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AN APPLICATION OF ULTRAPRODUCTS TO LATTICE-ORDERED GROUPS

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Using ultraproducts, N. R. Reilly proved that if G is a representable lattice-ordered group and J is an independent subset totally ordered by $\prec$, then the order on G can be extended to a total order which induces $\prec$ on J (see [**5**]). In [**4**], H. A. Hollister proved that a group G admits a total order if and only if it admits a representable order and, moreover, every lattice-order on a group is the intersection of right total orders. The purpose of this paper is to provide a partial converse, viz: if G is a lattice-ordered group and J is an independent subset totally ordered by $\prec$, then the order on G can be extended to a right total order which induces $\prec$ on J. In view of the above remarks, this is the best generalization of Reilly's result. The method of proof uses ultraproducts together with the idea used by H. A. Hollister in [**4**] and P. F. Conrad in [**2**] in his existence theorem for free lattice-ordered groups over a p.o. group.

For background material, see [**1**] and [**3**].

Notation and proof. Let G be a p.o. group with identity e. $G^\dagger = G\backslash\{e\}$ and $G^* = \{g \in G : g > e\}$. If $g \in G^\dagger$ and G is a lattice-ordered group, then there exists a convex l-subgroup of G maximal with respect to missing g. Such a convex l-subgroup is said to be a *value* of g in G. If G is a lattice-ordered group and $X \subseteq G$, then $O(X)$ will denote the l-ideal of G generated by X. If, in addition, X is a convex l-subgroup of G, $R(X)$ will denote the set of right cosets of X in G. $J \subseteq G^*$ is said to be *independent* if and only if for all $j \in J$, $j \notin O(J\backslash\{j\})$. $S_\omega(J)$ will denote the set of finite subsets of J.

THEOREM. *Let G be a lattice-ordered group and J an independent subset of G totally ordered by $\prec$. There exists a right total order on G which extends the lattice order and induces $\prec$ on J.*

Proof. Let $g \in J$. Then $g \notin O(J\backslash\{g\})$. Hence there exists a value C_g of g which contains $O(J\backslash\{g\})$. For $g \in G^\dagger\backslash J$, choose any value C_g of g. Let $T_g = R(C_g)$ be ordered by: $C_g x \geqq C_g$ if and only if there exists $y \in C_g$ such that $x \geqq y$. Each T_g is a totally ordered set. Let $T = \bigcup \{T_g : g \in G^\dagger\}$.

Let $i \in I = S_\omega(J)$, say $i = \{g_1, \ldots, g_n\}$ where $g_1 \prec \ldots \prec g_n$. Well-order $G^\dagger\backslash i = B$. Now well-order $G^\dagger$ by extending the order on B by:

$$h <' g_1 <' \ldots <' g_n$$

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for each $h \in B$. This induces a total ordering on T, namely: $t_1 < t_2$ if and only if $t_1 \in T_{f_1}$, $t_2 \in T_{f_2}$ and $f_1 <' f_2$ or $f_1 = f_2$ and $t_1 < t_2$ (in T_{f_1}). Let $A(T)$ be the set of all order-preserving permutations of the totally ordered set T. Let $\phi_i : G \to A(T)$ be given by: $(C_x y)(g\phi_i) = C_x yg$ for all $x \in G^\dagger$ and $y \in G$. For each $g \in G^\dagger$, well-order T_g, say by $<_1$. This gives rise to a well-ordering $<_1$ of T, namely: $t_1 <_1 t_2$ if and only if $t_1 \in T_{f_1}$, $t_2 \in T_{f_2}$ and $f_1 <' f_2$ or $f_1 = f_2$ and $t_1 <_1 t_2$ (in T_{f_1}). Define $A(T)^*$ by: $h \in A(T)^*$ if and only if $th > t$ where t is the least element (with respect to $<_1$) of $\{s \in T : sh \neq s\}$. Then $A(T)$ is a right totally ordered group, ϕ_i is an o-homomorphism and $e < g_1\phi_i < \ldots < g_n\phi_i$. Moreover, ϕ_i is 1-1 and onto $G_i = G\phi_i$.

Let D be a regular ultrafilter on I and $H = D\text{-prod } \lambda i G_i$. H is a right totally ordered group when ordered by: $h^\sim > e$ if and only if $\{i \in I : h_i > e\} \in D$ (the order is total since $\{i \in I : h_i \geqq e\} \cup \{i \in I : h_i \leqq e\} = I \in D$). Define $\phi : G \to H$ by: if $g\phi = f^\sim$, then $f \sim k$ where $k_i = g\phi_i$ for all $i \in I$. ϕ is an o-homomorphism of G onto $G\phi$ which is 1-1 and if $j, j' \in J$ and $j \prec j'$, then $j\phi < j'\phi$ since $\{i \in I : j\phi_i < j'\phi_i\} \supseteq \hat{j} \cap \hat{j}' \in D$ where $\hat{j} = \{i \in I : j \in i\}$.

Suppose that independence had been defined as follows: let G be a lattice-ordered group and $X \subseteq G$. $C(X)$ will denote the convex l-subgroup of G generated by X. $J \subseteq G^*$ is said to be *independent* if and only if for all $j \in J$, $j \notin C(J \backslash \{j\})$. The proof of the theorem goes through with this weaker definition of independence.

It should be noted that this application of ultraproducts is the same as that in a proof of the compactness theorem for first order theories and is closely related to that theorem.

References

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