to H^p$ onto the Hardy space $\{f \in L^p \mid \hat{f}(\xi) = 0; (\xi < 0)\}$ is bounded for $1 < p < \infty$; so to obtain almost sure convergence for solutions of (KdV), it suffices to apply Theorem 1.1 to R_+u and $(I - R_+)u$.

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Proof of Theorem 1.1. (i) We introduce

$$m(\lambda) = \frac{e^{i\lambda^3} - 1}{\lambda^\alpha} = g(\lambda)\lambda^{3-\alpha} \qquad (\lambda > 0), \tag{3.13}$$

where $g(\lambda)$ extends to define an entire function; and we proceed to calculate the Fourier transform of $h(x) = m(e^x)$. Let $\varphi_j(x)$ $(j = 1, 2)$ be smooth bump functions, supported on $[-1, 1]$ and $[1/2, \infty)$ respectively, with $1 = \varphi_1(x) + \varphi_2(x)$ for $x \geqslant 0$. Then we have absolutely convergent integrals

$$\hat{h}(\xi) = \int_0^1 \varphi_1(\lambda)g(\lambda)\lambda^{2-\alpha-i\xi}\,d\lambda + \int_{1/2}^\infty \varphi_2(\lambda)(e^{i\lambda^3} - 1)\lambda^{-\alpha-1-i\xi}\,d\lambda. \tag{3.14}$$

Integrating by parts, we show that, when $\alpha < 3$, the first integral is

$$\frac{-1}{(3-\alpha-i\xi)(4-\alpha-i\xi)(5-\alpha-i\xi)} \int_0^1 (\varphi_1 g)'''(\lambda)\lambda^{5-\alpha-i\xi}\,d\lambda; \tag{3.15}$$

this expression is clearly $O(1/|\xi|^3)$ as $|\xi| \to \infty$.

In this case the crucial integral to estimate is the second integral in (3.14), which we integrate by parts and write as

$$\frac{3i}{\alpha + i\xi} \int_{1/2}^\infty \lambda^{-\alpha-i\xi+2} e^{i\lambda^3} \varphi_2(\lambda)\,d\lambda + \frac{1}{\alpha + i\xi} \int_{1/2}^1 (e^{i\lambda^3} - 1)\varphi_2'(\lambda)\lambda^{-i\xi-\alpha}\,d\lambda. \tag{3.16}$$

The latest integral may be estimated after repeated integration by parts. In the first integral in (3.16), the phase function is $\phi(\lambda) = \lambda^3 - \xi \log \lambda$; so there is a point of stationary phase in the range of integration only when $\xi > (3/2)^3$, and it occurs at $\lambda = (\xi/3)^{1/3}$. At this point, $\phi'' = 3^{5/3}\xi^{1/3}$; consequently, we have an asymptotic expression for the modulus of first integral of (3.16) [**12**, p. 51] as $\xi \to \infty$, namely

$$\left| \sqrt{\pi}\frac{i-1}{\alpha + i\xi} e^{i(\xi/3)} \left(\frac{\xi}{3}\right)^{(2-\alpha-i\xi)/3-1/6} \right|. \tag{3.17}$$

Let p and α be as in the statement of the result. By Lemma 3.1, $\tau_p(u) = O(|u|^\beta)$ as $u \to \infty$, where $\beta < 1/6 + (\alpha - 2)/3$. We have chosen the constants such that the estimates (3.1) and (3.17) are good enough to ensure $\int |\hat{h}(\xi)| \|\Delta^{i\xi}\|\,d\xi < \infty$, as we require to show that the maximal operator (1.3) is bounded.

(ii) The last statement of the Theorem 1.1, involving almost sure convergence, follows from part (i) by a standard argument which will be given in § 5 below. □

4. Abstract maximal theorem for subspaces of L^p

In this section and the next we suppose that E is a closed linear subspace of the Lebesgue space $L^p(\Omega; \mu)$, where $1 < p < \infty$. Suppose further that iA is the generator of a bounded

C_0-group of operators on E, and the null space of A is zero. Before stating our maximal theorem, we recall the functional calculus of spectral integration, as developed in [**1–3**].

We write $\tau = \sup\{\|e^{itA}\|_{E\to E} \mid t \in \mathbb{R}\} < \infty$, and introduce a Stone-type spectral family for the group e^{itA} [**3**, Theorem 5.1]. There is a unique family of spectral projections $F(\lambda) \in B(E)$ with

(i) $\|F(\lambda)\|_{E\to E} \leqslant C_p\tau^2$;

(ii) $F(\lambda)F(\mu) = F(\mu)F(\lambda) = F(\lambda)$ when $\lambda \leqslant \mu$;

(iii) $\lambda \mapsto F(\lambda)$ is right-continuous, with left-hand limits, in the strong operator topology;

(iv) $F(\lambda) \to I$ as $\lambda \to \infty$, and $F(\lambda) \to 0$ as $\lambda \to -\infty$, strongly; and

(v) there is a well-defined integral with respect to the family $F(\lambda)$ and a functional calculus map, giving a bounded homomorphism

$$m \mapsto m(A) = \int_{-\infty}^{\infty} m(\lambda)F(\mathrm{d}\lambda), \tag{4.1}$$

from the Banach algebra of functions of bounded variation on $\mathbb{R}$ into $B(E)$. It is shown in [**2**] that the functional calculus map is bounded $\mathbb{M}^q(\mathbb{R}) \to B(E)$, for $|1/2 - 1/p| < 1/q$ and $1 \leqslant q < \infty$.

We can also use the functional calculus map to introduce closed (unbounded) and densely defined operators such as

$$|A|^z = \lim_{\substack{\varepsilon\to 0+\\ R\to\infty}} \int_{[-R,-\varepsilon]} + \int_{[\varepsilon,R]} |\lambda|^z F(\mathrm{d}\lambda) \qquad (z \in \mathbb{C}); \tag{4.2}$$

in this case we exploit the fact that the spectral projection for the null space of A is $F(0) - F(0-) = 0$. The function $\lambda \mapsto \lambda^{iu}$ has 1-variation $\leqslant 2|u|$ on each dyadic interval $[2^j, 2^{j+1}]$ $(j \in \mathbb{Z})$.

Lemma 4.1. *The imaginary powers $|A|^{iu}$ form a C_0-group of operators on E with modular function of at most linear growth; that is,*

$$\||A|^{iu}\|_{E\to E} \leqslant C(p,\tau)(1 + |u|) \qquad (u \in \mathbb{R}). \tag{4.3}$$

Proof. This follows from the Marcinkiewicz multiplier theorem of spectral integration (see [**1**] and [**2**, Theorem 1.2]), given our estimate on the $\mathbb{M}^1(\mathbb{R})$ norm of λ^{iu}. □

Theorem 4.2. *Let $3/2 < \alpha < 3$. Then the family of operators*

$$m(t^{1/3}|A|) = \frac{e^{-t|A|^3} - I}{(t^{1/3}|A|)^{\alpha}} \qquad (t > 0) \tag{4.4}$$

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on E is uniformly bounded, and the maximal function

$$f \mapsto \sup_{t>0} |m(t^{1/3}|A|)f| \qquad (f \in E) \tag{4.5}$$

defines a strongly bounded sublinear operator $E \to L^p(\Omega;\mu)$.

Proof. We now introduce

$$m(\lambda) = \frac{\mathrm{e}^{-\lambda^3}-1}{\lambda^\alpha} = g(\lambda)\lambda^{3-\alpha} \qquad (\lambda > 0), \tag{4.6}$$

where $g(\lambda)$ extends to define an entire function; and we proceed to calculate the Fourier transform of $h(x) = m(\mathrm{e}^x)$. Let $\varphi_j(x)$ $(j = 1,2)$ be smooth bump functions, supported on $[-1,1]$ and $[1/2,\infty)$ respectively, with $1 = \varphi_1(x) + \varphi_2(x)$ for $x \geqslant 0$. Then we have absolutely convergent integrals

$$\hat{h}(\xi) = \int_0^1 \varphi_1(\lambda)g(\lambda)\lambda^{2-\alpha-\mathrm{i}\xi}\,\mathrm{d}\lambda + \int_{1/2}^\infty \varphi_2(\lambda)(\mathrm{e}^{-\lambda^3}-1)\lambda^{-\alpha-1-\mathrm{i}\xi}\,\mathrm{d}\lambda. \tag{4.7}$$

Integrating by parts, we show that, when $\alpha < 3$, the first integral in (4.7) is

$$\frac{-1}{(3-\alpha-\mathrm{i}\xi)(4-\alpha-\mathrm{i}\xi)(5-\alpha-\mathrm{i}\xi)} \int_0^1 (\varphi_1 g)'''(\lambda)\lambda^{5-\alpha-\mathrm{i}\xi}\,\mathrm{d}\lambda; \tag{4.8}$$

this expression is clearly $O(1/|\xi|^3)$ as $|\xi| \to \infty$.

We also integrate the second integral in (4.7) by parts; the most threatening term so arising is

$$\frac{-1}{(\alpha+\mathrm{i}\xi)(1-\alpha-\mathrm{i}\xi)(2-\alpha-\mathrm{i}\xi)} \int_{1/2}^\infty \varphi_2(\lambda)(-3\lambda^2)^3\mathrm{e}^{-\lambda^3}\lambda^{2-\alpha-\mathrm{i}\xi}\,\mathrm{d}\lambda, \tag{4.9}$$

which is also $O(1/|\xi|^3)$ as $|\xi| \to \infty$.

Hence we can write

$$\frac{\mathrm{e}^{-\lambda^3}-1}{\lambda^\alpha} = \int_{-\infty}^\infty \hat{h}(\xi)\lambda^{\mathrm{i}\xi}\,\mathrm{d}\xi/2\pi \qquad (\lambda > 0), \tag{4.10}$$

an absolutely convergent integral; from whence we can write

$$\frac{\mathrm{e}^{-tA^3}-I}{(t^{1/3}|A|)^\alpha}f = \int_{-\infty}^\infty \hat{h}(\xi)t^{\mathrm{i}\xi/3}|A|^{\mathrm{i}\xi}f\,\mathrm{d}\xi/2\pi \qquad (t > 0, f \in E). \tag{4.11}$$

We deduce the key estimate

$$\left\|\sup_{t>0}\left|\frac{\mathrm{e}^{-t|A|^3}-I}{(t^{1/3}|A|)^\alpha}f\right|\right\|_{L^p(\Omega;\mu)} \leqslant \int_{-\infty}^\infty |\hat{h}(\xi)|\||A|^{\mathrm{i}\xi}\|_{E\to E}\,\mathrm{d}\xi/2\pi \times \|f\|_E \qquad (f \in E), \tag{4.12}$$

where the latest integral converges on account of our estimates on (4.8), (4.9) and (4.3). □

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5. Almost sure convergence for a cosine family

In this section we continue with the formalism of § 4 to obtain an almost sure convergence theorem for solutions of an abstract second-order equation. The Cauchy problem

$$\left.\begin{aligned} w_{tt} &= -|A|^6 w(x,t), \\ w(x,0) = f(x), &\qquad w_t(x,0) = 0 \end{aligned}\right\} \tag{5.1}$$

is well posed for $f \in E$ in the domain of $|A|^9$. Indeed, the solution is given by $w(x,t) = (\cos t|A|^3)f(x)$, as defined by the functional calculus of § 4. Simple estimates on the derivative of $\cos t\lambda^3$ and properties of the functional calculus map imply that $t \mapsto w(.,t) \in E$ is twice continuously differentiable, with $w(x,t) \to f(x)$ in the norm of E as $t \to 0$; further, $w_t(x,t) \to 0$ in E norm as $t \to 0$. For a more general class of initial values, the function $t \mapsto \cos t|A|^3 f$ is norm-continuous.

Theorem 5.1. *Suppose that $9/2 < \delta < 6$ and that $f \in E$ is in the domain of $|A|^\delta$. Then a (weak) solution $w = \cos(t|A|^3)f$ to (5.1) exists, and satisfies*

$$w(x,t) \to f(x) \quad \mu\text{-almost everywhere as } t \to 0+\,. \tag{5.2}$$

Proof. Arguing as in the proof of Theorem 1.1, we can show that

$$m(t^{1/3}|A|) = \frac{\cos t|A|^3 - I}{(t^{1/3}|A|)^\delta} \qquad (t > 0) \tag{5.3}$$

defines a uniformly bounded family of linear operators on E, and that the maximal function

$$f \mapsto \sup_{t>0} |m(t^{1/3}|A|)f| \qquad (f \in E) \tag{5.4}$$

defines a strongly bounded sublinear operator $E \to L^p(\Omega;\mu)$. (The value of δ may be chosen larger than in Theorems 1.1, 3.3 and 4.2, since here we have a cosine family rather than a (semi)-group. We appear to need this enlarged exponent to cope with the growth of the modular function in (4.3).)

Suppose that f is in the domain of $|A|^\delta$ and take $\varepsilon > 0$. Then a solution $w(x,t) = \cos(t|A|^3)f$ of (5.1) exists; on account of (1.4) and (5.3), we may define $w(x,t) = m(t|A|)(t^{1/3}|A|)^\delta f + f$. This weak solution satisfies, for $0 < t < \varepsilon$,

$$\begin{aligned} \|\sup_{0<t<\varepsilon} |w(x,t) - f(x)|\|_{L^p_x(\Omega;\mu)} &= \|(\cos t|A|^3 - I)f\|_{L^p_x L^\infty_t} \\ &= \|m(t^{1/3}|A|)(t^{1/3}|A|)^\delta f\|_{L^p_x L^\infty_t} \\ &\leqslant \varepsilon^{\delta/3}\|m(t^{1/3}|A|)(|A|^\delta f)\|_{L^p_x L^\infty_t} \end{aligned} \tag{5.5}$$

by functional calculus. Now the properties of the maximal function ensure that this is

$$\leqslant C(p,\delta)\varepsilon^{\delta/3}\||A|^\delta f\|_E. \tag{5.6}$$

Hence $u(x,t) \to f(x)$ almost everywhere as $t \to 0$. □

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Our next result concerns functions $f : \mathbb{R} \to X$, where X is a complex Banach space with the unconditional martingale difference property.

Definition. A Banach space X is said to be a UMD space, provided there is a uniform unconditionality constant C_X such that, for all L^2-martingale difference sequences (d_n) with values in X,

$$\mathbb{E}\left\|\sum_{n=1}^{N} \pm d_n\right\|_X^2 \leqslant C_X \mathbb{E}\left\|\sum_{n=1}^{N} d_n\right\|_X^2$$

for all choices of signs $\pm$. By results of Bourgain and Burkholder, summarized in [**3**, Theorem 2.7], this is equivalent to requiring that the Hilbert transform operator

$$H : f \mapsto \mathrm{PV}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)\,\mathrm{d}t}{x-t} \qquad (f \in L^2(\mathbb{R}; X)) \tag{5.7}$$

be bounded from $L^2(\mathbb{R}; X)$ to itself.

Many of the reflexive spaces of functions which arise in classical analysis have the UMD property; Hilbert space is an obvious example, the Lebesgue spaces L^p $(1 < p < \infty)$ and the von Neumann–Schatten ideals c^p $(1 < p < \infty)$ are examples presented in [**3**]. It is helpful to note that $L^p(\mathbb{R}; X)$ is a UMD space whenever X is a UMD space and $1 < p < \infty$.

Theorem 5.2. *Let X be a UMD space, $9/2 < \delta < 6$, $1 < p < \infty$ and let Δ be the Laplace operator in $L^p(\mathbb{R}; X)$. Then there is a uniformly bounded family of linear operators on $L^p(\mathbb{R}; X)$*

$$m(t^{1/3}\sqrt{\Delta}) = \frac{\cos t\Delta^{3/2} - I}{(t^{1/3}\Delta^{1/2})^{\delta}} \qquad (t > 0), \tag{5.8}$$

for which the maximal function

$$f \mapsto \sup_{t>0} \|m(t^{1/3}\sqrt{\Delta})f\|_X \qquad (f \in L^p(\mathbb{R}; X)) \tag{5.9}$$

defines a strongly bounded sublinear operator $L^p(\mathbb{R}; X) \to L^p(\mathbb{R})$.

Proof. By the Marcinkiewicz multiplier theorem for vector-valued functions, Δ^{iu} defines a C_0-group on $L^p(\mathbb{R}; X)$ with

$$\|\Delta^{iu}\|_{L^p(\mathbb{R};X)\to L^p(\mathbb{R};X)} \leqslant C_p(X)(1 + |u|) \qquad (u \in \mathbb{R}). \tag{5.10}$$

Indeed, the functional calculus map (4.1) is bounded $\mathbb{M}^1(\mathbb{R}) \to B(L^p(\mathbb{R}; X))$ (see [**1**, Theorem 1.2], [**6**, Theorem 4] and [**16**, Theorem 6]). Now we can repeat the proof of Theorem 1.1 to achieve the desired result. □

6. Concluding remarks

Problem 6.1. It is reasonable to hope that Theorem 5.2 should extend to the Laplace operator in $L^p(\mathbb{R}^d; X)$ with $d > 1$, possibly for a restricted range of p. The known estimates on the modular function of Δ^{iu} are not good enough to achieve this by our method. Note that (3.3) is only known to be valid when X is isomorphic (linearly homeomorphic) to a Hilbert space. In [**17**, p. 57], Stein obtains an estimate on the modular function of L^{iu} whenever $(-L)$ is the generator of a symmetric diffusion semigroup. His estimate is independent of dimension, but it grows exponentially in $|u|$.

Remark 6.2. We mention some previously known results on the almost everywhere convergence of solutions of the Cauchy problem $iw_t = \Delta w$, $w(x,0) = f(x)$ for the Schrödinger equation in free Euclidean space. Carleson has shown that, for $f \in L^2(\mathbb{R})$ with $\Delta^{1/8} f \in L^2(\mathbb{R})$, the Cauchy problem is well posed, and $w(x,t) \to f(x)$ almost everywhere as $t \to 0$ [**7**]. The Sobolev exponent of $f \in H^{1/4}(\mathbb{R})$ is optimal when $d = 1$. In higher space dimensions the optimal value of the index does not appear to be known; but Bourgain has achieved an almost sure convergence result for compactly supported initial data f in $H^\rho(\mathbb{R}^2)$, for some $\rho < 1/2$ [**5**]. The proofs of these results involve delicate estimates on the kernels of the operator TT^* on H^ρ, where $Tf(x) = e^{it(x)\partial_x^2} f(x)$ is the maximal function. Standard results described in [**19**] show that almost everywhere convergence is equivalent to weak (p,q) bounds on a suitable maximal function. The TT^* method may be used to yield almost everywhere convergence results for KdV-type equations. Nevertheless, it does not seem appropriate for vector-valued functions taking values in Banach spaces, other than Hilbert space [**18**, pp. 278, 317].

Our final result gives a characterization of those Lebesgue spaces which are isomorphic to Hilbert space, and it suggests that there are limits to what can be achieved using transference arguments for bounded groups of operators. See also [**11**, 1.5].

Proposition 6.3. *Let E be an infinite-dimensional and separable complex Banach space isomorphic to $L^p(\mu)$. Then p equals 2 if and only if, for each $K \geqslant 1$, there is $C(E,K) < \infty$ such that*

$$\mathbb{E}_\varepsilon \Big\| \sum_j \varepsilon_j T^j f_j \Big\|_E \leqslant C(E,K) \mathbb{E}_\varepsilon \Big\| \sum_j \varepsilon_j f_j \Big\|_E, \tag{6.1}$$

for all finite sequences $(f_j) \subset E$, and all bounded discrete (semi)-groups of operators on (linear subspaces of) E with $\|T^j\|_{E \to E} \leqslant K$ for all j.

Proof. By Kahane's inequality [**15**, 9.2], when E is isomorphic to Hilbert space there are positive constants $c(E)$ and $C(E)$ with

$$c(E) \Big(\sum_j \|f_j\|_E^2 \Big)^{1/2} \leqslant \mathbb{E}_\varepsilon \Big\| \sum_j \varepsilon_j f_j \Big\|_E \leqslant C(E) \Big(\sum_j \|f_j\|_E^2 \Big)^{1/2} \qquad ((f_j) \subset E), \tag{6.2}$$

from whence (6.1) follows.

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Our proof of the converse establishes rather more than the statement. Kwapień has shown that (6.2) characterizes those Banach spaces which are isomorphic to Hilbert space [**14**, Theorem 1.1(c)]. We suppose, with a view to obtaining a contradiction, that the maximal type of E is $p = \sup\{\rho \mid E \text{ has type } \rho\} < 2$. Since $E \sim L^p$, this supremum is attained. Then by the Maurey–Pisier Theorem [**15**, 13.2, 13.16], ℓ^p is finitely representable in E. For each n, we take E_n to be a subspace of E, 2-isomorphic to $\ell^p(n)$ and K-complemented in E. Now we let $(T^j)_{j\in\mathbb{Z}}$ be the cyclic group of linear operators on E_n associated with $T^j : e_{[r]} \mapsto e_{[j+r]}$, where $[r]$ the equivalence class of r modulo n, and e_j the standard unit basis of $\ell^p(n)$. Since E_n is K-complemented in E, $(T^j)_{j\in\mathbb{Z}}$ extends to a bounded C_0-group on E with $\|T^j\|_{E\to E} \leqslant 6K$. We take $f_j = e_1$ $(1 \leqslant j \leqslant n)$; then one calculates

$$\Big\|\sum_j \varepsilon_j T^{-j} f_j\Big\|_{\ell^p} = n^{1/p},$$

whereas

$$\mathbb{E}_\varepsilon \Big\|\sum_j \varepsilon_j e_1\Big\|_{\ell^p} \leqslant n^{1/2}. \tag{6.3}$$

But (6.1) cannot then be valid for all $n \geqslant 1$ with $p < 2$.

Further, if the minimal cotype of E is $q = \inf\{\eta \mid E \text{ has cotype } \eta\} > 2$, then copies of $\ell^q(n)$ embed uniformly in E. We use T^j as above, but this time we take $f_j = e_j$ for $1 \leqslant j \leqslant n$ and use Khintchine's inequality to show [**15**, 5.5]

$$\mathbb{E}_\varepsilon \Big\|\sum_{j=1}^n \varepsilon_j T^{-j} f_j\Big\|_{\ell^q} \geqslant n^{1/2}/\sqrt{2}, \tag{6.4}$$

whereas $\|\sum_{j=1}^n \varepsilon_j f_j\|_{\ell^q} = n^{1/q}$; so for (6.1) to persist for all n, we need $q \leqslant 2$.

Hence $p = q = 2$, and so (6.2) holds; consequently, by Kwapień's Theorem, E is isomorphic to the Hilbert space $L^2(\mu)$. □

References

1. E. Berkson and T. A. Gillespie, Spectral decompositions and harmonic analysis on *UMD* spaces, *Studia Math.* **112** (1994), 13–49.
2. E. Berkson and T. A. Gillespie, The q-variation of functions and spectral integration of Fourier multipliers, *Duke Math. J.* **88** (1997), 103–132.
3. E. Berkson, T. A. Gillespie and P. S. Muhly, Abstract spectral decompositions guaranteed by the Hilbert transform, *Proc. Lond. Math. Soc.* (3) **53** (1986), 489–517.
4. G. Blower, Multipliers for semigroups, *Proc. Edinb. Math. Soc.* **39** (1996), 241–252.
5. J. Bourgain, A remark on Schrödinger operators, *Israel J. Math.* **77** (1992), 1–16.
6. J. Bourgain, Vector-valued singular integrals and the H^1-BMO duality, in *Probability theory and harmonic analysis* (ed. J.-A. Chao and W. A. Woyczyński), pp. 1–19 (Marcel Dekker, New York, 1986).

7. L. CARLESON, Some analytic problems related to statistical mechanics, in *Euclidean harmonic analysis* (ed. J. J. Benedetto), pp. 5–45, Lecture Notes in Mathematics, vol. 779 (Springer, Berlin, 1980).
8. P. R. CHERNOFF, Essential self-adjointness of powers of generators of hyperbolic equations, *J. Funct. Analysis* **12** (1973), 401–414.
9. R. COIFMAN, J. L. R. DE FRANCIA AND S. SEMMES, Multiplicateurs de Fourier de $L^p(\mathbb{R})$ et estimations quadratiques, *C. R. Acad. Sci. Paris* Ser. 1 **306** (1988), 351–4.
10. M. G. COWLING, Harmonic analysis on semigroups, *Ann. Math.* **117** (1983), 267–283.
11. J. L. R. DE FRANCIA, A Littlewood–Paley inequality for arbitrary intervals, *Rev. Mat. Iberoamericana* **1** (1985), 1–14.
12. A. ERDÉLYI, *Asymptotic expansions* (Dover, New York, 1955).
13. E. HILLE AND R. S. PHILLIPS, *Functional analysis and semi-groups*, Colloquium Publications, vol. XXXI (AMS, Providence, RI, 1957).
14. S. KWAPIEŃ, Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients, *Studia Math.* **44** (1972), 583–595.
15. V. D. MILMAN AND G. SCHECHTMAN, *Asymptotic theory of finite-dimensional normed spaces*, Lecture Notes in Mathematics, vol. 1200 (Springer, Berlin, 1986).
16. E. M. STEIN, *Singular integrals and differentiability properties of functions* (Princeton University Press, Princeton, NJ, 1970).
17. E. M. STEIN, *Topics in harmonic analysis related to the Littlewood–Paley theory* (Princeton University Press, Princeton, NJ, 1970).
18. E. M. STEIN, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals* (Princeton University Press, Princeton, NJ, 1993).
19. P. WOJTASCZYCK, *Banach spaces for analysts* (Cambridge University Press, 1989).
20. Q. H. XU, Fourier multipliers for $L_p(\mathbb{R}^n)$ via q-variation, *Pacific J. Math.* **176** (1996), 287–296.
21. A. ZYGMUND, *Trigonometric series*, vol. 2 (Cambridge University Press, 1959).