J. Austral. Math. Soc. Ser. B 31(1990), 472–483

ON HEARING THE SHAPE OF AN ARBITRARY DOUBLY-CONNECTED REGION IN R^2

E. M. E. ZAYED[1]

(Received 5 September, 1988; revised 28 February, 1989)

Abstract

The basic problem in this paper is that of determining the geometry of an arbitrary doubly-connected region in R^2 together with an impedance condition on its inner boundary and another impedance condition on its outer boundary, from the complete knowledge of the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ for the two-dimensional Laplacian using the asymptotic expansion of the spectral function $\theta(t) = \sum_{j=1}^{\infty} \exp(-t\lambda_j)$ for small positive t.

1. Introduction

The underlying problem is to deduce the precise shape of a membrane from the complete knowledge of the eigenvalues λ_j for the Laplace operator $\Delta = \sum_{i=1}^{2}(\partial/\partial x^i)^2$ in the x^1x^2-plane.

Let $\Omega \subseteq R^2$ be a simply connected bounded domain with a smooth boundary $\partial\Omega$. Consider the impedance problem

$$(\Delta + \lambda)u = 0 \quad \text{in } \Omega, \qquad (\partial/\partial n + \gamma)u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\partial/\partial n$ denotes differentiation along the inward pointing normal to $\partial\Omega$, γ is a positive constant and $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Denote its eigenvalues, counted according to multiplicity, by

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots \to \infty \quad \text{as } j \to \infty. \tag{1.2}$$

The problem of determining the geometry of Ω and the impedance γ has been discussed recently in [4] from the asymptotic behaviour of the spectral

[1]Mathematics Department, University of Emirates, Faculty of Science, P. O. Box 15551, Al-Ain, United Arab Emirates.

https://doi.org/10.1017/S0334270000006792 Published online by Cambridge University Press

function

$$\theta(t) = \operatorname{tr}[\exp(-t\Delta)] = \sum_{j=1}^{\infty} \exp(-t\lambda_j), \quad \text{as } t \to 0. \tag{1.3}$$

Problem (1.1) has been investigated in [2], [3], [5] in the following special cases:

Case 1. $\gamma = 0$ (Neumann problem)

$$\theta(t) = \frac{|\Omega|}{4\pi t} + \frac{|\partial\Omega|}{8(\pi t)^{1/2}} + a_0 + \frac{7}{256}(t/\pi)^{1/2} \int_{\partial\Omega} (k(\sigma))^2 \, d\sigma + O(t), \quad \text{as } t \to 0. \tag{1.4}$$

Case 2. $\gamma \to \infty$ (Dirichlet problem)

$$\theta(t) = \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8(\pi t)^{1/2}} + a_0 + \frac{1}{256}(t/\pi)^{1/2} \int_{\partial\Omega} (k(\sigma))^2 \, d\sigma + O(t), \quad \text{as } t \to 0. \tag{1.5}$$

In these formulae, $|\Omega|$ is the area of Ω, $|\partial\Omega|$ is the total length of its boundary, σ is the arc length of the counterclockwise oriented boundary $\partial\Omega$ and $k(\sigma)$ is the curvature of $\partial\Omega$. The constant term a_0 has geometric significance, e.g., if Ω is smooth and convex, then $a_0 = 1/6$ and if Ω is permitted to have a finite number "H" of smooth convex holes, then $a_0 = (1 - H)/6$.

Furthermore, it has been shown by Gottlieb [1] that if L_N is the length of a part of the boundary $\partial\Omega$ with Neumann boundary condition and if L_D is the length of the remaining part of $\partial\Omega$ with Dirichlet boundary condition, then

$$\theta(t) = \frac{|\Omega|}{4\pi t} + \frac{L_N - L_D}{8(\pi t)^{1/2}} + a_0 + O(t^{1/2}) \quad \text{as } t \to 0. \tag{1.6}$$

The object of this paper is to discuss the following inverse problem: Let Ω be an arbitrary doubly-connected region in R^2 surrounding internally by a simply connected bounded domain Ω_1 with a smooth boundary $\partial\Omega_1$ and externally by a simply connected bounded domain Ω_2 with a smooth boundary $\partial\Omega_2$. Suppose that the eigenvalues (1.2) are given for the impedance problem

$$(\Delta + \lambda)u = 0 \quad \text{in } \Omega, \tag{1.7}$$

$$(\partial/\partial n_1 + \gamma_1)u = 0 \quad \text{on } \partial\Omega_1, \tag{1.8}$$

and

$$(\partial/\partial n_2 + \gamma_2)u = 0 \quad \text{on } \partial\Omega_2, \tag{1.9}$$

where $\partial/\partial n_1$ and $\partial/\partial n_2$ denote differentiations along the inward pointing normals to the boundaries $\partial\Omega_1$ and $\partial\Omega_2$ respectively, while γ_1 and γ_2 are positive constants. Determine the geometry of the arbitrary doubly-connected region Ω as well as the impedances γ_1 and γ_2 from the asymptotic behaviour of $\theta(t)$ for small positive t.

Note that problem (1.7)–(1.9) has been investigated recently by Zayed [6] in the special case where

$$\Omega = \{(r,\theta): a \le r \le b,\ 0 \le \theta \le 2\pi\}$$

is a circular annulus.

2. Statement of results

Suppose that the inner boundary $\partial\Omega_1$ of the region Ω is given locally by the equations $x^i = y^i(\sigma_1)$, $i = 1, 2$ in which σ_1 is the arc length of the counterclockwise oriented inner boundary $\partial\Omega_1$ and $y^i(\sigma_1) \in C^\infty(\partial\Omega_1)$. Suppose also that the outer boundary $\partial\Omega_2$ of Ω is given by the equations $x^i = y^i(\sigma_2)$, $i = 1, 2$ in which σ_2 is the arc length of the counterclockwise oriented outer boundary $\partial\Omega_2$ and $y^i(\sigma_2) \in C^\infty(\partial\Omega_2)$. Let L_1 and L_2 be the lengths of $\partial\Omega_1$ and $\partial\Omega_2$ respectively. Let $k_1(\sigma_1)$ and $k_2(\sigma_2)$ be the curvatures of $\partial\Omega_1$ and $\partial\Omega_2$ respectively, where $\int_{\partial\Omega_1} k_1(\sigma_1)\,d\sigma_1 = \int_{\partial\Omega_2} k_2(\sigma_2)\,d\sigma_2 = 2\pi$. Thus, the results of our main problem (1.7)–(1.9) which will be constructed in Section 7 can be summarized in the following cases:

Case 1. $(0 < \gamma_1 \ll 1,\ \gamma_2 \gg 1)$

$$\theta(t) = \frac{|\Omega|}{4\pi t} + \frac{1}{8(\pi t)^{1/2}}\{L_1 - (L_2 + 2\pi\gamma_2^{-1})\} - \frac{\gamma_1 L_1}{2\pi} + O(t^{1/2}), \quad \text{as } t \to 0. \tag{2.1}$$

Case 2. $(\gamma_1 \gg 1,\ 0 < \gamma_2 \ll 1)$

In this case the asymptotic expansion of $\theta(t)$ as $t \to 0$ follows from (2.1) with the interchanges $L_1 \leftrightarrow L_2$ and $\gamma_1 \leftrightarrow \gamma_2$.

Case 3. $(\gamma_1, \gamma_2 \gg 1)$

$$\theta(t) = \frac{|\Omega|}{4\pi t} - \frac{1}{8(\pi t)^{1/2}}\{(L_1 + 2\pi\gamma_1^{-1}) + (L_2 + 2\pi\gamma_2^{-1})\} + O(t^{1/2}), \quad \text{as } t \to 0. \tag{2.2}$$

Case 4. $(0 < \gamma_1, \gamma_2 \ll 1)$

$$\theta(t) = \frac{|\Omega|}{4\pi t} + \frac{L_1 + L_2}{8(\pi t)^{1/2}} - \frac{1}{2\pi}(\gamma_2 L_2 - \gamma_1 L_1) + O(t^{1/2}), \quad \text{as } t \to 0. \tag{2.3}$$

With reference to formulae (1.4)–(1.6) the asymptotic expansions (2.1)–(2.3) may be interpreted as follows:

(i) Ω is an arbitrary doubly-connected region in R^2 and we have the impedance boundary conditions (1.8), (1.9) with small/large impedances γ_1, γ_2 as indicated in the specifications of the four respective cases.

(ii) For the first three terms, Ω is an arbitrary doubly-connected region in R^2 of area $|\Omega|$.

In case 1, it has $H = (1 + (3\gamma_1 L_1)/\pi)$ holes, a part of the boundary of length L_1 with Neumann boundary condition and the other part of length $(L_2 + 2\pi\gamma_2^{-1})$ with Dirichlet boundary condition, provided H is an integer.

In case 3, it has only one hole ($H = 1$), a boundary of length $\{(L_1 + 2\pi\gamma_1^{-1}) + (L_2 + 2\pi\gamma_2^{-1})\}$ together with Dirichlet boundary conditions on $\partial\Omega_1$ and $\partial\Omega_2$.

In case 4, it has $H = 1 + 3(\gamma_2 L_2 - \gamma_1 L_1)/\pi$ holes, a boundary of length $L_1 + L_2$ together with Neumann boundary conditions on $\partial\Omega_1$ and $\partial\Omega_2$, provided H is an integer.

3. Formulation of the mathematical problem

With reference to [2] and using the same arguments of Section 1 in [4] and Section 2 in [6], we deduce that the spectral function $\theta(t)$ associated with our main problem (1.7)–(1.8) can be written in the form:

$$\theta(t) = |\Omega|/(4\pi t) + K(t), \tag{3.1}$$

and

$$K(t) = \iint_\Omega \chi(\underset{\sim}{x}, \underset{\sim}{x}; t)\, dx, \tag{3.2}$$

where $|\Omega|$ is the area of the region Ω, while $\chi(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)$ is a regular part of the Green's function $G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)$ for the heat equation $\Delta u = \partial u/\partial t$. In what follows, we shall use Laplace transforms with respect to t, and use s^2 as the Laplace transform parameter; thus

$$\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = \int_0^{+\infty} \exp^{-(s^2 t)}\, G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)\, dt. \tag{3.3}$$

Consequently, we deduce that $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ satisfies the membrance equation

$$(\Delta - s^2)\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = -\delta(\underset{\sim}{x}_1, -\underset{\sim}{x}_2) \quad \text{in } \Omega, \tag{3.4}$$

together with the impedence conditions (1.8), (1.9), where $\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)$ is the Dirac delta function located at the source point $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$.

The asymptotic expansion of $K(t)$ as $t \to 0$ may then be deduced directly from the asymptotic expansion of $\overline{K}(s^2)$ as $s \to \infty$, where

$$\overline{K}(s^2) = \iint_\Omega \overline{\chi}\underset{\sim}{x}, \underset{\sim}{x}; s^2)\, d\underset{\sim}{x}. \tag{3.5}$$

4. Construction of Green's functions

It is well known [6] that the membrane equation (3.4) has the fundamental solution $\overline{G}_0(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = K_0(sr_{\underset{\sim}{x}_1\underset{\sim}{x}_2})/2\pi$, where $r_{\underset{\sim}{x}_1\underset{\sim}{x}_2} = |\underset{\sim}{x}_1 - \underset{\sim}{x}_2|$ is the distance between the points $\underset{\sim}{x}_1 = (x_1^1, x_1^2)$, $\underset{\sim}{x}_2 = (x_2^1, x_2^2)$ of the region Ω and K_0 is the modified Bessel function of the second kind and of zero order. The existence of this solution enables us to construct integral equations for $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ satisfying the impedance boundary conditions (1.8), (1.9) for small/large impedances γ_1, γ_2. Therefore, Green's theorem gives:

Case 1. $(0 < \gamma_1 \ll 1,\ \gamma_2 \gg 1)$

$$\begin{aligned}\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) &= K_0(sr_{\underset{\sim}{x}_1\underset{\sim}{x}_2})/2\pi \\ &+ \frac{1}{\pi}\int_{\partial\Omega_1} \overline{G}(\underset{\sim}{x}_1, \underset{\sim}{y}; s^2)\{\partial/\partial n_{1\underset{\sim}{y}} K_0(sr_{\underset{\sim}{y}\underset{\sim}{x}_2}) + \gamma_1 K_0(sr_{\underset{\sim}{y}\underset{\sim}{x}_2})\}\, d\underset{\sim}{y} \\ &+ \frac{1}{\pi}\int_{\partial\Omega_2} \partial/\partial n_{2\underset{\sim}{y}} \overline{G}(\underset{\sim}{x}_1, \underset{\sim}{y}; s^2)\{(K_0(sr_{\underset{\sim}{y}\underset{\sim}{x}_2}) + \gamma_2^{-1}\partial/\partial n_{2\underset{\sim}{y}} K_0(sr_{\underset{\sim}{y}\underset{\sim}{x}_2})\}\, d\underset{\sim}{y}.\end{aligned} \tag{4.1}$$

Case 2. $(\gamma_1 \gg 1,\ 0 < \gamma_2 \ll 1)$

In this case $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ has the same form (4.1) with the interchange $\partial\Omega_1 \leftrightarrow \partial\Omega_2$, $\gamma_1 \leftrightarrow \gamma_2$ and $\underset{\sim}{n}_1 \leftrightarrow \underset{\sim}{n}_2$.

Case 3. $(\gamma_1, \gamma_2 \gg 1)$

In this case $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ has the same form (4.1) except its second term which is different from the second term of (4.1). In case 3, the second term of $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ is equal to the negative of the third term of (4.1) with the interchanges $\partial\Omega_1 \leftrightarrow \partial\Omega_2$, $\gamma_1 \leftrightarrow \gamma_2$ and $\underset{\sim}{n}_1 \leftrightarrow \underset{\sim}{n}_2$.

Case 4. $(0 < \gamma_1, \gamma_2 \ll 1)$

In this case $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ has the same form (4.1) except its third term which is different from the third term of (4.1). In case 4, the third term of $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ is equal to the negative of the second term of (4.1) with the interchanges $\partial\Omega_1 \leftrightarrow \partial\Omega_2$, $\gamma_1 \leftrightarrow \gamma_2$, and $\underset{\sim}{n}_1 \leftrightarrow \underset{\sim}{n}_2$.

On applying the iteration method (see [4]) to the integral equation (4.1), we obtain the Green's function $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ which has the regular part:

$$\begin{aligned}
\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) &= \frac{1}{2\pi^2}\int_{\partial\Omega_1} K_0(sr_{\underset{\sim}{x}_1\underset{\sim}{y}})\left\{\frac{\partial}{\partial n_{1\underset{\sim}{y}}}K_0(sr_{\underset{\sim}{y}\underset{\sim}{x}_2}) + \gamma_1 K_0(sr_{\underset{\sim}{y}\underset{\sim}{x}_2})\right\} d\underset{\sim}{y} \\
&+ \frac{1}{2\pi^2}\int_{\partial\Omega_2} \frac{\partial}{\partial n_{2\underset{\sim}{y}}}K_0(sr_{\underset{\sim}{x}_1\underset{\sim}{y}})\left\{K_0(sr_{\underset{\sim}{y}\underset{\sim}{x}_2}) + \gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}}}K_0(sr_{\underset{\sim}{y}\underset{\sim}{x}_2})\right\} d\underset{\sim}{y} \\
&+ \frac{1}{2\pi^2}\int_{\partial\Omega_1}\int_{\partial\Omega_1} K_0(sr_{\underset{\sim}{x}_1\underset{\sim}{y}})M_1(\underset{\sim}{y}, \underset{\sim}{y}') \\
&\qquad \times \left\{\frac{\partial}{\partial n_{1\underset{\sim}{y}'}}K_0(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2}) + \gamma_1 K_0(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})\right\} d\underset{\sim}{y}\, d\underset{\sim}{y}' \\
&+ \frac{1}{2\pi^2}\int_{\partial\Omega_2}\int_{\partial\Omega_2} \frac{\partial}{\partial n_{2\underset{\sim}{y}}}K_0(sr_{\underset{\sim}{x}_1\underset{\sim}{y}})M_2(\underset{\sim}{y}, \underset{\sim}{y}') \\
&\qquad \times \left\{K_0(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2}) + \gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}'}}K_0(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})\right\} d\underset{\sim}{y}\, d\underset{\sim}{y}' \\
&+ \frac{1}{2\pi^2}\int_{\partial\Omega_1}\left\{\int_{\partial\Omega_2} \frac{\partial}{\partial n_{2\underset{\sim}{y}}}K_0(sr_{\underset{\sim}{x}_1\underset{\sim}{y}})M_3(\underset{\sim}{y}, \underset{\sim}{y}')\, d\underset{\sim}{y}\right\} \\
&\qquad \times \left\{\frac{\partial}{\partial n_{1\underset{\sim}{y}'}}K_0(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2}) + \gamma_1 K_0(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})\right\} d\underset{\sim}{y}' \\
&+ \frac{1}{2\pi^2}\int_{\partial\Omega_2}\left\{\int_{\partial\Omega_1} K_0(sr_{\underset{\sim}{x}_1\underset{\sim}{y}})M_4(\underset{\sim}{y}, \underset{\sim}{y}')\, d\underset{\sim}{y}\right\} \\
&\qquad \times \left\{K_0(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2}) + \gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}'}}K_0(sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})\right\} d\underset{\sim}{y}', \qquad (4.2)
\end{aligned}$$

where

$$M_i(\underset{\sim}{y}, \underset{\sim}{y}') = \sum_{\nu=0}^{\infty} K_i^{(\nu)}(\underset{\sim}{y}', \underset{\sim}{y}), \qquad i = 1-4, \tag{4.3}$$

$$K_1(\underset{\sim}{y}', \underset{\sim}{y}) = \frac{1}{\pi}\left\{\frac{\partial}{\partial n_{1\underset{\sim}{y}}}K_0(sr_{\underset{\sim}{y}\underset{\sim}{y}'}) + \gamma_1 K_0(sr_{\underset{\sim}{y}\underset{\sim}{y}'})\right\}, \tag{4.4}$$

$$K_2(\underset{\sim}{y}', \underset{\sim}{y}) = \frac{1}{\pi}\left\{\frac{\partial}{\partial n_{2\underset{\sim}{y}'}}K_0(sr_{\underset{\sim}{y}\underset{\sim}{y}'}) + \gamma_2^{-1}\frac{\partial^2}{\partial n_{2\underset{\sim}{y}}\,\partial n_{2\underset{\sim}{y}'}}K_0(sr_{\underset{\sim}{y}\underset{\sim}{y}'})\right\}, \tag{4.5}$$

$$K_3(\underset{\sim}{y}',\underset{\sim}{y}) = \frac{1}{\pi}\left\{K_0(sr_{\underset{\sim}{y}\underset{\sim}{y}'}) + \gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}}}K_0(sr_{\underset{\sim}{y}\underset{\sim}{y}'})\right\}, \tag{4.6}$$

and

$$K_4(\underset{\sim}{y}',\underset{\sim}{y}) = \frac{1}{\pi}\left\{\frac{\partial^2}{\partial n_{1\underset{\sim}{y}}}\,\partial n_{2\underset{\sim}{y}'}K_0(sr_{\underset{\sim}{y}\underset{\sim}{y}'}) + \gamma_1\frac{\partial}{\partial n_{2\underset{\sim}{y}'}}K_0(sr_{\underset{\sim}{y}\underset{\sim}{y}'})\right\}. \tag{4.7}$$

Similarly, we can find $\overline{\chi}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)$ for the other three cases.

On the basis of (4.2) the function $\overline{\chi}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)$ will be estimated for large values of s together with small γ_1 and large γ_2. The case when $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ lie in the neighbourhood of the inner boundary $\partial\Omega_1$ or in the neighbourhood of the outer boundary $\partial\Omega_2$ is particularly interesting. To this end we shall use coordinates similar to those obtained in [4] as follows:

5. Differential geometry of the boundaries $\partial\Omega_1$ and $\partial\Omega_2$

Let n_1, n_2 be the minimum distances from a point $\underset{\sim}{x} = (x^1, x^2)$ of the region Ω to the boundaries $\partial\Omega_1$, $\partial\Omega_2$ respectively. Letters $\underset{\sim}{n}_1(\sigma_1)$, $\underset{\sim}{n}_2(\sigma_2)$ denote the inward drawn unit normals to $\partial\Omega_1$, $\partial\Omega_2$ respectively. We note that the coordinates in the neighbourhood of $\partial\Omega_2$ and its diagrams (see Figure 1(a), Figure 2) are in the same form as in Section 3 of [4] with the interchanges $\sigma \leftrightarrow \sigma_2$, $n \leftrightarrow n_2$, $h \leftrightarrow h_2$, $I \leftrightarrow I_2$, $C(I) \leftrightarrow C(I_2)$, $\delta \leftrightarrow \delta_2$. Thus, we have the same formula (3.1)–(3.4) of Section 3 in [4] with the interchanges $c(\sigma) \leftrightarrow k_2(\sigma_2)$, $n \leftrightarrow n_2$ and $\underset{\sim}{n}(\sigma) \leftrightarrow \underset{\sim}{n}_2(\sigma_2)$. Similarly, the coordinates in the neighbourhood of $\partial\Omega_1$ and its diagrams (see Figure 1(b), Figure 2) are similar to those obtained in Section 3 of [4] with the interchanges $\sigma \leftrightarrow \sigma_1$, $n \leftrightarrow n_1$, $h \leftrightarrow h_1$, $I \leftrightarrow I_1$, $C(I) \leftrightarrow C(I_1)$, $\delta \leftrightarrow \delta_1$. The only remark here is that the two unit normal vectors on $\partial\Omega_1$ and $\partial\Omega_2$ are in the opposite direction.

Therefore, we have the same formulas (3.1)–(3.4) of Section 3 in [4] with the following interchanges: $c(\sigma) \leftrightarrow k_1(\sigma_1)$, $n \leftrightarrow n_1$, $n(\sigma) \leftrightarrow \underset{\sim}{n}_1(\sigma_1)$, the plus sign of the second term of (3.1) by the minus sign, the minus sign of the second term in the third equation of (3.2) by the plus sign, the constant $-1/12$ in (3.3) by $+7/12$ and finally the minus sign in the second term of (3.4) by the plus sign.

https://doi.org/10.1017/S0334270000006792 Published online by Cambridge University Press

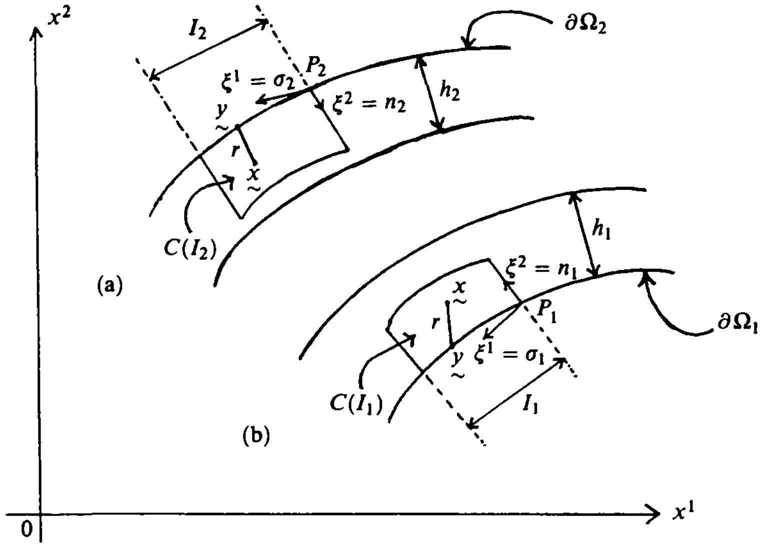

FIGURE 1

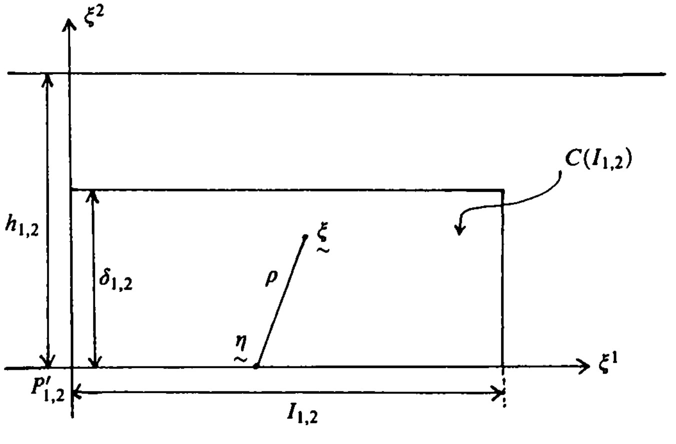

FIGURE 2

6. Some local expansions

It now follows that the local expansions of the functions

$$K_0(sr_{\underset{\sim}{x}\underset{\sim}{y}}),\ \frac{\partial}{\partial n_{1\underset{\sim}{y}}} K_0(sr_{\underset{\sim}{x}\underset{\sim}{y}}),\ \frac{\partial}{\partial n_{2\underset{\sim}{y}}} K_0(sr_{\underset{\sim}{x}\underset{\sim}{y}}), \tag{6.1}$$

https://doi.org/10.1017/S0334270000006792 Published online by Cambridge University Press

when the distance between $\underset{\sim}{x}$ and $\underset{\sim}{y}$ is small, are very similar to those obtained in Sections 4, 5 or [4]. Consequently, for small γ_1 and large γ_2 the local behaviour of the following kernels:

$$K_1(\underset{\sim}{y}', \underset{\sim}{y}),\ K_4(\underset{\sim}{y}', \underset{\sim}{y}), \tag{6.2}$$

$$K_2(\underset{\sim}{y}', \underset{\sim}{y}),\ K_3(\underset{\sim}{y}', \underset{\sim}{y}), \tag{6.3}$$

when the distance between $\underset{\sim}{y}$ and $\underset{\sim}{y}'$ is small, follows directly from the knowledge of the local expansions of the functions (6.1). This follows from the definition of e^{λ}-functions (see [3], [4]) in small domains $C(I_1)$ and $C(I_2)$. Thus, using methods similar to those obtained in Sections 6–10 of [4], we can show that the functions (6.1) are e^{λ}-functions with degrees $\lambda = 0, -1, -1$ respectively. Consequently, for small impedance γ_1 the functions (6.2) are e^{λ}-functions with degrees $\lambda = 0, -1$ while for large impedence γ_2 the functions (6.3) are e^{λ}-functions with degrees $\lambda = 0, 1$ respectively.

Definition. If $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ are points in a large domain $\Omega + \partial\Omega_1$ or $\Omega + \partial\Omega_2$, then we define

$$\hat{r}_{12} = \min_{\underset{\sim}{y}}(r_{\underset{\sim}{x}_1\underset{\sim}{y}} + r_{\underset{\sim}{x}_2\underset{\sim}{y}}) \quad \text{if } \underset{\sim}{y} \in \partial\Omega_1,$$

or

$$\widehat{R}_{12} = \min_{\underset{\sim}{y}}(r_{\underset{\sim}{x}_1\underset{\sim}{y}} + r_{\underset{\sim}{x}_2\underset{\sim}{y}}) \quad \text{if } \underset{\sim}{y} \in \partial\Omega_2.$$

An $E^{\lambda}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s)$-function is defined and infinitely differentiable with respect to $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ when these points belong to a large domain $\Omega + \partial\Omega_1$ or $\Omega + \partial\Omega_2$ except when $\underset{\sim}{x}_1 = \underset{\sim}{x}_2 \in \partial\Omega_1$ or $\partial\Omega_2$. Thus the E^{λ}-function has a similar local expansion of the e^{λ}-function (see [3], [4]).

By the help of Sections 8, 9 in [4] it is easily seen that formula (4.2) is an $E^0(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s)$-function and consequently

$$\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = O\{[1 + |\log s\hat{r}_{12}|]e^{-As\hat{r}_{12}}\} + O\{[1 + |\log s\widehat{R}_{12}|]e^{-Bs\widehat{R}_{12}}\}, \tag{6.4}$$

which is valid for $s \to \infty$ and for small γ_1 and large γ_2, where A and B are positive constants. Formula (6.4) shows that $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ is exponentially small for $s \to \infty$. Similar statements are true in the other three cases.

With reference to Section 10 in [4], if the e^{λ}-expansions of the functions (6.1)–(6.3) are introduced into (4.2) and if we use formulae similar to (6.4), (6.9) of Section 6 in [4], we obtain the following local behaviour of

$\widehat{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ when $\hat{r}_{12}$ or $\widehat{R}_{12}$ is small which is valid for $s \to \infty$ and for small γ_1 and large γ_2:

$$\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = \overline{\chi}_1(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) + \overline{\chi}_2(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2), \tag{6.5}$$

where if $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ belong to a sufficiently small domain $C(I_1)$, then

$$\overline{\chi}_1(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = -\frac{1}{2\pi}\left\{1 - \gamma_1\left(\frac{\partial}{\partial \xi_1^2}\right)^{-1}\right\} K_0(s\hat{\rho}_{12}) + O(s^{-1}e^{-As\hat{\rho}_{12}}), \tag{6.6}$$

while, if $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ belong to a sufficiently small domain $C(I_2)$, then

$$\overline{\chi}_2(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = -\frac{1}{2\pi}\left\{1 - \gamma_2^{-1}\left(\frac{\partial}{\partial \xi_1^2}\right)\right\} K_0(s\hat{\rho}_{12}) + O(s^{-1}e^{-Bs\hat{\rho}_{12}}). \tag{6.7}$$

When $\hat{r}_{12} \geq \delta_1 > 0$ or $\widehat{R}_{12} \geq \delta_2 > 0$ the function $\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ is of order $O(e^{-Ns})$, $s \to \infty$, $N > 0$. Thus, since $\lim \hat{r}_{12}/\hat{\rho}_{12} = 1$ or $\lim \widehat{R}_{12}/\hat{\rho}_{12} = 1$ when $\hat{r}_{12}$ or $\widehat{R}_{12}$ tends to zero, then we have the asymptotic formulae (6.6) and (6.7) with $\hat{\rho}_{12}$ in the small domains cases being replaced by $\hat{r}_{12}$ or $\widehat{R}_{12}$ in the large domain $\Omega + \partial\Omega_1$ or $\Omega + \partial\Omega_2$ respectively. Similar formulae for the other three cases can be found.

7. Construction of our results

Since for $\xi^2 \geq h_1 > 0$ or $\xi^2 \geq h_2 > 0$ the function $\overline{\chi}_1(\underset{\sim}{x}, \underset{\sim}{x}; s^2)$ is of $O(\exp -(2Ash_1))$ while the function $\overline{\chi}_2(\underset{\sim}{x}, \underset{\sim}{x}; s^2)$ is of $O(\exp(-2Bsh_2))$, the integral over the region Ω of the function $\overline{\chi}(\underset{\sim}{x}, \underset{\sim}{x}; s^2)$ can be approximated in the following way (see (3.5)):

$$\begin{aligned}\overline{K}(s^2) = &\int_{\xi^2=0}^{h_2}\int_{\xi^1=0}^{L_2} \overline{\chi}_2(\underset{\sim}{x}, \underset{\sim}{x}; s^2)\{1 - k_2(\xi^1)\xi^2\}\, d\xi^1\, d\xi^2 \\ &- \int_{\xi^2=0}^{h_1}\int_{\xi^1=0}^{L_1} \overline{\chi}_1(\underset{\sim}{x}, \underset{\sim}{x}; s^2)\{1 + k_1(\xi^1)\xi^2\}\, d\xi^1\, d\xi^2 \\ &+ O(\exp(-2Ash_1)) + O(\exp(-2Bsh_2)) \quad \text{as } s \to \infty\end{aligned} \tag{7.1}$$

If the e^{λ}-expansions of $\overline{\chi}_1(\underset{\sim}{x}, \underset{\sim}{x}; s^2)$ and $\overline{\chi}_2(\underset{\sim}{x}, \underset{\sim}{x}; s^2)$ are introduced into (7.1), one obtains an asymptotic series of the form:

$$\overline{K}(s^2) = \sum_{n=1}^{j} a_n s^{-n} + O(s^{-j-1}) \quad \text{as } s \to \infty, \tag{7.2}$$

where the coefficients a_n for all four cases are calculated from the e^λ-expansions by the help of formula (11.3) of Section 11 in [4].

Finally, on inverting Laplace transforms and using (3.1) we arrive at our results (2.1)–(2.3).

8. Discussions and conclusions

In this paper we note that the definitions and the local expansions of e^λ-functions and E^λ-functions in the small domain $C(I_2)$ and the large domain $\Omega + \partial\Omega_2$ respectively are exactly the same as in Pleijel [3] and Sleeman and Zayed [4]. But the definitions and the local expansions of these functions in the small domain $C(I_1)$ and the large domain $\Omega + \partial\Omega_1$ may be different from those obtained in [3], [4] in the inclusion of the terms $(\partial/\partial\xi^1)^l$ and $(\partial/\partial\xi^2)^m$ and because the two unit normal vectors on $\partial\Omega_1$ and $\partial\Omega_2$ are in the opposite direction.

Pleijel has introduced these functions to estimate $\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ as $s \to \infty$ for Neumann or Dirichlet problem when $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ lie in the neighbourhood of a smooth boundary of a general simply connected bounded domain, while in the present paper the author has used these functions to estimate $\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ as $s \to \infty$ for the impedance problem (1.7)–(1.9) with small/large impedances γ_1, γ_2, when $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ lie in the neighbourhood of the inner and outer boundaries of a general doubly-connected domain in R^2. From these discussions we conclude that the reference [3] plays an important role in the present paper. Because of that, I am deeply grateful to the Swedish mathematician Professor Å. Pleijel.

Acknowledgement

I would like to express my sincere thanks to the referees for their interesting suggestions and comments.

References

[1] H. P. W. Gottlieb, "Eigenvalues of the Laplacian with Neumann boundary conditions", *J. Austral. Math. Soc. Ser. B* **26** (1985) 293–309.

[2] M. Kac, "Can one hear the shape of a drum?", *Amer. Math. Monthly* 73 No. 4 Part II (1966) 1–23.

[3] Å Pleijel, "A study of certain Green's functions with applications in the theory of vibrating membranes," *Ark. Mat.* **2** (1954) 553–569.
[4] B. D. Sleeman and E. M. E. Zayed, "An inverse eigenvalue problem for a general convex domain," *J. Math. Anal. Appl.* **94** No. (1983) 78–95.
[5] K. Stewartson and R. T. Waechter, "On hearing the shape of a drum: further results", *Proc. Camb. Phil. Soc.* **69** (1971) 353–363.
[6] E. M. E. Zayed, "Eigenvalues of the Laplacian for the third boundary value problem", *J. Austral. Math. Soc. Ser. B* **29** (1987) 79–87.