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THE TOPOLOGICAL DEGREE OF A-PROPER MAPS

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1. Introduction. Recently several fixed-point theorems have been proved for new classes of non-compact maps between Banach spaces. First, Petryshyn [**15**] generalized the concept of compact and quasi-compact maps when he introduced the P-compact maps and proved a fixed-point theorem for this class of maps. Then in [**6**], de Figueiredo defined the notion of G-operator to unify his own work on fixed-point theorems and that of Petryshyn. He also proved that the class of G-operators was a fairly large one.

We notice the following facts: (i) The essential idea in the above cases is that if certain finite-dimensional "approximations" of the map have fixed points, then the map has a fixed point; (ii) One of the tools used in proving fixed-point theorems in the finite-dimensional case is the Brouwer degree and its generalization to maps of the type Identity + Compact in [**8**]. Furthermore, in the latter case, it was proved that the degree of finite-dimensional approximations of any map of the form Identity + Compact becomes constant after some step, and this limit is the degree. The next step then is to try to define the degree of maps of the type Identity + P-compact by considering the degree of finite-dimensional approximations; but in general we cannot expect the degree to be an integer. This was done in [**2**; **3**] for the class of A-proper maps first introduced by Petryshyn [**16**] under the heading "maps satisfying condition (H)".

Here in §§ 2 and 3 we improve on the work done in [**2**] by giving a new representation of the degree which allows us to prove the sum formula. The idea of this representation comes from the use of ultrapowers in non-standard analysis (see [**18**]). We also give a weaker "homotopy axiom" which proves more useful in computations. In § 4, we define a fixed-point index for P-compact maps and compute it in the differentiable case as it is done in [**8**; **7**, p. 136, Theorem 4.7].

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2. Preliminaries.

(A) *Basic facts concerning filters and ultrapowers.* Throughout this work,

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$\mathbf{N}$ will denote the set of natural numbers, $\mathbf{Z}$ the ring of integers, and $\mathbf{Z}^{\mathbf{N}}$ the ring of all sequences of integers with coordinatewise addition and multiplication. If E is a set, $\mathscr{P}(E)$ will denote the ordered family of all subsets of E.

Definition 1. A filter $\mathscr{F}$ on E is a non-void family of subsets of E such that
(i) $A_1, A_2 \in \mathscr{F}$ implies $A_1 \cap A_2 \in \mathscr{F}$,
(ii) If $A \in \mathscr{F}$ and $A \subset B$, then $B \in \mathscr{F}$,
(iii) $\emptyset \notin \mathscr{F}$.

An example of a filter on $\mathbf{N}$ is $\mathscr{F}_0 = \{A \subset \mathbf{N} | \mathbf{N}\backslash A \text{ is finite}\}$. On any set E, any non-void subset $A \subset E$ generates a filter $\{B \subset E \mid A \subset B\}$. This filter is called the principal filter generated by A.

If $\mathscr{F}$ is a filter, then $\mathscr{F} \subset \mathscr{P}(E)$ or $\mathscr{F} \in \mathscr{P}((\mathscr{P}(E))$; therefore on the class of all filters on E, we have an induced order relation $\leqq$. $\mathscr{F}_1 \leqq \mathscr{F}_2$ whenever for any A, $A \in \mathscr{F}_1$ implies $A \in \mathscr{F}_2$.

Definition 2. Any maximal element of the set of filters on E is called an ultrafilter on E.

PROPOSITION 1. *If a filter $\mathscr{F}$ on E is an ultrafilter, then if $A \cup B \in \mathscr{F}$ we have either $A \in \mathscr{F}$ or $B \in \mathscr{F}$.*

Proof. See [**1**, p. 65]. Thus $\mathscr{F}_0$ is not an ultrafilter on $\mathbf{N}$ since $\mathbf{N} \in \mathscr{F}_0$ but neither the subset of even integers nor the subset of odd integers belongs to $\mathscr{F}_0$.

By Zorn's lemma, every filter on a set E is contained in an ultrafilter on E. Let $\mathscr{F}$ be a filter on $\mathbf{N}$ such that $\mathscr{F}_0 \leqq \mathscr{F}$, and define a relation $\sim$ on $\mathbf{Z}^{\mathbf{N}}$ in the following way:

$$\{x_i\} \sim \{y_i\} \qquad \text{whenever } \{i | \, x_i = y_i\} \in \mathscr{F}.$$

This relation is compatible with the ring structure of $\mathbf{Z}^{\mathbf{N}}$; therefore the quotient set $^*\mathbf{Z}(\mathscr{F}) = \mathbf{Z}^{\mathbf{N}}/\sim$ with the induced operations is a ring. It contains the subring of classes of constant sequences isomorphic to $\mathbf{Z}$; from now on we shall identify $\mathbf{Z}$ with its isomorphic image.

Definition 3. If $\mathscr{F}$ is an ultrafilter on $\mathbf{N}$ such that $\mathscr{F}_0 \leqq \mathscr{F}$, then we say that the corresponding $^*\mathbf{Z}(\mathscr{F})$ is an ultrapower of $\mathbf{Z}$.

PROPOSITION 2. *If A is an infinite subset of* $\mathbf{N}$, *then there exists a filter $\mathscr{F}$ which contains A with $\mathscr{F}_0 \leqq \mathscr{F}$.*

Proof. Set $\mathscr{F} = \{B \subset \mathbf{N} | \, B \supset A \cap C \text{ for any } C \in \mathscr{F}_0\}$.

(B) *Some definitions.* Let X be a real Banach space. A projectional scheme for X is
(i) a nested sequence $\{X_n\}$ of finite-dimensional subspaces of X,
(ii) $\bigcup_n X_n$ is dense in X,
(iii) for each n there is a continuous linear projection $p_n: X \to X_n$ and $p_m p_n = p_n p_m = p_m$ if $n \geqq m$.

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The projectional scheme is complete if $\lim_n p_n x = x$ for each $x \in X$. X is said to have property $(\pi)_k$ for some $k \geqq 1$ if it has a projectional scheme such that $\|p_n\| \leqq k$ for all n. We restrict our study to real Banach spaces having property $(\pi)_k$ for some k. We suppose that the projectional schemes are fixed for each space and that whenever we have a map $f: A \to Y$ for some subset $A \subset X$, then $\dim X_n = \dim Y_n$ for each n with projections $p_n: X \to X_n$ and $q_n: Y \to Y_n$.

Following Petryshyn and de Figueiredo, we state the following definitions.

Definition 4. (a) Let G be a subset of the Banach space X and $f: G \to X$ a continuous map. f is said to be P-compact if for any $\alpha > 0$, the existence of a bounded sequence $\{x_{n_j} \in G \cap X_{n_j}\}$ such that $p_{n_j} f x_{n_j} - \alpha x_{n_j}$ converges to $y \in X$ implies the existence of a convergent subsequence $\{x_{n_{j_k}}\}$ with $\lim x_{n_{j_k}} = x \in G$ and $fx - \alpha x = y$.

As an example we have that any map f such that $f(G)$ is relatively compact is P-compact.

(b) Let X and Y be Banach spaces; $G \subset X$. $f: G \to Y$ is an A-proper map if for any bounded sequence $\{x_{n_j} \in G \cap X_{n_j}\}$ such that $q_{n_j} f x_{n_j}$ converges to y there exists a convergent subsequence $\{x_{n_{j_k}}\}$ with $\lim x_{n_{j_k}} = x \in G$ and $fx = y$.

Note that if f is P-compact, then $f - \lambda I$ is A-proper for any $\lambda > 0$.

(c) Let C be a closed convex subset of a Banach space X. A map $f: C \to X$ is Galerkin approximable (or is a G-operator) if $p_n f$ is continuous for n sufficiently large and if f has a fixed point in C whenever

$$\{n | p_n f \,|\, (C \cap X_n) \text{ has a fixed point in } (C \cap X_n)\} \in \mathscr{F}_0.$$

Any P-compact map is a G-operator. For more examples of P-compact maps see [**17**] and for G-operators see [**6**].

Our aim is to build a degree theory for a class of maps which includes the A-proper maps and for that purpose we state the following definition.

Definition 5. Let X and Y be Banach spaces, $\mathscr{F}$ a filter on $\mathbf{N}$ with $\mathscr{F}_0 \leqq \mathscr{F}$, $y \in Y$, and $G \subset X$. A map $f: G \to Y$ is a y-$\mathscr{F}$-operator if

(i) $q_n f$ is continuous when n is sufficiently large,

(ii) the existence of a bounded sequence $\{x_n \in G \cap X_n\}$ such that $\{n | \, q_n f x_n = q_n y\} \in \mathscr{F}$ implies that there exists an $x \in G$ for which $fx = y$.

Remarks. (1) If $Y = X$ and f is a G-operator, then $(I - f)$ is a 0-$\mathscr{F}_0$-operator.

(2) If we suppose that Y has a complete projectional scheme, then maps satisfying condition (h) introduced by Petryshyn [**14**, p. 340] are y-$\mathscr{F}$-operators for any $\mathscr{F} \geqq \mathscr{F}_0$ and any $y \in Y$.

(3) Therefore under the same condition on Y, an A-proper map is a y-$\mathscr{F}$-operator for any $y \in Y$ and any $\mathscr{F} \geqq \mathscr{F}_0$.

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3. Degree theory of A-proper maps. Throughout this section, G will be an open bounded subset of the Banach space X and $\partial G = \mathrm{cl}\, G \backslash G$. $\{X_n\}$ and $\{Y_n\}$ will denote the projectional schemes of X and Y, respectively, and we suppose that X_n and Y_n are oriented, with $\dim X_n = \dim Y_n$. Set $G_n = G \cap X_n$ for each n. G_n is an open bounded subset of X_n. If φ: $\mathrm{cl}\, G_n \to Y_n$ is a continuous map, then for any $a \in Y_n \backslash \varphi(\partial G_n)$ there is a well-defined integer $d(\varphi, G_n, a)$ called the degree of f at the point a (sometimes called the Brouwer degree). For a definition and properties of the degree see either [**4**] or [**9**].

Let $\mathscr{F}$ be an arbitrary (but fixed) filter on $\mathbf{N}$ with $\mathscr{F}_0 \leqq \mathscr{F}$ and set $^*\mathbf{Z}(\mathscr{F}_0) = {}^*\mathbf{Z}$. Suppose that f: $\mathrm{cl}\, G \to Y$ is a y-$\mathscr{F}$-operator for some $y \in Y$; then if $\{n |\, q_n y \notin q_n f(\partial G_n)\} \in \mathscr{F}$, the sequence $\{d(q_n f, G_n, q_n y)\}$ determines an element of $^*\mathbf{Z}(\mathscr{F})$ which we call the degree of f at y, denoted by $D(f, G, y)$.

Proposition 3. *Let f: $\mathrm{cl}\, G \to Y$ be a y-$\mathscr{F}$-operator. Then*

(a) *Whenever $D(f, G, y)$ is defined and $\{n |\, d(q_n f, G_n, q_n y) \neq 0\} \in \mathscr{F}$, there exists an $x \in \mathrm{cl}\, G$ such that $fx = y$. If $\mathscr{F}$ is an ultrafilter $\mathscr{F}_0 \leqq \mathscr{F}$, then $D(f, G, y) \neq 0$ implies that $\{n |\, d(q_n f, G_n, q_n y) \neq 0\} \in \mathscr{F}$;*

(b) *Suppose that g: $\mathrm{cl}\, G \to Y$ is another y-$\mathscr{F}$-operator such that*

$$\{n |\ \text{there exists a homotopy } F_n \text{ from } q_n f \text{ to } q_n g \text{ such that } q_n y \notin F_n(\partial G_n \times [0, 1])\} \in \mathscr{F};$$

then $D(f, G, y) = D(g, G, y)$.

Proof. Let us prove the second part of (a); the rest follows directly from the definitions and the properties of the Brouwer degree.

Let $\mathscr{F}$ be an ultrafilter and $D(f, G, y) \neq 0$. Suppose that

$$\{n |\, d(q_n f, G_n, q_n y) \neq 0\} \notin \mathscr{F},$$

then (its complement) $\{n |\, d(q_n f, G_n, q_n y) = 0\} \in \mathscr{F}$; thus $D(f, G, y) = 0$, a contradiction.

Corollary 1 (de Figueiredo). *Let C be an open convex bounded subset of a Banach space X. Suppose that $0 \in C$ and that f: $\mathrm{cl}\, C \to X$ is a G-operator such that except for finitely many ns,*

$$(*) \qquad p_n fx - \lambda x \neq 0 \quad \text{for all } \lambda \geqq 1 \text{ and } x \in \partial(C \cap X_n).$$

Then f has a fixed point in C.

Proof. If $p_n fx - \lambda x \neq 0$ for $\lambda \geqq 1$ and $x \in \partial(C \cap X_n)$, then $p_n(f - I)|C \cap X_n$ is homotopic to the identity and the homotopy is never zero on the boundary. Therefore $d(p_n(f - I), C \cap X_n, 0) = 1$ whenever $(*)$ is valid for n and using Proposition 3 (a), our proof is complete.

Let us now restrict our attention to A-proper maps. We suppose throughout the rest of this work that all projectional schemes are complete and except for Proposition 5, that $\mathscr{F}_0$ is the fixed filter; consequently, $^*\mathbf{Z}(\mathscr{F}) = {}^*\mathbf{Z}$.

PROPOSITION 4. *If* $f\colon \mathrm{cl}\, G \to Y$ *is* A-*proper, then whenever* $y \in Y - f(\partial G)$, *we can conclude that* $\{n \mid q_n y \notin q_n f(\partial G_n)\} \in \mathscr{F}_0$ *(the degree is therefore well-defined).*

For a proof see [**3**, Lemma 1].

THEOREM 1. *Let G be an open bounded subset of X and let* $f_1, f_2\colon \mathrm{cl}\, G \to Y$ *be* A-*proper maps with* $y \in Y \backslash f_i(\partial G)$, $i = 1, 2$.

(a) *If* $D(f_1, G, y) \neq 0 \in {}^*\mathbf{Z}$, *then there exists an* $x \in G$ *such that* $f_1 x = y$.

(b) (*Sum formula*). *If* $G = G' \cup G''$, G' *and* G'' *being open disjoint subsets of X such that* $y \in Y \backslash f_1(\partial G' \cup \partial G'')$, *then*

$$D(f_1, G, y) = D(f_1, G', y) + D(f_1, G'', y).$$

(c) *If there exists an open set* $G' \subset G$ *such that* $f_1^{-1}(\{y\}) \subset G'$, *then*

$$D(f_1, G', y) = D(f_1, G, y).$$

(d) *If* $F\colon \mathrm{cl}\, G \times [0, 1] \to Y$ *is a homotopy between* f_1 *and* f_2 *such that*

(i) *for each fixed* $t \in [0, 1]$, $F(\ , t)$ *is* A-*proper,*

(ii) $y \in Y \backslash F(\partial G \times [0, 1])$,

(iii) *for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for* $t_1, t_2 \in [0, 1]$, $|t_1 - t_2| < \delta$ *implies*

$$\|F(x, t_1) - F(x, t_2)\| < \epsilon \text{ for any } x \in \mathrm{cl}\, G,$$

Then $D(f_1, G, y) = D(f_2, G, y)$.

Proof. (a) $D(f_1, G, y) \neq 0 \in {}^*\mathbf{Z}$ implies that there is an infinite subsequence $\{n_j\}$ for which $d(q_{n_j} f_1, G_{n_j}, q_{n_j} y) \neq 0$; since f_1 is A-proper, we obtain the desired result.

(b) The hypothesis ensures that the three degrees are defined and

$$\begin{aligned} D(f_1, G, y) &= \{d(q_n f_1, G_n, q_n y)\} \\ &= \{d(q_n f_1, G_n' \cup G_n'', q_n y)\}, \quad G_n', G_n'' \text{ are disjoint and open in } X_n \\ &= \{d(q_n f_1, G_n', q_n y) + d(q_n f_1, G_n'', q_n y)\} \quad \text{by the sum formula for the Brouwer degree} \\ &= \{d(q_n f_1, G_n', q_n y)\} + \{d(q_n f_1, G_n'', q_n y)\} \\ &= D(f_1, G', y) + D(f_1, G'', y). \end{aligned}$$

(c) Since $f_1^{-1}(\{y\}) \subset G'$, we have $\{n \mid (q_n f_1)^{-1}\{q_n y\} \subset G_n'\} \in \mathscr{F}_0$; therefore

$$\{n \mid d(q_n f_1, G_n, q_n y) = d(q_n f_1, G_n', q_n y)\} \in \mathscr{F}_0,$$

which completes the proof.

(d) By Proposition 3 (b), it is sufficient to show that for n sufficiently large, $q_n y \notin q_n F(\partial G_n \times [0, 1])$, where $q_n F$ is the homotopy between $q_n f_1$ and $q_n f_2$. Suppose that this is not the case; then there exists an infinite sequence $\{n_j\}$, $n_j \to \infty$, such that

$$(x_{n_j}, t_{n_j}) \in (\partial G_{n_j}) \times [0, 1] \quad \text{and} \quad q_{n_j} F(x_{n_j}, t_{n_j}) = q_{n_j} y.$$

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Because [0, 1] is compact, we may assume without loss of generality that $\{t_{n_j}\}$ converges to t for some $t \in [0, 1]$. By hypothesis,

$$||F(x_{n_j}, t_{n_j}) - F(x_{n_j}, t)|| < \epsilon$$

if n_j is sufficiently large, whence $\{q_{n_j}F(x_{n_j}, t)\}$ converges to y and since $F(\ , t)$ is A-proper, this implies the existence of an $x \in \partial G$ such that $F(x, t) = y$ which contradicts hypothesis (ii).

Let us note that Theorem 1 is an improvement of [**2**, Theorem 1] as we have the sum formula with this representation of the degree. We now compare the definition of Browder and Petryshyn in [**2**] with ours. If f: cl $G \to Y$ is A-proper, let us denote by $D'(f, G, y)$ the degree defined in [**2**]; then $D'(f, G, y) \subset \mathbf{Z} \cup \{-\infty, +\infty\}$.

PROPOSITION 5. *Suppose that f: cl $G \to Y$ is A-proper and that $D'(f, G, y) \subset \mathbf{Z}$; if $n \in D'(f, G, y)$, there exists a filter $\mathscr{F}' \geqq \mathscr{F}_0$ such that*

$$D(f, G, y) = \{n, n, n, \ldots\} \in {}^*\mathbf{Z}(\mathscr{F}').$$

Proof. Since $\emptyset \neq D'(f, G, y) \subset \mathbf{Z}$, we have $D'(f, G, y) = \{n_1, n_2, \ldots, n_k\}$. Let $B_i = \{j | \, d(q_j f, G_j, q_j y) = n_i\}$, $i = 1, 2, \ldots, k$, and construct the filter $\mathscr{F}_i$ generated by $\mathscr{F}_0$ and B_i. Then $D(f, G, y) = \{n_i, n_i, \ldots\} \in {}^*\mathbf{Z}(\mathscr{F}_i')$.

To end this section let us compute the degree of linear injective A-proper maps. In [**12**], Petryshyn has shown the following.

PROPOSITION 6. *If f is a linear injective A-proper map from X to Y, then there exists a constant $c > 0$ and an integer N such that for each $n \geqq N$ we have $||q_n fx|| \geqq c||x||$ for each $x \in X_n$.*

COROLLARY 2. *If f is bounded, linear, A-proper, and injective, then for any bounded open set G in X, $D(f, G, y) = \{\pm 1, \pm 1, \ldots\}$ for any $y \in f(G)$.*

COROLLARY 3. *Under the same conditions as in Corollary 2, f is onto. For a proof see* [**12**, *Theorem* 5].

4. Fixed-point indices of P-compact maps. Before defining the fixed-point index, the following theorem is quoted from [**13**] to show that A-proper maps are essentially proper maps.

THEOREM 2. *Let G be an open bounded subset of X and f a continuous A-proper map from cl G into Y. Then for any closed subset $M \subset G$, the subset $M \cap f^{-1}(L)$ is compact if L is compact in Y.*

Proof. See [**13**, p. 141, Lemma 1].

We obtain the following as an easy corollary.

COROLLARY 4. *If f: cl $G \to X$ is P-compact and if the fixed points of f are in G and are isolated, then they are finite in number.*

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Let G be an open bounded subset of X, f: cl $G \to X$ a P-compact map having x_0 as isolated fixed point, and suppose that $x_0 \in G$. Let U_{x_0} be an open neighbourhood of x_0 in G such that U_{x_0} contains no other fixed point of f. Then we define the fixed-point index of f at x_0 to be $I(f, x_0) = D(f - I, U_{x_0}, 0)$. Because of Theorem 1(c), this definition is independent of the U_{x_0} chosen, provided it is small enough.

In [**8**], Leray and Schauder calculated explicitly the fixed-point index in the compact, differentiable case. Here we give an analogous theorem for the P-compact case.

THEOREM 3. *Let X be a Banach space with property $(\pi)_1$, f: cl $G \to X$ a P-compact map, differentiable at $x_0 \in G$ such that the derivative f_{x_0}' is also P-compact. Suppose that x_0 is a fixed point of f and that $+1$ is not an eigenvalue of f_{x_0}'. Then x_0 is an isolated fixed point of f and $I(f, x_0) = \{(-1)^{\beta_n}\} \in {}^*\mathbf{Z}$, where β_n is the sum of multiplicities of the eigenvalues of $p_n f_{x_0}'$: $X_n \to X_n$ which are greater than* 1.

Proof. The fact that x_0 is an isolated fixed point was proved in [**17**, Theorem 6.3].

The idea of the proof of the second part is to prove that $I(f, x_0) = I(f_{x_0}', 0)$ and then compute $I(f_{x_0}', 0)$. Since $+1$ is not an eigenvalue of f_{x_0}', we see that $f_{x_0}' - I$: $X \to X$ is injective, linear, and A-proper. By Proposition 6, there exists a $c > 0$ and an integer N_0 such that $n \geqq N_0$ implies that

$$||p_n f_{x_0}'x - x|| \geqq c||x|| \qquad \text{for } x \in X_n.$$

Since $p_n x \to x$, taking limits we have $||f_{x_0}'x - x|| \geqq c||x||$ for every $x \in X$.

Consider

$$\begin{aligned} ||f(x_0 + h) - (x_0 + h)|| &= ||f(x_0 + h) - f(x_0) - h|| \\ &= ||f(x_0 + h) - f(x_0) - f_{x_0}'(h) + f_{x_0}'(h) - h|| \\ &\geqq ||(f_{x_0}' - I)(h)|| - ||f(x_0 + h) - f(x_0) \\ &\quad - f_{x_0}'(h)|| \geqq c||h|| - ||\epsilon(x_0, h)||, \end{aligned}$$

where $||\epsilon(x_0, h)||\ ||h||^{-1} \to 0$ if $||h|| \to 0$. Therefore there exists a $\delta > 0$ such that $||h|| < \delta$ implies $||\epsilon(x_0, h)|| < 2^{-1}c||h||$.

Thus $||f(x_0 + h) - (x_0 + h)|| \geqq 2^{-1}c||h||$ when $||h|| < \delta$. Let U_{x_0} be the ball with centre x_0 and radius δ. Then

$$I(f, x_0) = D(f - I, U_{x_0}, 0) = \{d(p_n(f - I), U_{x_0} \cap X_n, 0)\}.$$

Choose an integer $N \geqq N_0$ such that

$$||p_N x_0 - x_0|| \leqq \min[8^{-1}\delta, 8^{-1}c\delta, 8^{-1}(||f_{x_0}'|| + 1)^{-1}c\delta].$$

Note that $p_N x_0 \in U_{x_0} \cap X_n$ if $n \geqq N$. Define the translation g: $U_{x_0} \to X$ by $g(x) = x - p_N x_0$, and note that $g(U_{x_0} \cap X_n) \subset X_n$ if $n \geqq N$. Let us consider the map H_n: $(U_{x_0} \cap X_n) \times [0, 1] \to X_n$ for $n \geqq N$ given by

$$H_n(x + x_0, t) = (1 - t)[p_n(f_{x_0}' - I) \cdot g](x + x_0) + tp_n(f - I)(x + x_0).$$

H_n is a homotopy between $p_n(f_{x0}' - I)g$ and $p_n(f - I)$. If we prove that for $n \geqq N$, $0 \notin H_n(\partial(U_{x0} \cap X_n), t)$ for every $t \in [0, 1]$, then we can conclude that

$$d(p_n(f_{x0}' - I)g, U_{x0} \cap X_n, 0) = d(p_n(f - I), U_{x0} \cap X_n, 0) \qquad \text{for } n \geqq N.$$

But since g is a translation, it is easy to see that if $n \geqq N$, then

$$d(p_n(f_{x0}' - I) \cdot g, U_{x0} \cap X_n, 0) = d(p_n(f_{x0}' - I), B \cap X_n, 0),$$

where B is a ball of radius δ and centre at the origin. Thus we would obtain: $I(f, x_0) = I(f_{x0}', 0)$. We now prove that $H_n(x + x_0, t) \neq 0$ for

$$(x + x_0) \in \partial(U_{x0} \cap X_n), t \in [0, 1]$$

and $n \geqq N$.

$$\begin{aligned}||H_n(x + x_0, t)|| &= ||[p_n(f_{x0}' - I) \cdot g](x + x_0) - tp_n[(f_{x0}' - I) \cdot g \\ &\qquad - (f - I)](x + x_0)|| \\ &\geqq ||p_n(f_{x0}' - I)(x + x_0 - p_N x_0)|| \\ &\qquad - ||(f_{x0}' - I)(x + x_0 - p_N x_0) - (f - I)(x + x_0)||,\end{aligned}$$

since $||p_n|| \leqq 1$ and $t \in [0, 1]$.

$$\begin{aligned}||p_n(f_{x0}' - I)(x + x_0 - p_N x_0)|| &\geqq c||x + x_0 - p_N x_0|| \\ &\geqq c||x - (p_N x_0 - x_0)|| \\ &\geqq c||x|| - c\delta/8 \\ &\geqq 7(c\delta/8)\end{aligned}$$

if $(x + x_0) \in \partial(U_{x0} \cap X_n)$. On the other hand, if $(x + x_0) \in \partial(U_{x0} \cap X_n)$, then $||x|| = \delta$ and

$$\begin{aligned}||(f_{x0}' - I)(x + x_0 - p_N x_0) - (f - I)(x + x_0)|| &= ||(-x_0 + p_N(x_0)) + x_0 - f(x + x_0) + f_{x0}'(x) + f_{x0}'(x_0 - p_N x_0)|| \\ &\leqq ||x_0 - p_N(x_0)|| + ||\epsilon(x_0, x)|| + ||f_{x0}'|| \, ||x_0 - p_N(x_0)|| \\ &\leqq 8^{-1}c\delta + 2^{-1}c\delta + 8^{-1}c\delta \\ &= 8^{-1} \cdot 6c\delta.\end{aligned}$$

Therefore $H_n(x + x_0, t) \neq 0$ for $(x + x_0) \in \partial(U_{x0} \cap X_n)$ and $t \in [0, 1]$. The proof of the theorem is completed by applying [**7**, p. 133, Theorem 4.6] to $p_n f_{x0}': X_n \to X_n$ for $n \geqq N$.

We now prove a theorem concerning the effect of a slight perturbation of a P-compact operator on its fixed points and the result is analogous to [**7**, Theorem 4.8]. In this theorem, full use is made of the sum formula.

Definition 6. If f is a continuous and differentiable map from cl G into X, then the derivative f' defines a map from G into the space of linear maps $L(X, X)$. If f' is continuous, we say that f is continuously differentiable.

THEOREM 4. *Suppose that the hypotheses of Theorem* 3 *are still valid and, furthermore, let f be continuously differentiable on a neighbourhood U of the fixed* *point x_0, with f_y' P-compact for every $y \in U$. Then there exists a neighbourhood V of x_0 and an ϵ_0 with $0 < \epsilon_0 < 1$ such that if $|\epsilon| \leqq \epsilon_0$, then $(1 + \epsilon)f$ has a unique fixed point in V.*

Proof. Since 1 is not an eigenvalue of f_{x_0}', we have, as before, a constant $c > 0$ such that $||f_{x_0}'(x) - x|| \geqq c||x||$ for every $x \in X$. Because of the hypotheses, we can choose an open ball V of radius δ_0 and centre x_0 such that the following statements are true:

(1) x_0 is the only fixed point of f in cl V,

(2) $||f_x' - f_{x_0}'|| \leqq 3^{-1}c$ and $||f_x'|| \leqq ||f_{x_0}'|| + 1 = K$ for any $x \in$ cl V,

(3) $||f(x)|| \leqq ||f(x_0)|| + 1 = ||x_0|| + 1 = M$, $x \in$ cl V,

(4) $||(x + x_0) - f(x + x_0)|| \geqq 2^{-1}c||x||$ if $(x + x_0) \in$ cl V.

Let $0 < \epsilon_0 < 1$ be such that $\epsilon_0 \leqq \min((4M)^{-1}c\delta_0, (3K)^{-1}c)$. If $|\epsilon| < \epsilon_0$, consider $H(\ , t) = H_t = f + t\epsilon f - I$: cl $V \to X$. Then

(a) H_t is A-proper for each $t \in [0, 1]$,

(b) Given $\xi > 0$, there exists $\eta > 0$ such that $||H_{t_1}(x) - H_{t_2}(x)|| < \xi$ whenever $|t_1 - t_2| < \eta$ for every $x \in$ cl V,

(c) $H(x, 0) = (f - I)(x)$ for $x \in$ cl V,
$H(x, 1) = (f + \epsilon f - I)(x)$ for $x \in$ cl V,

(d) If $t \in [0, 1]$ and $x \in \partial V$, then

$$\begin{aligned} ||H(x, t)|| &= ||(f + t\epsilon f - I)(x)|| \\ &\geqq ||(f - I)(x)|| - ||\epsilon f(x)|| \\ &\geqq 2^{-1}c\delta_0 - |\epsilon| \, ||f(x)|| \\ &\geqq 2^{-1}c\delta_0 - (4M)^{-1}(c\delta_0)M; \end{aligned}$$

thus $||H(x, t)|| > 0$.

Whence $D(f + t\epsilon f - I, V, 0) = D(f - I, V, 0) = I(f, x_0)$,

(**) $D(f + t\epsilon f - I, V, 0) = D(f - I, V, 0) = I(f, x_0) = \gamma \neq 0$ (by Theorem 3),

i.e. there exists $y \in V$ such that $(1 + \epsilon)f(y) = y$. Let y_0 be such a point. The derivative of $(1 + \epsilon)f$ at y_0 is $(1 + \epsilon)f_{y_0}'$ and it is easily verified that 1 is not an eigenvalue of $(1 + \epsilon)f_{y_0}'$.

We can then use Theorem 3 to conclude that y_0 is an isolated fixed point of $(1 + \epsilon)f$ and that $I((1 + \epsilon)f, y_0) = I((1 + \epsilon)f_{y_0}', 0)$. Next, it is easily shown that for n large, $p_n(f_{x_0}' - I)$ and $p_n[(1 + \epsilon)f_{y_0}' - I]$ are homotopic and satisfy the conditions of Theorem 1(b), which implies that

$$I([1 + \epsilon]f_{y_0}', 0) = I(f_{x_0}', 0) = \gamma.$$

Thus

$$I((1 + \epsilon)f, y_0) = I((1 + \epsilon)f_{y_0}', 0) = \gamma.$$

By Corollary 4, the fixed points of $(1 + \epsilon)f$ in V are finite in number, let $y_1, \dots, y_r$ be these fixed points; then

$$\begin{aligned} D((1 + \epsilon)f - I, V, 0) &= \sum_{i=1}^{r} I([1 + \epsilon]f, y_i) \\ &= r\gamma \end{aligned}$$

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but $D((1 + \epsilon)f - I, V, 0) = \gamma$; see (**). Therefore $r = 1$. We can conclude that $(1 + \epsilon)f$ has only one fixed point in V.

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