6

Moduli Problems with Flat Divisorial Part

So far we have identified stable pairs (X, Δ) as the basic objects of our moduli problem, the one-parameter families that we want to allow, and worked out the reduced part of the moduli spaces. Now we come to the next step of identifying the stable families over an arbitrary base scheme.

In this chapter we consider several special cases that are easier to handle, since we are able to treat the underlying variety X and the boundary divisor Δ as separate objects that are both flat over the base. This is achieved by imposing one of four different types of restrictions on the coefficients occurring in Δ.

- (No boundary) Stable varieties X with $\Delta = 0$.
- (Standard coefficients) The coefficients in Δ are in the "diminished standard coefficient" set $\{1 - \frac{1}{3}, 1 - \frac{1}{4}, 1 - \frac{1}{5}, \ldots, 1\}$.
- (Major coefficients) The coefficients in Δ are all $> \frac{1}{2}$.
- (Generic coefficients) The coefficients in Δ are $\mathbb{Q}$-linearly independent.

These examples cover many cases; the most jarring omission is that none of these allow $\frac{1}{2}$ as a coefficient.

After a general discussion of moduli problems in Section 6.1, we treat two notions of stability for stable varieties in Sections 6.2–6.3. The first of these – introduced in Kollár and Shepherd-Barron (1988) – starts with the proposal that all plurigenera should be deformation invariant. The second – introduced in Viehweg (1995) – posits that all sufficiently divisible plurigenera should be deformation invariant. The two versions agree over reduced base schemes.

Both of these versions can be extended to pairs (X, Δ), as long as Δ is a standard or major boundary as above.

In Section 6.4 we discuss another variant – due to Alexeev (2006, 2015) – that works if the coefficients in Δ are sufficiently general. This is especially natural when the boundary arises as a small perturbation of a basic situation.

The infinitesimal deformation theory of stable varieties is not yet well understood, but a large part of the first order theory for surfaces is treated in Altmann and Kollár (2019). After a general discussion of first order deformations of singular varieties in Section 6.5, we work out in detail the theory for cyclic quotient surface singularities in Section 6.6. These are the simplest noncanonical singularities and they show that the two versions outlined in Sections 6.2–6.3 differ from each other already over Spec $k[\varepsilon]$.

Assumptions In this chapter we work over a $\mathbb{Q}$-scheme, but the definitions are set up in full generality. See Section 8.8 for a discussion of some problems in positive characteristic.

6.1 Introduction to Moduli of Stable Pairs

Based on the outline in Section 1.2, we discuss the plan that we use to treat many moduli problems in algebraic geometry. The following version is designed to work best for the moduli of stable pairs (X, Δ).

The method first deals with stable pairs with an embedding into a fixed projective space and then removes the effect of the embedding.

Step 6.1 (Objects of the moduli problem) At the beginning we have to decide which objects and families our moduli problem should cover. This is usually done in three stages.

6.1.1 (Interior objects over algebraically closed fields) As the very first step, we have to decide what kind of objects we want to parametrize. Probably the first nonlinear moduli problem considered was elliptic curves, followed by smooth projective curves of higher genus and their close relatives, abelian varieties. The study of the moduli of higher dimensional smooth projective varieties was systematically undertaken first by Matsusaka. His approach focuses on polarized pairs (X, L), where X is a variety and L an ample divisor or divisor class. Here our main aim is to study canonical models of varieties and pairs of general type.

It is expected that, once we understand the moduli of varieties, it should be relatively easy to work out the moduli theory of related compound objects. For example varieties with a group action, pointed varieties, maps between varieties, or various combinations of these.

6.1.2 (Boundary objects over algebraically closed fields) By now the answers are mostly well established, but historically this was a difficult and very nontrivial step. The compactification of the moduli of smooth curves by stable curves was discovered by Deligne and Mumford (1969).

For surfaces, the need to work with canonical models (instead of minimal models) seems to have become clear early, but the choice of stable surfaces for boundary points was proposed only in Kollár and Shepherd-Barron (1988).

It should be noted that the distinction between interior and boundary points is not always clear cut. While everyone agrees that smooth curves give the interior points and nodal curves the boundary points of $\overline{\mathrm{M}}_g$, for surfaces one may view either canonical models or only smooth canonical models as interior points.

Although historically the development went in the other direction, for a logical treatment of a moduli problem it is better to settle on the right class of interior and boundary objects at the beginning. Then gradually prove that they have the required properties.

6.1.3 (Objects over arbitrary fields) For stable pairs, the definitions of (6.1.1–2) carry over to arbitrary fields, but in a few examples new questions emerge.

For pointed schemes $(X, p_1, \dots, p_r)$ it may be better to replace the set of closed points $\{p_1, \dots, p_r\}$ by a 0-dimensional subscheme $Z \subset X$ of length r. A more subtle problem appears for polarizations, due to the difference between $\mathrm{Pic}(X_k)$ and $\mathbf{Pic}(X_k)(k)$, where $\mathbf{Pic}(X_k)(k)$ is the set of k-points of the Picard scheme of X_k; see Bosch et al. (1990, sec.8.1) for a discussion. This will not be a major issue for us. There are also problems caused by inseparable extensions in positive characteristic.

Conclusion 6.1.4 We are working with stable varieties (1.41) and, more generally, with stable pairs (X, Δ) as defined in (2.1). There seems to be full agreement about these being the right objects in characteristic 0.

Step 6.2 (Families of the moduli problem) In many moduli problems, it is considered obvious that the families are determined by the objects: one should work with flat families whose fibers are among our objects. Then the traditional approach is to determine families over $\operatorname{Spec} k[\varepsilon]$, and, more generally, over Artinian base schemes. This is usually called *obstruction theory*; see Artin (1976), Sernesi (2006), and Hartshorne (2010) for introductions to various cases.

However, for stable varieties and pairs, flat families with stable fibers do *not* give a sensible moduli theory. We need to proceed differently.

6.2.1 (Families over DVRs) In Chapter 2, we defined and described stable families over smooth curves and one-dimensional regular schemes. The advantage of this setting is that the total space of a family is also a locally stable pair, so minimal model theory can be applied both to the fibers and to the total space.

6.2.2 (*Families over reduced bases*) For stable varieties, we proved in (3.1) that stable families over DVRs determine stable families over reduced base schemes. We needed to work quite a bit harder to extend the theory to stable families of pairs over reduced base schemes in Chapter 4, but the end result is the same, at least in characteristic 0: the families over DVRs determine the families over reduced base schemes.

6.2.3 (Families over arbitrary bases) This is where the picture becomes rather complicated. For stable varieties, there have been different proposals for about 30 years; we discuss these in Sections 6.2–6.3. These were proved to be non-equivalent in Altmann and Kollár (2019); see Section 6.6.

We believe that the notion of KSB stability – to be treated in Section 6.2 – gives the optimal answer for stable varieties.

For pairs, the problem is that, while KSB stability has a natural generalization to pairs, not all stable families over smooth curves satisfy it; see (2.41). Thus insisting on it frequently gives nonproper moduli spaces. Still, the strongest version of KSB stability is expected to work well for pairs (X, Δ) if all the coefficients in Δ are $> \frac{1}{2}$; we discuss these in (6.24) and (6.29).

Another approach, outlined in Alexeev (2006, 2015), gives a good theory if the coefficients in Δ are sufficiently general real numbers; see (6.40).

However, there was not even a plausible proposal for the general theory before Kollár (2019). We work out the details of it in Chapter 7.

Conclusion 6.2.4 We are not aware of any other proposed definition that might work in general, but it is too soon to tell whether the theory of Chapter 7 is the final word on the subject. We comment on some of the issues next.

Once we have settled on the right objects and families, we need to start working on producing all families and constructing the moduli spaces.

We would like to have a “sensible” way to obtain all stable varieties, pairs, and their stable families. It is not a priori clear what this means.

For example, every variety of dimension n is obtained as the normalization of a hypersurface in $\mathbb{P}^{n+1}$. We can thus start working through all hypersurfaces and describe their normalizations.

For curves, this is not a bad approach. Classical authors developed much of the theory by thinking of smooth curves as normalizations of plane curves with nodes. However, this becomes harder as the genus increases. The problem is that even if a curve is general, the nodal sets of its plane representatives are always in special position.

There are some cases of surfaces where such a description is useful. For example, Enriques obtained his namesake surfaces in 1896 as sextics in $\mathbb{P}^3$

that are double along the edges of a tetrahedron. However, for most surfaces, projection to $\mathbb{P}^3$ introduces very complicated singular sets that hide the geometry of the surface. There is no "optimal" representation and it is quite hard to decide when the normalizations of two hypersurfaces are isomorphic to each other. This approach does not seem very helpful in general; see, however, the proof of Noether's formula in Griffiths and Harris (1978, sec.4.6).

Thus we aim to find projective embeddings of varieties that do not depend on too many auxiliary choices.

Step 6.3 (Rigidification by embedding) A global coordinate system on a space V is a way of associating a string of numbers (called coordinates) to any point of V. Equivalently, a choice of a map from V to $\mathbb{R}^n$ or $\mathbb{C}^n$. We prefer to work with projective objects, so for us the natural choice is to use homogeneous coordinates. Equivalently, we fix an algebraic morphism $X \to \mathbb{P}^N$. (There is a slight notational issue here. Although we almost always construct $\mathbb{P}^N$ as $\operatorname{Proj} k[x_0, \dots, x_N]$, we usually emphasize that there are no natural coordinates on it. By contrast, with rigidification we do think of the target $\mathbb{P}^N$ as having fixed coordinates.)

For varieties, the most frequently used approach is to use an embedding $(X \hookrightarrow \mathbb{P}^N)$, though sometimes finite maps $X \to \mathbb{P}^N$ or maps to other targets – weighted projective spaces or $\mathbb{P}^N$-bundles over curves – give better insight.

Thus we choose a very ample line bundle L on X, a subspace $V^{N+1} \subset H^0(X, L)$ and a basis of V^{N+1} (up to a multiplicative constant). In practice, it is much better to eliminate the second of these choices by taking $V = H^0(X, L)$. That is, we work with embeddings $(X \hookrightarrow \mathbb{P}^N)$ whose image is *linearly normal.* The rigidification involves two types of choices.

6.3.1 (Discrete choice) A very ample line bundle L. (We use this terminology although $\mathbf{Pic}(X)$ is not always discrete).

If C is a stable curve, then ω_C^r is very ample for $r \geq 3$. If S is a canonical model of a surface of general type, then ω_S is an ample line bundle and ω_S^r is very ample for $r \geq 5$ by Bombieri (1973), and Ekedahl (1988). Thus again we get an embedding of S into a projective space whose dimension depends only on the coefficients of the Hilbert polynomial $\chi(\omega_S^r)$, namely (K_S^2) and $\chi(\mathscr{O}_S)$.

The situation is more complicated for stable surfaces. These can have singularities where ω_S is not locally free. Even worse, for any $m \in \mathbb{N}$ there are stable surfaces S_m and canonical 3-folds X_m such that $\omega_{S_m}^{[m]}$ (resp. $\omega_{X_m}^{[m]}$) is not locally free at some point $x_m \in S_m$. Thus every section of $\omega_{S_m}^{[m]}$ vanishes at x_m and $H^0(X, \omega_{S_m}^{[m]})$ gives a rational map that is not defined at x_m.

We skirt this problem by fixing $m > 0$ and aiming to construct a moduli space for those stable varieties for which $\omega_S^{[m]}$ is locally free, very ample and has no higher cohomologies. Similarly, if (X, Δ) is a stable pair and Δ is a $\mathbb{Q}$-divisor, we can take $L = \omega_X^{[m]}(m\Delta)$ for some $m > 0$. Thus L is indeed a discrete choice for us.

Then we show in Step 6.8 that, if m is sufficiently divisible (depending on other numerical invariants), then the theory we get is independent of m.

There does not seem to be a similarly natural choice of L if Δ is an $\mathbb{R}$-divisor. We have to work around this in Section 8.2.

6.3.2 (Continuous choice) Different bases in $H^0(X, L)$ are equivalent to each other under the natural group action by $\mathrm{GL}(H^0(X, L))$. We eliminate the effect of this choice in Step 6.5.

Aside For smooth varieties over $\mathbb{C}$, the use of topological rigidifiers can be very powerful; leading to the Teichmüller space for curves and to Griffiths's theory of period domains. These work well for smooth varieties, but have many problems for their degenerations. For flat families of stable varieties $f: X \to S$, the topological type, or even dimension of $H^*(X_s(\mathbb{C}), \mathbb{C})$ need not be a locally constant function on S. It does not seem to be possible to make sense of a topological rigidification in general.

6.3.3 (Moduli of embedded varieties) Once we have a rigidification, we construct moduli spaces of more general embedded objects. Instead of embedded stable varieties $(X \hookrightarrow \mathbb{P}^N)$ of dimension n, one can work either with n-cycles (Cayley–Chow approach) or, which works better for us, with all subschemes $(X \subset \mathbb{P}^N)$ (Hilbert–Grothendieck approach). Thus we start with the universal family over the Hilbert scheme of n-dimensional subschemes

$$\pi\colon\ \mathrm{Univ}_n(\mathbb{P}^N) \to \mathrm{Hilb}_n(\mathbb{P}^N).$$

We encounter a severe difficulty when we try to work with pairs (X, Δ).

6.3.4 (Moduli of embedded pairs) We need to construct the universal family of relative Mumford divisors (6.13)

$$\mathrm{MDiv}(\mathrm{Univ}_n(\mathbb{P}^N)/\,\mathrm{Hilb}_n(\mathbb{P}^N)) \to \mathrm{Hilb}_n(\mathbb{P}^N).$$

The traditional approaches try to obtain this as a subscheme of either

- $\mathrm{Hilb}_{n-1}(\mathrm{Univ}_n(\mathbb{P}^N)/\,\mathrm{Hilb}_n(\mathbb{P}^N))$, or of
- $\mathrm{Chow}_{n-1}(\mathrm{Univ}_n(\mathbb{P}^N)/\,\mathrm{Hilb}_n(\mathbb{P}^N))$.

By (4.76), the Chow version works over reduced schemes, but neither works in general.

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Conclusion 6.3.5 We have the universal family of embedded varieties, but we hit a problem with pairs. This was a long-standing conundrum in the theory; K-flatness – to be worked out in Chapter 7 – was introduced to solve it.

Here we take an easier path, and in Sections 6.2–6.4 we consider several cases when the Hilbert scheme variant works in (6.3.4); see (6.13) for details.

Assume now that the above steps have been completed. Then, instead of our original moduli problem, we have solved a related one that also includes a rigidification and has many more objects. In order to get back to our original problem, we need to remove the nonstable objects and then see how to undo the effects of rigidification.

Step 6.4 (Representability) Assume first that $\Delta = 0$ and let us go back to $\mathrm{Hilb}_n(\mathbb{P}^N)$ as in (6.3.3). As we saw in Section 3.5, the set of stable fibers of π is not even a locally closed subset of $\mathrm{Hilb}_n(\mathbb{P}^N)$. Nonetheless, as we proved in Section 3.5, stable families are parametrized by a locally closed partial decomposition of $\mathrm{Hilb}(\mathbb{P}^N)$. By the choice we made in Step 6.3.1, we aim to work only with those stable subvarieties $X \subset \mathbb{P}^N$ for which $\mathscr{O}_X(1) \simeq \omega_X^{[m]}$. This is again a representable condition by (9.42). Thus we get the moduli space of m-canonically embedded stable subvarieties of dimension n in $\mathbb{P}^N$

$$\mathrm{C^mESV}(n, *, \mathbb{P}^N) \to \mathrm{Hilb}_n(\mathbb{P}^N). \tag{6.4.1}$$

(Here $*$ stands for the not-yet-specified volume.)

For pairs, we start with the case when Δ is a $\mathbb{Q}$-divisor, which we write as $\Delta = \sum a_i D_i$ for some fixed $\mathbf{a} := (a_1, \dots, a_r)$, where the D_i are effective $\mathbb{Z}$-divisors. (This will be called a *marking* in Section 8.1; see (8.21) for real coefficients.) Once we solve the questions raised in Step 6.3.4, the results of Section 4.6 give the moduli space of m-canonically embedded stable pairs

$$\mathrm{C^mESP}(\mathbf{a}, n, *, \mathbb{P}^N) \to \mathrm{Hilb}_n(\mathbb{P}^N). \tag{6.4.2}$$

Conclusion 6.4.3 For each $m > 0$, we have obtained universal families of m-canonically embedded stable varieties and pairs. However, m and the embedding are artificial choices; we still need to undo their effect.

(In practice we need to be more precise here and control various properties of the embedding – like linear normality, vanishing of certain cohomology groups – but these turn out to be technical issues; see Section 8.4.)

Step 6.5 (Quotients by group actions) Let us deal next with the continuous choice in the rigidification, which is a basis in $H^0(X, L)$. As we noted in (6.3.2), the different continuous choices are equivalent to each other under a GL-action.

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This gives a group action on the moduli of rigidified objects, and the moduli space of the nonrigidified objects is the space of orbits of this action

$$\mathrm{C^mESP}(\mathbf{a}, n, *, \mathbb{P}^N)/\mathrm{PGL}_{N+1}. \tag{6.5.1}$$

We discuss in Section 8.6 that such quotients have a natural algebraic space structure. So, aside from the slight difference between schemes and algebraic spaces, we consider the quotient problem solved.

6.6 (Conclusion of Steps 6.1–6.5) As in Step 6.4, fix a rational coefficient vector $\mathbf{a}$. Let $\mathcal{SP}(\mathbf{a}, n, v)$ denote the functor of stable pairs $(X, \Delta = \sum a_i D_i)$ of dimension n and volume v.

We check in Step 6.8 that there is an $m = m(\mathbf{a}, n, v)$ such that $\omega_X^{[m]}(m\Delta)$ is locally free, very ample, and has no higher cohomologies. Thus $P_m(X, \Delta) := H^0(X, \omega_X^{[m]}(m\Delta))$ is a locally constant function on stable families in $\mathcal{SP}(\mathbf{a}, n, v)$. Using (8.62), we obtain the coarse moduli space of $\mathcal{SP}(\mathbf{a}, n, v)$ as the union of geometric quotients

$$\mathrm{SP}(\mathbf{a}, n, v) = \amalg_i \, \mathrm{C^mESP}(\mathbf{a}, n, v, \mathbb{P}^{N_i})//\mathrm{PGL}_{N_i+1},$$

where $N_i + 1$ runs through the possible values of $P_m(X, \Delta)$. (See (8.21) for real coefficients.)

Now that we have constructed our moduli spaces $\mathcal{SP}(\mathbf{a}, n, v)$, we should study their properties.

Step 6.7 (Separatedness and valuative-properness) Since these notions depend only on families over DVRs, these will always hold for us. The discussion in (1.20) needs no amplification.

The next two topics merit a treatment of their own; here we give only a few comments and the main references to the literature.

Step 6.8 (Boundedness) We aim to prove that $\mathrm{SP}(\mathbf{a}, n, v)$ is actually of finite type, hence proper. Equivalently, that $\mathrm{SP}(\mathbf{a}, n, v) = \mathrm{SP}(\mathbf{a}, n, v, m)$ for some m (depending on $\mathbf{a}, n, v$).

We discussed stable varieties in (1.21), but there are some changes for pairs. The Hilbert function $\chi(X, \omega_X^{[r]}(\lfloor r\Delta \rfloor))$ is no longer deformation invariant, but its (rescaled) leading coefficient $\mathrm{vol}(X, \Delta) = (K_X + \Delta)^{\dim X}$, and the constant coefficient $\chi(X, \mathcal{O}_X)$ are. This is why we use only the volume in the definition of $\mathrm{SP}(\mathbf{a}, n, v)$ in (6.5.1).

An infinite union is of finite type only if it eventually stabilizes, so one can formulate our question independent of moduli theory as follows. It was proved by Alexeev (1993) for surfaces and by Hacon et al. (2018) in general.

6.8.1 (Boundedness theorem, rational coefficients) Assume that the a_i are rational. Then there is an $m = m(\mathbf{a}, n, v)$ such that $mK_X + m\Delta$ is a very ample Cartier divisor for every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)$.

If some of the a_i are irrational, then usually $mK_X + m\Delta$ is never a $\mathbb{Z}$-divisor. The natural correction would be to use $mK_X + \lfloor m\Delta \rfloor$, but there are examples when it is never Cartier (11.50.3). Thus we need a different form.

6.8.2 (Boundedness theorem, real coefficients) Assume that the a_i are real. Fix an algebraically closed field k of characteristic 0. Then there is a k-scheme of finite type S and a stable morphism $p\colon (X^S, \Delta^S) \to S$ such that every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)(k)$ appears among the fibers of p.

The two versions are equivalent for rational coefficients by (6.14).

The following variant is much easier to prove (4.60) and is sufficient for most applications.

6.8.3 (Weak boundedness theorem) Every irreducible component of $\mathrm{SP}(\mathbf{a}, n, v)$ is of finite type. □

6.8.4 (Hints to the proof for real coefficients) (Based on suggestions of C. Xu.) Hacon et al. (2014) proves that there is a smooth k-scheme of finite type S and a projective, log smooth morphism $p\colon (Y, E + D) \to S$ such that, for every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)(k)$, there is a log resolution $(X', E' + \Delta') \to (X, \Delta)$ and an $s \in S$ such that $(Y_s, E_s + D_s) \simeq (X', E' + \Delta')$. Therefore, if $p\colon (Y, E + D) \to S$ has a simultaneous canonical model $p^c\colon (Y^c, D^c) \to S$, then every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)(k)$ appears among the fibers of p^c, proving boundedness. If $\mathbf{a} \subset \mathbb{Q}$, the latter is proved in Hacon et al. (2018).

In the irrational case, we argue as follows. Pick any (X, Δ) and choose convex rational approximations (X, Δ_j) for $j = 1, \dots, r$ as in (11.47), so that they have the same dlt modifications (11.47.9).

Choose $s \in S$ such that $(Y_s, E_s + D_s) \simeq (X', E' + \Delta')$. Working in an étale neighborhood of s, there is a bijection between the irreducible components of D and the irreducible components of D_s, hence the irreducible components of Δ. Thus the Δ_j determine $\mathbb{Q}$-divisors D_j.

The aim is to show that applying Hacon et al. (2018) to any one of the $p\colon (Y, E + D_j) \to S$, we get $p^c\colon (Y^c, D^c) \to S$.

To see this, note that the fiber of $p\colon (Y, E+D_1) \to S$ over s is a log resolution of (X, Δ_1). Thus Hacon et al. (2018) gives a simultaneous, minimal, $\mathbb{Q}$-factorial model $p^m\colon (Y^m, E^m + D_1^m) \to S$.

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By our choice, the fiber over s is also a $\mathbb{Q}$-factorial, dlt model for the other (X, Δ_j). Since Y^{m} is $\mathbb{Q}$-factorial, the other $p^{\mathrm{m}}\colon (Y^{\mathrm{m}}, E^{\mathrm{m}} + D_j^{\mathrm{m}}) \to S$ are also locally stable, possibly after shrinking S.

The contraction $Y_s^{\mathrm{m}} \to X$ now extends to a neighborhood of Y_s^{m}, giving a morphism $p^{\mathrm{c}}\colon Y^{\mathrm{c}} \to S$ such that, $p^{\mathrm{c}}\colon (Y^{\mathrm{c}}, D_j^{\mathrm{c}}) \to S$ is stable for every j, again possibly after shrinking S. Thus all the $K_{Y^{\mathrm{c}}/S} + D_j^{\mathrm{c}}$ are $\mathbb{Q}$-Cartier.

Since Δ is a convex linear combination of the Δ_j, $K_{Y^{\mathrm{c}}/S} + \Delta^{\mathrm{c}}$ is $\mathbb{R}$-Cartier, hence $p^{\mathrm{c}}\colon (Y^{\mathrm{c}}, \Delta^{\mathrm{c}}) \to S$ is stable by (11.4.4), as needed.

This takes care of an open neighborhood of $s \in S$; we finish by Noetherian induction. □

Step 6.9 (Projectivity) Once we know that the connected (or irreducible) components are proper, we would like to show that they are projective. In cases when GIT works, it gives (quasi)projectivity right away, but the general quotient theorems of Kollár (1997) and Keel and Mori (1997) do not give projectivity; in fact there are many quotients that are not quasi-projective Kollár (2006).

So we need to find some ample line bundles on our moduli spaces. Let $f\colon X \to S$ be a stable morphism. The only divisorial sheaves that we can always write down on X are $\omega_{X/S}^{[m]}$; these give the sheaves $\det f_*\omega_{X/S}^{[m]}$ on S. It is not hard to work out that these are actually line bundles, so let us hope that some of these are ample.

It was Iitaka who realized that the sheaves $f_*\omega_{X/S}^{[m]}$ should always have semi-positivity properties, at least in characteristic 0, Iitaka (1972). These properties were established and applied to prove Iitaka's conjectures by many authors; see Mori (1987) for a survey. These methods were used to prove projectivity statements for the moduli of stable surfaces in Kollár (1990). Extending these results to higher dimensions turned out to be quite difficult. It was done by Fujino (2018) for stable varieties and by Kovács and Patakfalvi (2017) for stable pairs. The situation is more complicated in positive characteristic, but the surface case was settled by Patakfalvi (2014, 2017).

Conclusion 6.9.1 In all cases, the outcome is that every proper subset of the moduli space is projective. Thus we consider the projectivity question solved.

Let us now summarize the properties that we would like to see.

6.10 (Good moduli theories) A moduli theory **M** is given by specifying the objects over fields and the families. We are mainly studying those cases whose objects are various subsets of all stable pairs.

For example, the most classical example is $\mathbf{M}$ = Curves, whose objects are stable curves and whose families are all flat, proper morphisms with stable curves as fibers.

In Chapter 4, we established the optimal definitions for families of stable pairs over reduced base spaces and proved many properties. However, unlike for curves, there seem to be several natural, but nonequivalent, moduli theories of stable pairs over nonreduced base schemes.

We say that $\mathbf{M}$ is a *good moduli theory* if the following hold:

(6.10.1) $\mathbf{M}$ is separated (1.20). Since this depends only on families over DVRs, this always holds for us by (2.50).

(6.10.2) $\mathbf{M}$ is valuative-proper (1.20). The positive answer is given by (2.51), but we need to check that the central fiber also satisfies the additional assumptions that we have in $\mathbf{M}$.

(6.10.3) Embedded moduli spaces exist (6.3.3–4). Having a flat divisorial part makes this much simpler; see (6.12–6.13) for details.

(6.10.4) Representability as in (6.4).

(6.10.5) Boundedness in the weaker form (6.8.3). Together with valuative-properness, this means that the irreducible components of the corresponding moduli spaces are proper.

For the main results of this chapter we work with the following set-up, which is a slight generalization of (3.28) and (4.2).

6.11 (Basic set-up for Chapter 6) We consider flat families of demi-normal schemes with flat families of Mumford divisors. That is, our objects are proper morphisms $f\colon X \to S$ of pure relative dimension n and subschemes $\{D_i \subset X\colon i \in I\}$ satisfying the following conditions:

(6.11.1) f is flat with demi-normal (11.36) fibers,

(6.11.2) the D_i are relative Mumford divisors (4.68), and

(6.11.3) the $D_i \to S$ are flat with divisorial subschemes (4.16) as fibers.

Next, fix distinct, positive real numbers $\{a_i\colon i \in I\}$. Then $f\colon (X, \sum a_i D_i) \to S$ is family of pairs as in (5.2).

We already treated stable families over reduced bases in Chapter 4, so assume that $f\colon (X, \sum a_i D_i) \to S$ is stable or locally stable over $\operatorname{red} S$. The main question we aim to address is the following.

Question 6.11.4 If S is nonreduced, what additional restrictions should be imposed in order to get a stable (resp. locally stable) family over S?

Comments 6.11.5 There may be several different good answers to this question. These in turn give different moduli spaces, though all of them have the same underlying reduced subspace.

Also, as we noted in (2.41–2.44), requiring the D_i to be flat over S means that we do not even get all stable families over smooth curves when $a_i < \frac{1}{2}$. So, while our answers cover many important special cases, substantially new ideas will be needed to get the full theory.

6.12 (Advantages of flat divisorial parts) The cases considered in this chapter have four major technical advantages. The first three come from using the flatness option for the divisorial part in (6.3.4).

(6.12.1) One can define the families using only flatness; thus we avoid the notion of K-flatness, which is defined and studied in Chapter 7.

(6.12.2) Hilbert schemes give a quick way to write down the universal family of Mumford divisors.

(6.12.3) The pluricanonical sheaves commute with base change, as in (2.79) and (4.33). This is not crucial, but it helps us avoid some artificial choices.

The last one may be an accidental consequence of our choices.

(6.12.4) There is a natural way of writing the boundary as a linear combination of $\mathbb{Z}$-divisors, thus we avoid the notion of marking, to be introduced in Section 8.5.

The key advantage turns out to be (6.12.2), which takes care of Step 6.3.4. So let us discuss it in detail.

6.13 (Universal family of flat Mumford divisors) Let $g\colon X \to S$ be a flat, projective morphism. Consider the relative Hilbert scheme $\mathrm{Hilb}(X/S)$. It parametrizes flat families of closed subschemes of $X \to S$. Thus it has a largest open subscheme that parametrizes subschemes $B_s \subset X_s$ of pure codimension 1, without embedded points, such that X_s is regular at the generic points of B_s. This is the universal family of flat, Mumford divisors on X/S, denoted by

$$\mathrm{MDiv}(X/S) \to S.$$

When we wish to parametrize r such divisors, the universal family is given by the r-fold fiber product

$$\mathrm{MDiv}(X/S) \times_S \cdots \times_S \mathrm{MDiv}(X/S),$$

which we abbreviate as $\times_S^r \mathrm{MDiv}(X/S)$.

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We want to apply this to the Hilbert scheme of n-dimensional subschemes of $\mathbb{P}^N$, with its universal family

$$u\colon\ \mathrm{Univ}_n(\mathbb{P}^N_S) \to \mathrm{Hilb}_n(\mathbb{P}^N_S).$$

Although not strictly necessary, it is convenient to pass to the largest open subscheme $\mathrm{Hilb}^\circ(\mathbb{P}^N_S) \subset \mathrm{Hilb}(\mathbb{P}^N_S)$ over which the fibers of u are demi-normal and of pure dimension n. Thus we have

$$u^\circ\colon\ \mathrm{Univ}^\circ_n(\mathbb{P}^N_S) \to \mathrm{Hilb}^\circ_n(\mathbb{P}^N_S). \tag{6.13.1}$$

The universal family of flat, Mumford divisors is

$$\mathrm{MDiv}(\mathrm{Univ}^\circ_n(\mathbb{P}^N_S)/\,\mathrm{Hilb}^\circ_n(\mathbb{P}^N_S)) \to \mathrm{Hilb}^\circ_n(\mathbb{P}^N_S). \tag{6.13.2}$$

If we need r such divisors, the universal family we want is given by the r-fold fiber product

$$\times^r_{\mathrm{Hilb}^\circ_n(\mathbb{P}^N_S)} \mathrm{MDiv}(\mathrm{Univ}^\circ_n(\mathbb{P}^N_S)/\,\mathrm{Hilb}^\circ_n(\mathbb{P}^N_S)) \to \mathrm{Hilb}^\circ_n(\mathbb{P}^N_S). \tag{6.13.3}$$

As in (6.3.3), we can now use (4.43) to show that the functor of stable pairs is representable by a monomorphism. (9.42) takes care of the condition of being embedded by a given multiple of $K_X + \Delta$.

The schemes in (6.13.3) have infinitely many irreducible components, but once we bound the degrees of the underlying varieties and of the divisors, we get a quasi-projective parameter space.

The following was used in (6.8).

Lemma 6.14 *Fix n, v, a rational vector* $\mathbf{a}$ *and the characteristic $p \geq 0$. For $\mathcal{SP}(\mathbf{a}, n, v)$, the following are equivalent.*

(6.14.1) *There is an $m = m(\mathbf{a}, n, v)$ such that $m(K_X + \Delta)$ is very ample for every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)(k)$ where* $\mathrm{char}\, k = p$.

(6.14.2) *There are $N = N(\mathbf{a}, n, v)$ and $D = D(\mathbf{a}, n, v)$ such that every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)(k)$ is isomorphic to an embedded pair (X, Δ) in $\mathbb{P}^N$ satisfying* $\deg X \leq D$ *and* $\deg \Delta \leq D$.

(6.14.3) *Then there is a $\mathbb{Q}$-scheme (resp. $\mathbb{F}_p$-scheme) of finite type S and a stable morphism $\pi\colon (X^S, \Delta^S) \to S$ such that every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)(k)$ is obtained from π be base change.*

Proof Assume (1). Then $\dim |mK_X + m\Delta| \leq m^n v + n =: N$ by Matsusaka's inequality (11.52.3). Hence all pairs in $\mathcal{SP}(\mathbf{a}, n, v)$ are isomorphic to an embedded pair (X, Δ) in $\mathbb{P}^N$ such that $\deg X = m^n v$. We also know that $\deg(K_X + \Delta) =$

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$m^{n-1}v$. A lower bound for $\deg K_X$ can be obtained by looking at a general curve section. This gives an upper bound for $\deg \Delta$.

(2) $\Rightarrow$ (3) was treated in (6.13).

Finally assume (3) and let $g \in S$ be a generic point. By assumption there is an m_g such that $m_g(K_{X_g} + \Delta_g)$ is very ample. Then the same holds over an open neighborhood $g \in S^\circ \subset S$. We finish by Noetherian induction. □

In the next three sections, we give various stability notions and then check that they all give a good moduli theory as in (6.10).

6.2 Kollár–Shepherd-Barron Stability

This notion of stability is obtained by imposing the strongest possible properties that are satisfied by one-parameter stable families. For surfaces, this was accomplished in Kollár and Shepherd-Barron (1988). There were two reasons why the original paper dealt only with surfaces. First, the existence of stable limits relies on the minimal model program, which was only available for families of surfaces at that time. It was, however, clear that this part should work in all dimensions. Second, the proof of the representability (6.18) relied on detailed properties of lc singularities of surfaces. The theory of hulls and husks, to be discussed in Chapter 9, was developed to prove representability.

We discuss three versions. First, the classical setting of stable varieties without boundary divisors, then a generalization where we allow standard coefficients, and finally arbitrary coefficients in $(\frac{1}{2}, 1]$.

Kollár–Shepherd-Barron Stability without Boundary

6.15 (Stable objects) The stable objects are geometrically reduced, proper k-schemes X with slc singularities such that K_X is ample.

6.16 (Stable families) A family $f: X \to S$ is *KSB-stable* if

(6.16.1) f is flat with slc fibers,

(6.16.2) $\omega_{X/S}^{[m]}$ is a flat family of divisorial sheaves (3.25) for $m \in \mathbb{Z}$,

(6.16.3) f is proper and $\omega_{X/S}^{[M]}$ is an f-ample line bundle for some $M > 0$.

The first two of these conditions define *locally KSB-stable* families.

6.17 (Explanation) This definition restates (3.40). It imposes the strongest restrictions on stable families, thus it gives the smallest scheme structure on the moduli space of stable varieties.

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We see in Section 6.3 that assumption (6.16.2) can be weakened, leading to a moduli space with the same underlying reduced space, but with a larger nilpotent structure. The difference between the two versions is explored in Section 6.6.

Theorem 6.18 *KSB-stability, as in (6.15–6.16) is a good moduli theory (6.10)*

Proof As we already noted, only the conditions (6.10.2–4) need checking. For valuative-properness, the stable extension exists by (2.51), and (2.79.2) shows that it satisfies (6.10.3). The existence of embedded moduli spaces is a trivial special case of (6.13). Representability is a restatement of (3.3).

The coarse moduli space exists by (6.6). □

Let us also note another good property of this case.

Proposition 6.19 *For KSB-stable families as in (6.15–6.16), the Hilbert function $\chi(X, \omega_X^{[m]})$ and the plurigenera $h^0(X, \omega_X^{[m]})$ are deformation invariant.*

Proof For the Hilbert function, this follows from the assumption (6.16.2).

If $m \geq 2$ then the higher cohomologies of $\omega_X^{[m]}$ vanish by (11.34). For $m = 1$ we use (2.69). □

Kollár–Shepherd-Barron stability with standard coefficients

Definition 6.20 Let Δ be an effective $\mathbb{R}$-divisor such that coeff $\Delta \subset (\frac{1}{2}, 1]$, that is, $\frac{1}{2} < \operatorname{coeff}_D \Delta \leq 1$ for every irreducible $D \subset \operatorname{Supp} \Delta$. There is a unique way of writing $\Delta = \sum_i a_i D_i$ where the D_i are effective $\mathbb{Z}$-divisors, $a_i > \frac{1}{2}$ for every i and $a_i \neq a_j$ for $i \neq j$. We call this the *reduced normal form* of Δ.

6.21 (Stable objects) We parametrize pairs $(X, \Delta = \sum_i a_i D_i)$ in reduced normal form such that

(6.21.1) (X, Δ) is slc,

(6.21.2) $a_i \in \{1 - \frac{1}{3}, 1 - \frac{1}{4}, \dots, 1\}$ (diminished standard coefficient set),

(6.21.3) X is projective and $K_X + \Delta$ is ample.

6.22 (Stable families) A family $f\colon (X, \Delta = \sum_i a_i D_i) \to S$ is *KSB-stable* if

(6.22.1) $f\colon (X, \Delta) \to S$ is a flat family of pairs as in (6.11),

(6.22.2) the fibers (X_s, Δ_s) satisfy (6.21.1–2),

(6.22.3) the $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor - B)$ are flat families of divisorial sheaves (3.25) for every $m \in \mathbb{Z}$ and for every $B = \sum_{j \in J} D_j$ where $a_j = 1$ for $j \in J$, and

(6.22.4) f is proper and $\omega_{X/S}^{[M]}(M\Delta)$ is an f-ample line bundle for some $M > 0$.

The first three of these conditions define *locally KSB-stable* families.

6.23 (Explanation) These conditions are rather straightforward generalizations of (6.16.1–3), but why the restriction on the coefficients?

It follows from (2.79.5) and (4.33) that the $B = 0$ parts of condition (6.22.3) are satisfied if the coefficients of Δ are all $1 - \frac{1}{m}$ and S is reduced. For the $B \neq 0$ cases we use (2.79.8) and (4.33). Note that the conditions on B imply that $B \leq \lfloor \Delta \rfloor$. If S is unibranch, then we could have required (6.22.3) to hold for every $B \leq \lfloor \Delta \rfloor$. However, B has to be a generically Cartier divisor; this is assured if B is a sum of some of the D_i. This is the reason of the somewhat awkward formulation of (6.22.3).

We proved in (2.82) that, if the coefficients are $> \frac{1}{2}$, then the scheme-theoretic specializations of the boundary divisors are reduced and the different $(D_i)_s$ have no common irreducible components. In particular, $\lfloor m\Delta \rfloor_s = \lfloor m\Delta_s \rfloor$ for every $s \in S$. That is, valuative-properness holds. Imposing both of these restrictions gives the coefficient set $\{1 - \frac{1}{3}, 1 - \frac{1}{4}, \dots, 1\}$.

Pairs satisfying $\frac{1}{2} < a_i \leq 1$ are studied in (6.26–6.27).

Theorem 6.24 *KSB-stability with standard coefficients, as defined in (6.21–6.22) is a good moduli theory (6.10).*

Proof As before, only (6.10.2–4) need checking. We already noted that valuative-properness holds. The existence of embedded moduli spaces follows from (6.13). For representability, the proof of (3.3) – given in (3.42) – carries over with minor changes.

We apply (3.31) with $N_i := \omega_{X/S}^{[i]}(\lfloor i\Delta \rfloor)$ for $1 \leq i < M$ and $L_1 := \omega_{X/S}^{[M]}$. We get $S^{NL} \to S$ such that all the $\omega_{X^{NL}/S^{NL}}^{[m]}(\lfloor m\Delta^{NL} \rfloor)$ are flat families of divisorial sheaves and $\omega_{X^{NL}/S^{NL}}^{[M]}(\lfloor M\Delta^{NL} \rfloor)$ is invertible.

Then (4.45) shows that S^{KSB} is an open subscheme of S^{NL}. □

Proposition 6.25 (Kollár, 2018a, Cor.3) *For KSB-stable families with standard coefficients as in (6.21–6.22), the Hilbert function* $\chi(X, \omega_X^{[m]}(\lfloor m\Delta \rfloor))$ *and the plurigenera* $h^0(X, \omega_X^{[m]}(\lfloor m\Delta \rfloor))$ *are deformation invariant.*

Proof For the Hilbert function, this follows from (6.22.3). For the plurigenera, write $mK_X + \lfloor m\Delta \rfloor = K_X + (\lfloor m\Delta \rfloor - (m-1)\Delta) + (m-1)(K_X + \Delta)$. Since the coefficients are standard, $0 \leq \lfloor m\Delta \rfloor - (m-1)\Delta \leq \Delta$, hence (11.34) applies, so the higher cohomologies vanish for $m \geq 2$. For $m = 1$ we use (2.69). □

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Kollár–Shepherd-Barron stability with major coefficients

6.26 (Stable objects) We parametrize pairs $(X, \Delta = \sum_i a_i D_i)$ in reduced normal form (6.20) such that

(6.26.1) (X, Δ) is slc,

(6.26.2) $a_i \in (\frac{1}{2}, 1]$,

(6.26.3) X is projective and $K_X + \Delta$ is ample.

6.27 (Stable families) A family $f\colon (X, \Delta = \sum_i a_i D_i) \to S$ is *KSB-stable* if

(6.27.1) $f\colon (X, \Delta) \to S$ is a flat family of pairs as in (6.11),

(6.27.2) the fibers (X_s, Δ_s) satisfy (6.26.1–2),

(6.27.3) the $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor - B)$ are flat families of divisorial sheaves (3.25) for every $m \in \mathbb{Z}$ and for every $B = \sum_{j\in J} D_j$ where $a_j = 1$ for $j \in J$, and

(6.27.4) f is proper and $K_{X/S} + \Delta$ is an f-ample $\mathbb{R}$-divisor.

The first three of these conditions define *locally KSB-stable* families.

For technical reasons we introduce a weakening of (3):

(6.27.3') The $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$ are flat families of divisorial sheaves over S for $m \in M(a_1, \dots, a_r, n) \subset \mathbb{Z}$; a set of positive density defined in (11.49).

6.28 (Explanation) The restriction that the coefficients be in $(\frac{1}{2}, 1]$ is dictated by (2.82). Example (2.41) shows that flatness of the divisorial part fails with coefficient $= \frac{1}{2}$. The requirement (6.27.3) is dictated by (2.83). The choice of B is discussed in (6.23).

We conjecture that (6.27.3) is always the right assumption. However, (2.83) is known only if the general fiber is normal, so we cannot guarantee that (6.27.3) holds for all families of relative dimension ≥ 3.

Theorem 6.29 *KSB-stability with major coefficients, as defined in (6.26–6.27) is a good moduli theory (6.10), satisfying (6.27.3').*

Furthermore, (6.27.3) is satisfied in relative dimension 2 and on those irreducible components that generically parametrize normal varieties.

Proof The proof closely follows (6.24). We proved in (2.82) that, if the coefficients are $> \frac{1}{2}$, then the scheme-theoretic specializations of the boundary divisors are reduced, so assuming that the D_i are flat divisorial sheaves is correct. Following the proofs in (3.42) and (6.24), we can guarantee the requirements (6.27.1–2) and (6.27.4).

The difficulty is with proving that (6.27.3) holds. Following Kollár (2018a), we outlined a proof in (2.83) when the general fibers are normal. Kollár

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(2018b) treats all families of surfaces. Thus (6.29) holds for surfaces and for those irreducible components that generically parametrize normal varieties.

For the version (6.27.3') we use (11.50).

The construction of the moduli space works as before if the a_i are rational. We leave the irrational case to the general theory in Chapter 8; see (8.15). □

Complement 6.29.1 The Hilbert function $\chi(X, \omega_X^{[m]}(\lfloor m\Delta \rfloor))$ is deformation invariant if (6.27.3) holds. Unlike in the earlier cases, the plurigenera need not be deformation invariant; see (Kollár, 2018a, 40–43).

6.3 Strict Viehweg Stability

6.30 (Stable objects) The same as in (6.15): reduced, proper k-schemes X with slc singularities such that K_X is ample.

6.31 (Stable families) A family $f : X \to S$ is *V^+-stable* if the following hold:

(6.31.1) f is flat with slc fibers.

(6.31.2) For every $m \in \mathbb{Z}$ and $x \in X$, $\omega_{X/S}^{[m]}$ is locally free at x iff $\omega_{X_s}^{[m]}$ is locally free at x, where $s = f(x)$.

(6.31.3) f is proper and $\omega_{X/S}^{[M]}$ is an f-ample line bundle for some $M > 0$.

The first two of these conditions define *locally V^+-stable* families.

6.32 (Explanation) The original version in Viehweg (1995) assumes (6.31.2) only for some $m > 0$. By (4.37) the latter is equivalent to V$^+$-stability in characteristic 0, but not in positive characteristic, see Section 8.8.

Already for families of surfaces with quotient singularities this definition gives a large nilpotent structure on the moduli space of stable varieties, even when KSB-stability gives a smooth moduli space, see Section 6.6.

Strict Viehweg Stability with Major Coefficients

6.33 (Stable objects) We parametrize pairs $(X, \Delta = \sum_i a_i D_i)$ in reduced normal form such that

(6.33.1) (X, Δ) is slc,

(6.33.2) $a_i \in (\frac{1}{2}, 1] \cap \mathbb{Q}$ for every i,

(6.33.3) X is projective and $K_X + \Delta$ is ample.

The first two of these conditions define *locally stable* pairs.

6.34 (Stable families) A family $f : (X, \Delta = \sum_i a_i D_i) \to S$ is *V^+-stable* if the following hold:

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(6.34.1) $f\colon X \to S$ is flat and the fibers of $f|_{D_i}\colon D_i \to S$ are reduced subschemes of pure codimension 1 for every i.
(6.34.2) The fibers (X_s, Δ_s) are stable as in (6.33).
(6.34.3) $\omega_{X/S}^{[m]}(m\Delta)$ is locally free along X_s iff $\omega_{X_s}^{[m]}(m\Delta_s)$ is locally free.
(6.34.4) f is proper and $\omega_{X/S}^{[M]}(M\Delta)$ is an f-ample line bundle for some $M > 0$.

The first three of these conditions define *locally* V^+*-stable* families.

6.35 (Explanation) These conditions are rather straightforward generalizations of (6.31) and (6.27).

Theorem 6.36 *V^+-stability with major coefficients, as defined in (6.33–6.34) is a good moduli theory (6.10).*

Proof The arguments given in (6.29) work since we no longer require the condition (6.27.3) that gave us trouble there.

Representability is actually simpler, since we work only with the locally free $\omega_{X/S}^{[M]}(M\Delta)$ and ignore the other $\omega_{X/S}^{[m]}(\lfloor m\Delta\rfloor)$. □

6.4 Alexeev Stability

6.37 (Stable objects) We parametrize pairs $(X, \Delta = \sum_i a_i D_i)$ in reduced normal form (6.20) such that
(6.37.1) (X, Δ) is slc,
(6.37.2) $1, a_1, \dots, a_r$ are $\mathbb{Q}$-linearly independent,
(6.37.3) X is projective and $K_X + \Delta$ is ample.

6.38 (Stable families) A family $f\colon (X, \Delta = \sum_i a_i D_i) \to S$ is *A-stable* if the following hold:
(6.38.1) $f\colon (X, \Delta) \to S$ is a flat family of pairs as in (6.11).
(6.38.2) The fibers (X_s, Δ_s) are stable as in (6.37).
(6.38.3) The $\omega_{X/S}^{[m_0]}(\sum m_i D_i)$ are flat families of divisorial sheaves (3.25) over S for every $m_i \in \mathbb{Z}$.
(6.38.4) f is proper and $K_{X/S} + \Delta$ is an f-ample $\mathbb{R}$-divisor.

The first three of these conditions define *locally A-stable* families.

6.39 (Explanation) The two new features are the $\mathbb{Q}$-linear independence in (6.37) and (6.38.3).

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Let us start with $\mathbb{Q}$-linear independence. As a simple example, let X be a smooth, projective variety and $\sum D_i$ an snc divisor with index set $\{i \in I\}$. Then $(X, \sum a_i D_i)$ is an lc pair for every $a_i \in [0,1]$. So we can ask how the answers to various questions – for example the ampleness of $K_X + \sum a_i D_i$, or the steps of the MMP – depend on the a_i.

In many cases, the answer is that $[0,1]^I$ admits a rational chamber decomposition such that the answers depend only on the chamber we are in, not the particular choice of the $\{a_i : i \in I\}$ inside the chamber. There is reason to expect that if a point $\{a'_i : i \in I\}$ lies in an open chamber, then $K_X + \sum a'_i D_i$ exhibits generic – hence simplest – behavior.

Since the chambers are polyhedra with rational vertices, a point $\{a'_i : i \in I\}$ whose coordinates are $\mathbb{Q}$-linearly independent, must lie in an open chamber. Thus assumption (6.37.2) is a convenient way to guarantee that we encounter the generic behavior.

By (6.38.4), $K_{X/S} + \sum_i a_i D_i$ is $\mathbb{R}$-Cartier. By (11.43), the $\mathbb{Q}$-linear independence assumption implies that $K_{X/S}$ and the D_i are $\mathbb{Q}$-Cartier. Thus all the $m_0 K_{X/S} + \sum m_i D_i$ are $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisors. Therefore all the sheaves in (6.38.3) should be flat over S with S_2 fibers by (2.79.1). This gives a moduli space with many flat universal sheaves, and, as we see next, it also helps with the proof of existence.

Finally note that, since the D_i are not assumed irreducible, $\lfloor m \sum_i a_i D_i \rfloor$ may not be a linear combination of the D_i, so we do not assume anything about the sheaves $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$. If the $a_i < \frac{1}{2}$, then these frequently do not have S_2 fibers (2.41–2.44). Although (11.50) shows that infinitely many of them do, it is not clear how to predict which ones.

Theorem 6.40 *A-stability, as in (6.37–6.38) is a good moduli theory (6.10).*

Proof As before, separatedness and valuative-properness holds. The idea of the proof of the existence of embedded moduli spaces is the following. The chamber structure mentioned in (6.39) suggests that, if we pick a rational point $(a'_1, \dots, a'_r)$ in the interior of the chamber, then the pairs $(X, \sum a_i D_i)$ and $(X, \sum a'_i D_i)$ have the same moduli theory. We can thus work with the rational-coefficient pairs $(X, \sum a'_i D_i)$ as in (6.13). This is basically what we do, but the details are more complicated. See (8.21) for a full treatment.

Representability needs a somewhat different proof. The set of slc fibers is constructible by (4.44), hence there are $M_i > 0$ such that $M_0 K_{X_s}$ and the $M_i D_i|_{X_s}$ are Cartier whenever (X_s, Δ_s) is slc.

We apply (3.31) where the set $\{N\}$ consists of the sheaves $\omega_{X/S}^{[m_0]}(\sum m_i D_i)$ for $0 \le m_i \le M_i$ and the set $\{L\}$ of the sheaves $\omega_{X/S}^{[m_0]}, \mathscr{O}_X(M_1 D_1), \dots, \mathscr{O}_X(M_r D_r)$.

We get $S^{NL} \to S$ such that the $\omega^{[m_0]}_{X^{NL}/S^{NL}}(\sum m_i D_i^{NL})$ are flat families of divisorial sheaves for all $m_i \in \mathbb{Z}$ and $\omega^{[M_0]}_{X^{NL}/S^{NL}}, \mathscr{O}_{X^{NL}}(M_1 D_1^{NL}), \dots, \mathscr{O}_{X^{NL}}(M_r D_r^{NL})$ are all invertible. Then (4.45) shows that S^{A} is an open subscheme of S^{NL}. □

6.5 First Order Deformations

In this section, we study first order infinitesimal deformations of normal varieties. We describe the deformations of the smooth locus and then try to understand when a deformation of the smooth locus extends to a deformation of the whole variety. The final aim is to get an explicit obstruction theory for lifting sections of powers of the dualizing sheaf. This turns out to be given by the classical notion of divergence.

6.41 (First order thickening) Let k be a field and R a k-algebra. Consider the algebra $R[\varepsilon]$ where ε is a new variable satisfying $\varepsilon^2 = 0$. It is flat over $k[\varepsilon]$ and $R[\varepsilon] \otimes_{k[\varepsilon]} k \simeq R$. We think of $R[\varepsilon]$ as the trivial first order deformation of R.

Let $v\colon R \to R$ be a k-linear derivation. Then

$$\alpha_v\colon r_1 + \varepsilon r_2 \mapsto r_1 + \varepsilon\big(v(r_1) + r_2\big) \tag{6.41.1}$$

defines an automorphism of $R[\varepsilon]$ that is trivial modulo (ε). Conversely, every automorphism of $R[\varepsilon]$ that is trivial modulo (ε) arises this way. (The product (or Leibniz) rule for v is equivalent to the multiplicativity of α_v.)

Let X be a k-scheme. The trivial first order deformation of X is

$$X[\varepsilon] := X \times_k \operatorname{Spec}_k k[\varepsilon]. \tag{6.41.2}$$

As in (6.41.1), every derivation $v\colon \mathscr{O}_X \to \mathscr{O}_X$ defines an automorphism α_v of $X[\varepsilon]$ that is trivial modulo (ε). This gives an exact sequence

$$0 \to \operatorname{Hom}(\Omega^1_X, \mathscr{O}_X) \to \operatorname{Aut}(X[\varepsilon]) \to \operatorname{Aut}(X) \to 1. \tag{6.41.3}$$

If X is smooth, or at least normal, then $\mathcal{H}om(\Omega^1_X, \mathscr{O}_X)$ is the tangent sheaf T_X of X, hence we can rewrite the sequence as

$$0 \to H^0(X, T_X) \xrightarrow{\alpha} \operatorname{Aut}(X[\varepsilon]) \to \operatorname{Aut}(X) \to 1. \tag{6.41.4}$$

Aside On a differentiable manifold M one can identify the Lie algebra of all vector fields with the Lie algebra of the automorphism group. If X is a smooth variety, then this identification works if X is proper, but not otherwise. For instance, an affine curve C of genus ≥ 1 has only finitely many automorphisms, but $H^0(C, T_C)$ is infinite dimensional. Infinitesimal thickenings restore the connection between vector fields and automorphisms.

6.42 (Locally trivial first order deformations) Let k be a field and X a k-scheme. A *deformation* of X over $A := \operatorname{Spec}_k k[\varepsilon]$ is a flat A-scheme X' together with an isomorphism $X' \times_A \operatorname{Spec} k \simeq X$. The set of isomorphism classes of first order deformations is denoted by $T^1(X)$. It is easy to see that $T^1(X)$ is naturally a k-vector space whose zero is the trivial deformation $X[\varepsilon]$, but this is not very important for us now. See Artin (1976) or Hartshorne (2010) for detailed discussions.

We say that X' is *locally trivial* if there is an affine cover $X = \cup_i X_i$ such that each X'_i is a trivial deformation of X_i. We aim to classify all locally trivial first order deformations of arbitrary k-schemes X, but our main interest is in cases when X is smooth and quasi-projective.

Let $X = \cup_i X_i$ be an affine cover. This gives an affine cover $X' = \cup_i X'_i$ and we assume that each X'_i is a trivial deformation of X_i. Fix trivializations $\phi_i \colon X'_i \simeq X_i[\varepsilon]$. Over $X'_{ij} := X'_i \cap X'_j$ we have two trivializations, these differ by an automorphism

$$\alpha_{ij} := \phi_j^{-1} \circ \phi_i \colon X'_{ij} \to X'_{ij}, \tag{6.42.1}$$

which is the identity on X_{ij}. By (6.41.1), the automorphisms α_{ij} correspond to $v_{ij} \in \operatorname{Hom}(\Omega^1_{X_{ij}}, \mathscr{O}_{X_{ij}})$ and these form a 1-cocycle $D := \{v_{ij}\}$. Changing the trivializations changes the cocyle by a coboundary. Thus we get a well defined

$$D = D(X') \in H^1(X, \mathcal{H}om(\Omega^1_X, \mathscr{O}_X)). \tag{6.42.2}$$

The construction can be reversed. It is left to the reader to check that $D(X')$ is independent of the choices we made. The final outcome is the following.

Claim 6.42.3 Let X be a k-scheme. There is a one-to-one correspondence, denoted by $D \mapsto X_D$, between

(a) elements of $H^1(X, \mathcal{H}om(\Omega^1_X, \mathscr{O}_X))$, and

(b) locally trivial deformations of X over $\operatorname{Spec}_k k[\varepsilon]$, up-to isomorphism.

Furthermore, if X is normal then $H^1(X, \mathcal{H}om(\Omega^1_X, \mathscr{O}_X)) = H^1(X, T_X)$. □

Next we check that every first order deformation of a smooth variety Y is locally trivial. To see this, we may assume that Y is affine. Then Y' is also affine and we can fix a vector space isomorphism $k[Y'] \simeq k[Y] \otimes k[\varepsilon]$. Pick a point $p \in Y$, local coordinates $y_1, \dots, y_n$. Then $k(Y)$ is separable over $k(y_1, \dots, y_n)$. Choose arbitrary lifts $y'_1, \dots, y'_n \in k[Y']$. Any other $z \in k[Y]$ satisfies a monic, separable equation $F(z, \mathbf{y}) = 0$. We claim that z has a unique lift $z' \in k(Y')$ such that $F(z', \mathbf{y}') = 0$. To see this pick any lift z^*. Then $F(z^*, \mathbf{y}') = \varepsilon G(z)$ for some $G(z) \in k[Y]$. We are looking for z' in the form $z' = z^* + \varepsilon g$ where $g \in k[Y]$. Since $F(z^* + \varepsilon g, \mathbf{y}') = \varepsilon G(z) + \varepsilon g \cdot \partial F(z, \mathbf{y})/\partial z$, we see that $g = -G(z)(\partial F(z, \mathbf{y})/\partial z)^{-1}$

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is the unique solution. We do this for a finite set of generators $\{z_i\}$ of $k[Y]$ to get a trivialization in a neighborhood where all the $\partial F_i(z, \mathbf{y})/\partial z$ are invertible.

Combining with (6.42.3), this shows that every deformation of a smooth, affine variety over $k[\varepsilon]$ is trivial. (See Hartshorne (1977, exc.II.8.6) for a slightly different proof.)

6.43 (Arbitrary first order deformations) Let k be a field and X a normal k-variety. Let $U \subset X$ be the smooth locus, $Z \subset X$ the singular locus, and $j\colon U \hookrightarrow X$ the natural injection.

Let $X' \to \operatorname{Spec}_k k[\varepsilon]$ be a flat deformation of X. By restriction, it induces a flat deformation U' of U. Note that U' uniquely determines X'. Indeed, $\operatorname{depth}_Z \mathscr{O}_X \geq 2$ since X is normal, hence $\operatorname{depth}_Z \mathscr{O}_{X'} \geq 2$ since $\mathscr{O}_{X'}$ is an extension of two copies of $\mathscr{O}_X$. Therefore $\mathscr{O}_{X'} = j_*\mathscr{O}_{U'}$ by (10.6). Thus we have an injection $T^1(X) \hookrightarrow T^1(U) = H^1(U, T_U)$.

Following Schlessinger (1971), our plan is to study $T^1(X)$ by first describing $T^1(U)$ and then understanding which $D \in H^1(U, T_U)$ correspond to a deformation of X; see also von Essen (1990). The second step is in (6.46).

Definition 6.44 Let X be a k-scheme. Given $v \in \operatorname{Hom}(\Omega^1_X, \mathscr{O}_X)$, *differentiation* by v is defined as the composite

$$v(\ )\colon \mathscr{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{v} \mathscr{O}_X. \tag{6.44.1}$$

Let $x_1, \dots, x_n$ be (analytic or étale) local coordinates at a smooth point of X and write $v = \sum_i v_i \frac{\partial}{\partial x_i}$. Then the maps are

$$v\colon f \mapsto \textstyle\sum_i \frac{\partial f}{\partial x_i} dx_i \mapsto \sum_i v_i \frac{\partial f}{\partial x_i}.$$

Thus if X is smooth and v is identified with a section of T_X, then (6.44.1) agrees with the usual definition.

Next let $D \in H^1(X, \mathcal{H}om(\Omega^1_X, \mathscr{O}_X))$ and choose a representative 1-cocycle $D = \{v_{ij}\}$ using an affine cover $X = \cup X_i$. For any $s \in H^0(X, \mathscr{O}_X)$ the derivatives $\{v_{ij}(s|_{X_{ij}})\}$ form a 1-cocycle with values in $\mathscr{O}_X$. This defines $D(s) \in H^1(X, \mathscr{O}_X)$. We think of it either as a *cohomological differentiation* map

$$D\colon H^0(X, \mathscr{O}_X) \to H^1(X, \mathscr{O}_X), \tag{6.44.2}$$

or as a k-bilinear map

$$H^1(X, \mathcal{H}om(\Omega^1_X, \mathscr{O}_X)) \times H^0(X, \mathscr{O}_X) \to H^1(X, \mathscr{O}_X). \tag{6.44.3}$$

If X is normal, then we can rewrite this as

$$H^1(X, T_X) \times H^0(X, \mathscr{O}_X) \to H^1(X, \mathscr{O}_X). \tag{6.44.4}$$

Let X_D be the deformation of X corresponding to D. Its structure sheaf sits in an exact sequence

$$0 \to \varepsilon\mathscr{O}_X \to \mathscr{O}_{X_D} \to \mathscr{O}_X \to 0. \tag{6.44.5}$$

Taking cohomology, we see that D in (6.44.2) is the connecting map

$$H^0(X_D, \mathscr{O}_{X_D}) \to H^0(X, \mathscr{O}_X) \xrightarrow{D} H^1(X, \mathscr{O}_X). \tag{6.44.6}$$

Warning 6.44.7 Since the constant $1_X \in H^0(X, \mathscr{O}_X)$ always lifts, $D(1_X) = 0$. Thus D is an $H^0(X, \mathscr{O}_X)$-module homomorphism iff it is identically 0.

We can summarize the above considerations as follows.

Lemma 6.45 *Let X be a k-scheme, $D \in H^1(X, \mathcal{H}om(\Omega^1_X, \mathscr{O}_X))$ and X_D the corresponding deformation of X. Then a global section $s \in H^0(X, \mathscr{O}_X)$ lifts to $s_D \in H^0(X_D, \mathscr{O}_{X_D})$ iff $D(s) \in H^1(X, \mathscr{O}_X)$ is* 0. □

Corollary 6.46 *Let X be a normal, affine variety and $U \subset X$ its smooth locus. Let U_D be the deformation of U corresponding to $D \in H^1(U, T_U)$. Then*

(6.46.1) *U_D extends to a flat deformation X_D of X iff D (as in (6.44.2)) is identically 0.*

(6.46.2) *$T^1(X)$ is the left kernel of $H^1(U, T_U) \times H^0(U, \mathscr{O}_U) \to H^1(U, \mathscr{O}_U)$.*

Proof Assume that U_D extends to a flat deformation X_D of X. Since X is affine, so is X_D and so $H^0(X_D, \mathscr{O}_{X_D}) \to H^0(X, \mathscr{O}_X)$ is surjective. Thus D is identically 0 by (6.45).

Conversely, if D is identically 0, then $H^0(U_D, \mathscr{O}_{U_D}) \to H^0(U, \mathscr{O}_U)$ is surjective and $H^0(U, \mathscr{O}_U) = H^0(X, \mathscr{O}_X)$ since X is normal. We can then take $X_D := \operatorname{Spec}_k H^0(U_D, \mathscr{O}_{U_D})$. This proves the first claim and the second is a reformulation of it. □

Remark 6.47 If X is not affine, then $D \in H^1(U, T_U)$ gives a k-linear map $D\colon \mathscr{O}_X = j_*\mathscr{O}_U \to R^1 j_*\mathscr{O}_U \simeq \mathscr{H}^2_Z(\mathscr{O}_X)$ where $Z := X \setminus U$ is the singular locus. Then U_D extends to a flat deformation X_D of X iff $D\colon \mathscr{O}_X \to \mathscr{H}^2_Z(\mathscr{O}_X)$ is identically 0.

6.48 (Lie derivative) Let M be a smooth, real manifold and v a vector field on M. By integrating v we get a 1-parameter family of diffeomorphisms ϕ_t of M. The *Lie derivative* of a covariant tensor field S is defined as

$$L_v S := \tfrac{d}{dt}(\phi_t^* S)_{t=0}. \tag{6.48.1}$$

In local coordinates $\{y_i\}$, write $v = \sum_i v_i \frac{\partial}{\partial y_i}$. The Lie derivatives of a function s and of a 1-form dy_j are given by the formulas

$$L_v s = v(s) = \textstyle\sum_i v_i \frac{\partial s}{\partial y_i} \quad \text{and} \quad L_v(dy_j) = dv_j. \tag{6.48.2}$$

Since functions and 1-forms generate the algebra of covariant tensors, the Lie derivative is uniquely determined by the formulas (6.48.2). One can extend the definition to all tensors by duality.

We can transplant this definition to algebraic geometry as follows.

Let Y be a smooth variety over a field k and $v \in H^0(Y, T_Y)$ a vector field. By (6.41.4) v can be identified with an automorphism α_v of $Y[\varepsilon]$. We write Ω_Y for the module of derivations (frequently denoted by Ω^1_Y). The covariant tensors are sections of the algebra $\sum_{m\geq 0} \Omega_Y^{\otimes m}$.

Let $S \in H^0(Y, \sum_{m\geq 0} \Omega_Y^{\otimes m})$ be a covariant tensor on Y. It has a trivial extension to $Y[\varepsilon]$; denote it by $S[\varepsilon]$. Thus $\alpha_v^*(S[\varepsilon])$ is a global section of $\sum_{m\geq 0} \Omega^{\otimes m}_{Y[\varepsilon]}$. Since α_v is the identity on Y, $\alpha_v^*(S[\varepsilon]) - S[\varepsilon]$ is divisible by ε and we can define the *Lie derivative* of S by the formula

$$\alpha_v^*(S[\varepsilon]) = S[\varepsilon] + \varepsilon L_v S. \tag{6.48.3}$$

Expanding the identity $\alpha_v^*(S_1[\varepsilon] \otimes S_2[\varepsilon]) = \alpha_v^*(S_1[\varepsilon]) \otimes \alpha_v^*(S_2[\varepsilon])$ shows that the Lie derivative is a k-linear derivation of the tensor algebra

$$L_v \colon \oplus_{m\geq 0} \Omega_Y^{\otimes m} \to \oplus_{m\geq 0} \Omega_Y^{\otimes m}. \tag{6.48.4}$$

The Lie derivative preserves natural quotient bundles of $\Omega_Y^{\otimes m}$. Thus we get similar maps L_v for symmetric and skew-symmetric tensors. Our main interest is in powers of ω_Y. The corresponding map

$$L_v \colon \omega_Y^m \to \omega_Y^m \tag{6.48.5}$$

is obtained using the identification $\Omega_Y^{\otimes n} \twoheadrightarrow \Omega_Y^n = \omega_Y$ where $n = \dim Y$.

From (6.41.1), we see that

$$\alpha_v^*(s[\varepsilon]) = s[\varepsilon] + \varepsilon v(s) \quad \text{and} \quad \alpha_v^*(dy_j) = d(\alpha_v^*(y_j)) = dy_j + \varepsilon dv_j. \tag{6.48.6}$$

Comparing with (6.48.2), we see that the algebraic definition coincides with the differential geometry definition.

6.49 (Cartan formula) This is an identity which holds for exterior forms S

$$L_v(S) = d(v \lrcorner S) + v \lrcorner dS, \tag{6.49.1}$$

where $\lrcorner$ denotes *contraction* or *inner product* by a vector field $v \in H^0(Y, T_Y)$ obtained as follows. We have the contraction map $T_Y \otimes \Omega_Y^m \to \Omega_Y^{m-1}$, thus every $v \in H^0(Y, T_Y)$ gives the $\mathscr{O}_Y$-linear map

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$$v\lrcorner\colon \Omega_Y^m \to \Omega_Y^{m-1}. \tag{6.49.2}$$

In (analytic or étale) local coordinates $y_1, \dots, y_n$, write $v = \sum_i v_i \frac{\partial}{\partial y_i}$. Then

$$v\lrcorner(dy_1 \wedge \cdots \wedge dy_m) = \textstyle\sum_r (-1)^{r-1} v_r \cdot dy_1 \wedge \cdots \wedge \widehat{dy_r} \wedge \cdots \wedge dy_m, \tag{6.49.3}$$

where the hat indicates that we omit that term.

To prove (6.49.1), one first checks that $S \mapsto d(v\lrcorner S) + v\lrcorner dS$ is also a derivation. Thus it is sufficient to verify (6.49.1) for a generating set of exterior forms. For functions and for dy_j we recover the identities (6.48.2).

6.50 As in (6.44), let Y be a smooth k-variety. Pick $D \in H^1(Y, T_Y)$ and choose a representative 1-cocycle $D = \{v_{ij}\}$ using an affine cover $Y = \cup Y_i$. For any $S \in H^0(Y, \Omega_Y^{\otimes m})$ the Lie derivatives $\{L_{v_{ij}}(S|_{Y_{ij}})\}$ form a 1-cocycle with values in $\Omega_Y^{\otimes m}$. This defines

$$L_D(S) \in H^1(Y, \Omega_Y^{\otimes m}), \tag{6.50.1}$$

which we view as a *cohomological differentiation* map

$$L_D\colon \oplus H^0(Y, \Omega_Y^{\otimes m}) \to \oplus H^1(Y, \Omega_Y^{\otimes m}). \tag{6.50.2}$$

As we noted in (6.48), the map L_D respects natural quotient bundles of $\Omega_Y^{\otimes m}$. Thus we get similar maps for symmetric and skew-symmetric tensors and for powers of ω_Y

$$L_D\colon \oplus H^0(Y, \omega_Y^m) \to \oplus H^1(Y, \omega_Y^m). \tag{6.50.3}$$

For $m = 0$, the map L_D agrees with the map D defined in (6.44.2).

As in (6.44.7), L_D is a k-linear differentiation which is usually not $H^0(Y, \mathscr{O}_Y)$-linear. However, if $D\colon H^0(Y, \mathscr{O}_Y) \to H^1(Y, \mathscr{O}_Y)$ is 0, then L_D is $H^0(Y, \mathscr{O}_Y)$-linear; this holds both for (6.50.2) and (6.50.3).

Arguing as in (6.45), we obtain the following lifting criterion.

Lemma 6.51 *Let Y be a smooth k-variety and Y_D a first order deformation of Y. Then $S \in H^0(Y, \Omega_Y^{\otimes m})$ lifts to $S_D \in H^0(Y_D, \Omega_{Y_D}^{\otimes m})$ iff $L_D(S) \in H^1(Y, \Omega_Y^{\otimes m})$ is* 0. *The same holds for all natural quotient bundles of $\Omega_Y^{\otimes m}$.* □

Next we consider what the previous method gives for ω_Y and its powers.

On $M := \mathbb{R}^n$ with coordinates y_i, the divergence of a vector field $v = \sum v_i \frac{\partial}{\partial y_i}$ is $\nabla \cdot v := \sum \frac{\partial v_i}{\partial y_i}$. Note that the y_i give an n-form $\sigma = dy_1 \wedge \cdots \wedge dy_n$, which gives isomorphisms $T_M \simeq \mathcal{H}om(\Omega_M^n, \Omega_M^{n-1}) \simeq \Omega_M^{n-1}$. This identifies the divergence with exterior derivation $d\colon \Omega_M^{n-1} \to \Omega_M^n$.

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6.52 (Divergence) More generally, let Y be a smooth k-variety, $\sigma \in H^0(Y, \omega_Y^m)$ and $v \in H^0(Y, T_Y)$. Then σ and $L_v\sigma$ (6.48.5) are both sections of the line bundle ω_Y^m, hence their quotient is a rational function, called the *divergence* of v with respect to σ,

$$\nabla_\sigma v := \frac{L_v\sigma}{\sigma}. \tag{6.52.1}$$

(Most books seem to use this terminology only when σ is a nowhere 0 section of ω_Y, and σ is frequently suppressed in the notation.)

In order to compute this, start with a section σ of ω_Y. Since $d\sigma = 0$, Cartan's formula (6.49) shows that $L_v\colon \omega_Y \to \omega_Y$ is the composite map

$$L_v\colon \omega_Y = \Omega_Y^n \xrightarrow{v\lrcorner} \Omega_Y^{n-1} \xrightarrow{d} \Omega_Y^n = \omega_Y. \tag{6.52.2}$$

In local coordinates $y_1, \dots, y_n$, assume that $\sigma = dy_1 \wedge \cdots \wedge dy_n$ and $v = \sum_i v_i \frac{\partial}{\partial y_i}$. Contraction by v sends σ to

$$\textstyle\sum_i (-1)^{i-1} v_i\, dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n. \tag{6.52.3}$$

Exterior differentiation now gives that

$$L_v\sigma = d(v\lrcorner\sigma) = \textstyle\sum_i \frac{\partial v_i}{\partial y_i} \cdot \sigma, \tag{6.52.4}$$

which is the usual formula for the divergence. Thus, if σ is a nowhere 0 section of ω_Y^m, then we get the divergence as a k-linear map $\nabla_\sigma\colon T_Y \to \mathscr{O}_Y$. Thus it induces a map on cohomologies; we are especially interested in

$$\nabla_\sigma\colon H^1(Y, T_Y) \to H^1(Y, \mathscr{O}_Y). \tag{6.52.5}$$

For powers of ω_Y, we get the next formula.

Lemma 6.53 *Let Y be a smooth k-variety of dimension n. Let $v \in H^0(Y, T_Y)$ be a vector field, $s \in H^0(Y, \mathscr{O}_Y)$ a function, and $\sigma \in H^0(Y, \omega_Y)$ an n-form. Then*

$$\nabla_{(s\sigma^m)} v = \tfrac{v(s)}{s} + m\nabla_\sigma v. \tag{6.53.1}$$

Proof This is really just the assertion that the Lie derivative is a derivation, but it is instructive to do the local computations.

The claimed identities are local, so we use local coordinates $y_1, \dots, y_n$ and assume that $\sigma = dy_1 \wedge \cdots \wedge dy_n$. Write $v = \sum_i v_i \frac{\partial}{\partial y_i}$. We need to compute how the isomorphism α_v acts on $s\sigma^m$. It sends y_i to $y_i + \varepsilon v(y_i) = y_i + \varepsilon v_i$, thus

$$\alpha_v^*(dy_i) = (1 + \varepsilon \tfrac{\partial v_i}{\partial y_i}) dy_i + \varepsilon(\textstyle\sum_{j\neq i} \frac{\partial v_i}{\partial y_j} dy_j). \tag{6.53.2}$$

Next we wedge these together. Any two epsilon terms wedge to 0 since $\varepsilon^2 = 0$. Thus $\varepsilon(\sum_{j\neq i} \frac{\partial v_i}{\partial y_j} dy_j)$ gets killed unless it is wedged with all the other dy_j, but the result is then 0 in the exterior algebra. The only term that survives is

$$\begin{aligned} \textstyle\prod_i (1 + \varepsilon \frac{\partial v_i}{\partial y_i}) \cdot dy_1 \wedge \cdots \wedge dy_n &= (1 + \varepsilon \textstyle\sum_i \frac{\partial v_i}{\partial y_i}) \cdot dy_1 \wedge \cdots \wedge dy_n \\ &= (1 + \varepsilon \nabla_{\mathbf{y}} v) \cdot dy_1 \wedge \cdots \wedge dy_n. \end{aligned} \tag{6.53.3}$$

Thus we get that $s\sigma^m$ is mapped to

$$\begin{aligned} (s + \varepsilon v(s))(1 + m\varepsilon \nabla_{\mathbf{y}} v) \cdot \sigma^m &= (s + \varepsilon v(s) + m\varepsilon s \nabla_{\mathbf{y}} v) \cdot \sigma^m \\ &= s\sigma^m + \varepsilon \cdot (\tfrac{v(s)}{s} + m \nabla_{\mathbf{y}} v) \cdot s\sigma^m. \quad \square \end{aligned}$$

Notation 6.54 Let X be a normal, affine k-variety and X_D a flat deformation of X over $k[\varepsilon]$ corresponding to $D \in T^1(X)$. Let $U \subset X$ be the smooth locus. By (6.43), we can think of D as a cohomology class $D \in H^1(U, T_U)$. By (6.44.2), D induces a map

$$D\colon H^0(U, \mathscr{O}_U) \to H^1(U, \mathscr{O}_U) \tag{6.54.1}$$

which is identically 0 by (6.46.2). There is a natural exact sequence

$$0 \to \varepsilon \cdot \omega_U^m \to \omega_{U_D}^m \to \omega_U^m \to 0. \tag{6.54.2}$$

Taking cohomologies gives an exact sequence

$$H^0(U_D, \omega_{U_D}^m) \to H^0(U, \omega_U^m) \stackrel{\delta_m}{\to} H^1(U, \omega_U^m). \tag{6.54.3}$$

As we noted in (6.50), δ_m is $H^0(U, \mathscr{O}_U)$-linear since D in (6.54.1) is 0.

It was observed in Stevens (1988) that, for cyclic quotients, the deformation obstruction equals the divergence. The next result shows that this is a general phenomenon.

Theorem 6.55 *Let X, $U \subset X$, $D = \{v_{ij}\} \in H^1(U, T_U)$ and X_D be as in (6.54). Assume that ω_U^m has a nowhere 0 section σ_m for some $m > 0$ such that* $\operatorname{char} k \nmid m$. *As in (6.52.5), we get $\nabla_{\sigma_m} D := \{\nabla_{\sigma_m}(v_{ij})\} \in H^1(U, \mathscr{O}_U)$. Then*

(6.55.1) *$\nabla D := \frac{1}{m} \nabla_{\sigma_m} D \in H^1(U, \mathscr{O}_U)$ is independent of m and σ_m.*

(6.55.2) *The boundary map $\delta_m\colon H^0(U, \omega_U^m) \to H^1(U, \omega_U^m)$ defined in (6.54.3) is multiplication by $m\nabla D$.*

(6.55.3) *$\omega_{U_D}^m$ is free $\Leftrightarrow \nabla D = 0$ in $H^1(U, \mathscr{O}_U)$.*

Proof Choose affine charts $\{U_i\}$ on U such that $D = \{v_{ij}\}$ and $\sigma_m|_{U_{ij}} = s_{ij}\sigma_{ij}^m$ for some $\sigma_{ij} \in H^0(U_{ij}, \omega_{U_{ij}})$. Any other section of ω_U^m can be written as $g\sigma_m$ where $g \in H^0(U, \mathscr{O}_U)$. Using (6.53), we obtain that

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$$\nabla_{\sigma_m} D = \{\nabla_{\sigma_m}(v_{ij})\} = \left\{\frac{v_{ij}(s_{ij})}{s_{ij}} + m\nabla_{\sigma_{ij}}(v_{ij})\right\}. \tag{6.55.4}$$

Similarly, we get that

$$\nabla_{g\sigma_m} D = \left\{\frac{v_{ij}(gs_{ij})}{gs_{ij}} + m\nabla_{\sigma_{ij}}(v_{ij})\right\}. \tag{6.55.5}$$

Since

$$\frac{v_{ij}(gs_{ij})}{gs_{ij}} = \frac{v_{ij}(g)}{g} + \frac{v_{ij}(s_{ij})}{s_{ij}}, \tag{6.55.6}$$

subtracting (6.55.4) from (6.55.5) yields

$$\nabla_{g\sigma_m} D - \nabla_{\sigma_m} D = \tfrac{1}{g} D(g) \in H^1(U, \mathscr{O}_U). \tag{6.55.7}$$

As we noted in (6.54), $D(g) = 0$ in $H^1(U, \mathscr{O}_U)$. Thus $\nabla_{g\sigma_m} D = \nabla_{\sigma_m} D$ (as classes in $H^1(U, \mathscr{O}_U)$). Independence of the choice of m is shown by the formula

$$\nabla_{(\sigma_m^r)} D = \left\{\frac{v_{ij}(s_{ij}^r)}{s_{ij}^r} + rm\nabla_{\sigma_{ij}}(v_{ij})\right\} = r \cdot \left\{\frac{v_{ij}(s_{ij})}{s_{ij}} + m\nabla_{\sigma_{ij}}(v_{ij})\right\}. \tag{6.55.8}$$

Thus ∇D is well defined and this proves (1–2).

Finally, $\omega_{U_D}^m$ is free iff σ_m lifts to a section of $\omega_{X_D}^m$, and $\nabla D \cdot \sigma_m$ is the lifting obstruction. This implies (3). □

Remark 6.56 Let $x \in X$ be an isolated normal singularity and $U := X \setminus \{x\}$. Then $H^1(U, \mathscr{O}_U) = H_x^2(X, \mathscr{O}_X)$ and $H^1(U, T_U) = H_x^2(X, T_X)$. Thus if $\omega_U^m \simeq \mathscr{O}_U$ for some $m > 0$ then the divergence in (6.52.5) becomes a map

$$\nabla\colon T^1(X) \to H_x^2(X, \mathscr{O}_X).$$

If $\operatorname{depth}_x \mathscr{O}_X \geq 3$, then $H_x^2(X, \mathscr{O}_X) = 0$ by Grothendieck's vanishing theorem (10.29.5), thus in this case the divergence vanishes and sections of ω_U^m lift to all first order deformations. This, however, already follows from (6.54.3) since $H^1(U, \omega_U^m) = H^1(U, \mathscr{O}_U) = H_x^2(X, \mathscr{O}_X) = 0$.

If X is lc and ω_X is locally free, then sections of ω_X lift to any deformation by Kollár and Kovács (2020); see also (2.67). By (6.55), this implies that $\nabla\colon T^1(X) \to H^1(U, \mathscr{O}_U)$ is the zero map.

This should either have a direct proof or some interesting consequences.

Next we give explicit forms of the maps in the general theory for $X := \mathbb{A}^2$ and $U := \mathbb{A}^2 \setminus \{(0,0)\}$. At first this seems quite foolish to do since we already know that a smooth affine variety has only trivial infinitesimal deformations. However, we will be able to use these computations to get very

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detailed information about deformations of two-dimensional cyclic quotient singularities.

Notation 6.57 Let k be a field, $X = \mathbb{A}^2_{xy}$ and $U := X \setminus \{(0,0)\}$. Using the affine charts $U_0 := U \setminus (x = 0)$, $U_1 := U \setminus (y = 0)$ and $U_{01} := U \setminus (xy = 0)$, we compute that

$$H^1(U, \mathscr{O}_U) = \left\langle \tfrac{1}{x^i y^j} : i, j \geq 1 \right\rangle \tag{6.57.1}$$

and also that

$$H^1(U, T_U) = \left\langle \tfrac{1}{x^i y^j} \cdot \tfrac{\partial}{\partial x},\ \tfrac{1}{x^i y^j} \cdot \tfrac{\partial}{\partial y} : i, j \geq 1 \right\rangle.$$

Note that $H^1(U, \mathscr{O}_U)$ is naturally a quotient of

$$H^0(U_{01}, \mathscr{O}_{U_{01}}) = k[x^{-i}y^{-j} : i, j \in \mathbb{Z}];$$

but the basis in (6.57.1) depends on the choice of coordinates x, y. Similarly, $H^1(U, T_U)$ is naturally a quotient of $H^0(U_{01}, T_{U_{01}})$.

It is very convenient computationally that the diagonal subgroup $\mathbb{G}_m^2 \subset \mathrm{GL}_2$ acts on these cohomology groups and subsequent constructions are $\mathbb{G}_m^2$-equivariant. In order to keep track of this action, it is better to use the $\mathbb{G}_m^2$-invariant differential operators

$$\partial_x := x\tfrac{\partial}{\partial x} \quad \text{and} \quad \partial_y := y\tfrac{\partial}{\partial y}. \tag{6.57.2}$$

Thus $\partial_x(x^r y^s) = r x^r y^s$, $\partial_y(x^r y^s) = s x^r y^s$ and

$$H^1(U, T_U) = \left\langle \tfrac{\partial_x}{x^i y^j} : i \geq 2, j \geq 1 \right\rangle \bigoplus \left\langle \tfrac{\partial_y}{x^i y^j} : i \geq 1, j \geq 2 \right\rangle. \tag{6.57.3}$$

The $\mathbb{G}_m^2$-eigenspaces in $H^1(U, T_U)$ are usually two-dimensional

$$\left\langle \tfrac{\partial_x}{x^i y^j}, \tfrac{\partial_y}{x^i y^j} \right\rangle \quad \text{for} \quad i, j \geq 2. \tag{6.57.4.a}$$

The one-dimensional eigenspaces are

$$\left\langle \tfrac{\partial_x}{x^i y} \right\rangle \quad \text{and} \quad \left\langle \tfrac{\partial_y}{x y^j} \right\rangle \quad \text{for} \quad i, j \geq 2. \tag{6.57.4.b}$$

The pairing $H^1(U, T_U) \times H^0(U, \mathscr{O}_U) \to H^1(U, \mathscr{O}_U)$ defined in (6.44.3) is especially transparent using the bases (6.57.1–4), since

$$\tfrac{a\partial_x - b\partial_y}{x^i y^j}(x^r y^s) = (ar - bs) \cdot x^{r-i} y^{s-j}, \tag{6.57.5}$$

where $a, b \in k$ and $i, j \geq 1$. This is identically 0 as an element of $H^0(U_{01}, \mathscr{O}_{U_{01}})$ iff $ar - bs = 0$. It is more important to know when this is 0 as an element of $H^1(U, \mathscr{O}_U)$. The latter holds iff

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(6.a) either $ar - bs = 0$, or

(6.b) $r \geq i$, or $s \geq j$.

This easily implies that the left kernel of $H^1(U, T_U) \times H^0(U, \mathscr{O}_U) \to H^1(U, \mathscr{O}_U)$ is trivial, hence $T^1(\mathbb{A}^2) = 0$ by (6.46.2); but this we already knew.

Combining (6.51) and (6.53) gives the following.

Lemma 6.58 *Using the notation of (6.57), let $D \in H^1(U, T_U)$ and U_D the corresponding deformation. Then $f(dx \wedge dy)^m$ lifts to a section of $\omega^m_{U_D}$ iff*

$$D(f) + mf\nabla D \in H^1(U, \mathscr{O}_U) \quad \textit{vanishes.} \qquad \square$$

We are thus interested in computing the kernels of the operators

$$(D, f) \mapsto D(f) + mf\nabla D.$$

We start by describing the kernel of ∇.

6.59 (Computing the divergence) Set $D := (a\partial_x - b\partial_y)x^{-i}y^{-j}$. By explicit computation,

$$\nabla\left(\tfrac{a\partial_x - b\partial_y}{x^i y^j}\right) = -\tfrac{a(i-1) - b(j-1)}{x^i y^j}. \tag{6.59.1}$$

Thus ∇D is identically 0 iff $a(i-1) - b(j-1) = 0$. If D is a nonzero element of $H^1(U, T_U)$ then $i, j > 0$ and then ∇D is 0 as an element of $H^1(U, \mathscr{O}_U)$ iff it is identically 0.

If $(i, j) = (1, 1)$, then $\nabla D = 0$, but then D vanishes in $H^1(U, T_U)$. If $\nabla D = 0$ and $i = 1, j > 1$ then $b = 0$ and again D vanishes in $H^1(U, T_U)$. Thus we conclude that

$$\ker[H^1(U, T_U) \xrightarrow{\nabla} H^1(U, \mathscr{O}_U)] = \left\langle \tfrac{(j-1)\partial_x - (i-1)\partial_y}{x^i y^j} : i, j \geq 2 \right\rangle. \tag{6.59.2}$$

Corollary 6.60 *Let $D \in H^1(U, T_U)$. Then $D(xy), \nabla D \in H^1(U, \mathscr{O}_U)$ are both 0 iff D is contained in the subspace*

$$K_{VW} := \left\langle \tfrac{\partial_x - \partial_y}{(xy)^i} : i \geq 2 \right\rangle \subset H^1(U, T_U).$$

Proof Corresponding to the two cases in (6.57.6.a–b), the kernel of the map $D \mapsto D(xy) \in H^1(U, \mathscr{O}_U)$ is a direct sum of two subspaces

$$K_1 := \left\langle \tfrac{\partial_x - \partial_y}{x^i y^j} : i, j \geq 2 \right\rangle \quad \text{and} \quad K_2 := \left\langle \tfrac{\partial_y}{xy^j}, \tfrac{\partial_x}{x^i y} : i, j \geq 2 \right\rangle. \tag{6.60.1}$$

Combining this with (6.59.2) gives the claim. $\square$

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6.6 Deformations of Cyclic Quotient Singularities

We use the methods of the previous section to understand first order deformations of cyclic quotient singularities. It is based on Altmann and Kollár (2019), which uses toric geometry. For cyclic quotients the two approaches are equivalent, but they suggest different generalizations.

Notation 6.61 X is a pure dimensional, S_2 scheme over a field k such that ω_X is locally free outside a closed subset $Z \subset X$ of codimension ≥ 2 and $\omega_X^{[m]}$ is locally free for some $m > 0$. The smallest such $m > 0$ is called the *index* of ω_X. Both of these conditions are satisfied by schemes with slc singularities.

Let $(0, T)$ be a local scheme such that $k(0) \simeq k$ and $p\colon X_T \to T$ a flat deformation of $X \simeq X_0$. As in (2.5), for every $r \in \mathbb{Z}$ we have maps

$$\mathcal{R}^{[r]}\colon \omega_{X_T/T}^{[r]}|_{X_0} \to \omega_{X_0}^{[r]}. \tag{6.61.1}$$

These maps are isomorphisms over $X \setminus Z$ and we are interested in understanding those cases when $\mathcal{R}^{[r]}$ is an isomorphisms over X.

By (9.17), if T is Artinian, then $\mathcal{R}^{[r]}$ is an isomorphism $\Leftrightarrow$ $\mathcal{R}^{[r]}$ is surjective $\Leftrightarrow$ $\omega_{X_T/T}^{[r]}$ is flat over T.

Definition 6.62 Let $p\colon X_T \to T$ be a flat deformation as in (6.61).

(6.62.1) We call $p\colon X_T \to T$ a *KSB-deformation* if $\mathcal{R}^{[r]}$ is an isomorphism for every r. It is enough to check these for $r = 1, \ldots, \operatorname{index}(\omega_X)$. (These are also called qG-deformations. The letter are short for "quotient of Gorenstein," but this is misleading if $\dim X \geq 3$.) These appear on KSB-stable families (6.16).

(6.62.2) We call $p\colon X_T \to T$ a *Viehweg-type deformation* (or V-deformation) if $\mathcal{R}^{[r]}$ is an isomorphism for every r divisible by $\operatorname{index}(\omega_X)$. It is enough to check this for $r = \operatorname{index}(\omega_X)$. These appear on V^+-stable families (6.31).

(6.62.3) We call $p\colon X_T \to T$ a *Wahl-type deformation* (or W-deformation) if $\mathcal{R}^{[r]}$ is an isomorphism for $r = -1$. These deformations were considered in Wahl (1980, 1981) and called ω^*-constant deformations there.

(6.62.4) We call $p\colon X_T \to T$ a VW-deformation if it is both a V-deformation and a W-deformation.

It is clear that every KSB-deformation is also a VW-deformation. Understanding the precise relationship between these four classes has been a long-standing open problem, especially for quotient singularities of surfaces. For reduced base spaces we have the following, which is a combination of (4.33) and (3.1).

Theorem 6.63 *A flat deformation of an slc variety over a reduced, local scheme of characteristic 0 is a V-deformation iff it is a KSB-deformation.*

This raised the possibility that every V-deformation of an slc singularity is also a KSB-deformation over arbitrary base schemes. It would be enough to check this for Artinian bases. Here we focus on first order deformations and prove that these two classes are quite different from each other.

Definition 6.64 Let X be a scheme satisfying the assumptions of (6.61). Let $T^1(X)$ be the set of isomorphism classes of deformations of X over $k[\varepsilon]$. This is a (possibly infinite dimensional) k-vector space. Let $T^1_{KSB}(X) \subset T^1(X)$ denote the space of first order KSB-deformations, $T^1_V(X)$ the space of first order V-deformations, $T^1_W(X)$ the space of first order W-deformations and $T^1_{VW}(X)$ the space of first order VW-deformations. We have obvious inclusions

$$T^1_{KSB}(X) \subset T^1_{VW}(X) \subset T^1_V(X), T^1_W(X) \subset T^1(X),$$

but the relationship between $T^1_V(X)$ and $T^1_W(X)$ is not clear.

These $T^1_*(X)$ are the tangent spaces to the corresponding miniversal deformation spaces; we denote these by $\mathrm{Def}_{KSB}(X), \mathrm{Def}_V(X)$ and so on. See Artin (1976) or Looijenga (1984) for precise definitions and introductions, or (2.25–2.29) for details on surface quotient singularities.

6.65 (Cyclic quotient singularities) Let $\frac{1}{n}(1,q)$ denote the cyclic group action $g\colon (x,y) \mapsto (\eta x, \eta^q y)$, where η is a primitive nth root of unity. We always assume that $\operatorname{char} k \nmid n$ and $(n,q) = 1$; then the action is free outside the origin on $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$. The ring of invariants is

$$R_{nq} := k[x,y]^G = k[x^i y^j \colon i,j \geq 0,\ i + qj \equiv 0 \mod n], \tag{6.65.1}$$

and the corresponding quotient singularity is

$$S_{n,q} := \mathbb{A}^2/\tfrac{1}{n}(1,q) = \operatorname{Spec}_k R_{nq}. \tag{6.65.2}$$

While we work with this affine model, all the results apply to its localization, Henselisation, or completion at the origin.

We can also choose $\eta' = \eta^q$ as our primitive nth root of unity. This shows the isomorphism $S_{n,q} \simeq S_{n,q'}$ if $qq' \equiv 1 \mod n$.

Various ways of studying such singularities go back a long time. The first relevant work might be Jung (1908). See also Brieskorn (1967/1968).

In (6.70), we give an algorithm that yields an explicit, minimal generating set of R_{nq}. The number of generators is the embedding dimension.

For us, the embedding dimension is the most natural invariant, but traditionally the multiplicity is considered the basic one. For cyclic quotients, more generally, for rational surface singularities, these are related by the formula

$$\operatorname{embdim}(S_{n,q}) = \operatorname{mult}(S_{n,q}) + 1. \tag{6.65.3}$$

We completely describe first order KSB-, V- and W-deformations of cyclic quotient singularities. The main conclusion is that KSB-deformations and V-deformations are quite different over Artinian bases; see (6.82).

The A_{n-1}-singularity $\mathbb{A}^2/\frac{1}{n}(1, n-1)$ has embedding dimension 3, and all of its deformations are KSB. In the other cases, we have the following.

Theorem 6.66 *Let* $S_{n,q} := \mathbb{A}^2/\frac{1}{n}(1, q)$ *be as in (6.65) with* $q \neq n-1$. *Then*

$$\dim T^1_V(S_{n,q}) - \dim T^1_{VW}(S_{n,q}) = \operatorname{embdim}(S_{n,q}) - 4 \quad \textit{or} \quad \operatorname{embdim}(S_{n,q}) - 5.$$

In particular, if $\operatorname{embdim}(S_{n,q}) \geq 6$ *then* $S_{n,q}$ *has V-deformations that are not VW-deformations, hence also not KSB-deformations.*

Complement 6.66.1 In (6.85) we list all $S_{n,q}$ for which every V-deformation is a KSB-deformation.

By contrast, KSB-deformations and VW-deformations are quite close to each other, as shown by the next result, proved in (6.84).

Theorem 6.67 *Let* $S_{n,q} := \mathbb{A}^2/\frac{1}{n}(1, q)$ *be as in (6.65).*

(6.67.1) *If* $(n, q+1) = 1$*, then* $\operatorname{Def}_{KSB}(S_{n,q}) = \operatorname{Def}_{VW}(S_{n,q}) = \{0\}$.

(6.67.2) *If* $S_{n,q}$ *admits a KSB-smoothing, then* $\operatorname{Def}_{KSB}(S_{n,q}) = \operatorname{Def}_{VW}(S_{n,q})$.

(6.67.3) *In general,* $\dim T^1_{KSB}(S_{n,q}) \leq \dim T^1_{VW}(S_{n,q}) \leq \dim T^1_{KSB}(S_{n,q}) + 1$.

Next we discuss what the general theory of the previous section says about deformations of two-dimensional quotient singularities.

6.68 (Deformation of quotients) Let k be a field, X an affine k-scheme that is S_2, $x \in X$ a closed point and $U := X \setminus \{x\}$. Let G be a finite group acting on X such that x is a G-fixed point and the action is free on U. The quotient map $\pi_U \colon U \to U/G$ is finite and étale. This extends to a finite map $\pi_X \colon X \to X/G$ which is ramified at x.

$\mathscr{O}_{U/G}$ is identified with the G-invariant subsheaf $(\pi_* \mathscr{O}_U)^G$ and similarly $\omega_{U/G}$ is identified with $(\pi_* \omega_U)^G$. (For the latter we need that the action is free). Thus

$$\begin{aligned} H^0(U/G, \mathscr{O}_{U/G}) &= H^0(U, \mathscr{O}_U)^G = H^0(X, \mathscr{O}_X)^G, \quad \text{and} \\ H^0(U/G, \omega^{[m]}_{U/G}) &= H^0(U, \omega^{[m]}_U)^G = H^0(X, \omega^{[m]}_X)^G. \end{aligned} \tag{6.68.1}$$

If $\operatorname{char} k \nmid |G|$ then the G-invariant subsheaf is a direct summand, hence by taking cohomologies we similarly see that

$$H^1(U/G, \mathcal{O}_{U/G}) = H^1(U, \mathcal{O}_U)^G \quad \text{and} \quad H^1(U/G, T_{U/G}) = H^1(U, T_U)^G.$$

If $D \in H^1(U, T_U)$ is G-invariant, then U_D descends to a deformation $(U/G)_D$ of U/G; these give all first order deformations. If $H^0(U/G, \mathcal{O}_{U/G})$ is flat over $k[\varepsilon]$, then its spectrum gives a flat deformation of X/G and every flat deformation that is locally trivial on U/G arises this way.

Thus, using (6.46) we get the following fundamental observation.

Theorem 6.69 Schlessinger (1971) *Let k be a field, X a smooth, affine k-variety, $x \in X$ a closed point and $U := X \setminus \{x\}$. Let G be a finite group acting on X such that x is a G-fixed point, the action is free on U and* $\operatorname{char} k \nmid |G|$. *Then $T^1(X/G)$ is the left kernel of the pairing*

$$H^1(U, T_U)^G \times H^0(U, \mathcal{O}_U)^G \to H^1(U, \mathcal{O}_U)^G \tag{6.69.1}$$

defined in (6.44). More generally, if X is normal, the left kernel corresponds to those flat deformations of X/G that are locally trivial on U/G. □

Next we compute the terms in (6.69.1) for cyclic quotient singularities.

Notation 6.70 Our aim is to describe the generators of $R_{n,q}$ as in (6.65.1). We assume that $\operatorname{char} k \nmid n$ and $(n, q) = 1$.

Most of the following formulas can be found in Riemenschneider (1974); see Stevens (2013) for an introduction and many examples.

The group action preserves the monomials, hence R_{nq} has a generating set consisting of monomials. A nonminimal generating set can be constructed as follows. For any $0 < j < n$ let $0 < \gamma_j < n$ be the unique integer such that $\gamma_j + qj \equiv 0 \mod n$. Then

$$x^n, x^{\gamma_1}y, x^{\gamma_2}y^2, \ldots, x^{\gamma_{n-1}}y^{n-1}, y^n$$

is a generating set of R_{nq}. We know that $\gamma_1 = n - q$ and $\gamma_{n-1} = q$. This is a minimal generating set of R_{nq} as a $k[x^n, y^n]$-module, but usually not as a k-algebra. Indeed, $x^{\gamma_i}y^i$ divides $x^{\gamma_j}y^j$ if $\gamma_i < \gamma_j$ and $i < j$. In any concrete case one can use this observation to get a minimal set of algebra generators.

We label the monomials of the minimal algebra generators as $M_i = x^{a_i}y^{b_i}$, ordered by increasing y-powers

$$M_0 = x^n, M_1 = x^{n-q}y = x^{a_1}y^{b_1}, M_2 = x^{a_2}y^{b_2}, \ldots, M_r = y^n. \tag{6.70.1}$$

At the same time, the a_i form a decreasing sequence. Indeed, if $b_i < b_j$ and $a_i \leq a_j$, then M_i divides M_j so the sequence would not be minimal.

From (6.71.2), we obtain that there are relations of the form

$$M_i^{c_i} = M_{i-1}M_{i+1} \quad \text{for} \quad i = 1, \dots, r-1. \tag{6.70.2}$$

This tells us that the a_i and the c_i are recursively defined by

$$a_0 = n, a_1 = n-q, c_i = \lceil a_{i-1}/a_i \rceil, a_{i+1} = c_i a_i - a_{i-1}. \tag{6.70.3}$$

Similarly, $b_0 = 0, b_1 = 1$ and $b_{i+1} = c_i b_i - b_{i-1}$. These imply that $(a_i, a_{i+1}) = (b_i, b_{i+1}) = 1$ for every i and that the $c_i \geq 2$ are computed by the modified continued fraction expansion

$$\frac{n}{n-q} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cdots}}. \tag{6.70.4}$$

The following observations about the a_i, b_i, c_i are quite useful. The first two follow from the original construction of the M_i, the third from (6.70.5) and the last one is equivalent to (6.71.3).

$a_{i-1} = \min\{\alpha > 0 \colon \exists x^\alpha y^\beta \in R_{nq}$ such that $\beta < b_i\}$ for $i > 0$.
$b_{i+1} = \min\{\beta > 0 \colon \exists x^\alpha y^\beta \in R_{nq}$ such that $\alpha < a_i\}$ for $i < r$.
$c_i - 1 = \lfloor \frac{a_{i-1}}{a_i} \rfloor = \lfloor \frac{b_{i+1}}{b_i} \rfloor$ for $0 < i < r$.
$a_i b_{i+1} - a_{i+1} b_i = n$ for $0 \leq i < r$.

Note that $r+1$ is the embedding dimension of S_{nq} and r is its multiplicity. Thus $r = 2$ iff $M_1 = M_{r-1} = xy$ and hence we have the A_{n-1}-singularity $\mathbb{A}^2/\frac{1}{n}(1,-1)$. These are exceptional for many of the subsequent formulas, so we assume from now on that $r \geq 3$.

6.71 (Cones and semigroups) Let $v_0, v_1 \in \mathbb{Z}^2$ be primitive vectors and $C := \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_1 \subset \mathbb{R}^2$ the closed cone spanned by them. Let $\bar{C}(\mathbb{Z})$ be the closed, convex hull of $(\mathbb{Z}^2 \cap C) \setminus \{(0,0)\}$ and $N(C)$ the part of the boundary of $\bar{C}(\mathbb{Z})$ that connects v_0 and v_1. Let $m_0 = v_0, m_1, \dots, m_{r-1}, m_r = v_1$ be the integral points in $N(C)$ as we move from v_0 to v_1. We leave it to the reader to prove that

(6.71.1) the m_i generate the semigroup $\mathbb{Z}^2 \cap C$,
(6.71.2) there are $c_1, \dots, c_{r-1} \geq 2$ such that $c_i m_i = m_{i-1} + m_{i+1}$, and
(6.71.3) the triangles with vertices $\{(0,0), m_i, m_{i+1}\}$ all have the same area.

Thus $R(C)$, the semigroup algebra of $\mathbb{Z}^2 \cap C$, is generated by $m_0, \dots, m_s$.

For $1 \leq q < n$ and $(n,q) = 1$, consider the cone C_{nq} spanned by $v_0 = (1,0)$ and $v_1 = (q,n)$. Then

$$\mathbb{Z}^2 \cap C_{nq} = \langle \tfrac{i}{n}v_0 + \tfrac{j}{n}v_1 \colon i, j \geq 0, i + qj \equiv 0 \mod n \rangle.$$

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Thus we see that the semigroup algebra $R(C_{nq})$ is isomorphic to the algebra of invariants R_{nq} defined in (6.65). (It is not hard to see that, up to the action of $\mathrm{SL}(2,\mathbb{Z})$, every rational cone in $\mathbb{R}^2$ is of the form C_{nq}.)

6.72 (Computing $T^1(S_{nq})$) Continuing with the notation of (6.68–6.70), we see that $D \in H^1(U, T_U)^G$ is in $T^1(S_{nq})$ iff $D(M_i) = 0 \in H^1(U, \mathscr{O}_U)$ for every i.

Since the pairing (6.69.1) is $\mathbb{G}_m^2$-equivariant, it is sufficient to consider one eigenspace at a time. As in (6.57.4.a–b), the eigenspaces in $H^1(U, T_U)^G$ are usually two-dimensional and of the form

$$\left\langle \tfrac{\partial_x}{M}, \tfrac{\partial_y}{M} \right\rangle, \tag{6.72.1}$$

where M is a monomial in the M_i-s involving both x, y. The exceptions are one-dimensional subspaces. For every $s \geq 0$ we have two of them

$$\left\langle \tfrac{\partial_x}{M_0^s M_1} \right\rangle \quad \text{and} \quad \left\langle \tfrac{\partial_y}{M_{r-1} M_r^s} \right\rangle. \tag{6.72.2}$$

Thus we can write $D = (\alpha\partial_x - \beta\partial_y)/M$. Note that

$$D(x^a y^b) = (\alpha a - \beta b)\tfrac{x^a y^b}{M}, \tag{6.72.3}$$

hence if $a < \operatorname{ord}_x M$ and $b < \operatorname{ord}_y M$, then this is 0 in $H^1(U, \mathscr{O}_U)$ iff $\beta/\alpha = a/b$. Thus if M is divisible by at least two different monomials M_i, M_j for $0 < i, j < r$ then $D(M_i) = 0$ and $D(M_j) = 0$ imply that we need to satisfy both of the equations $\beta/\alpha = a_i/b_i$ and $\beta/\alpha = a_j/b_j$, a contradiction. We get a similar contradiction for the eigenspaces (6.72.2) if $s > 0$. We are left with the cases when $M = M_i^s$ for some $0 < i < r$. If $s \geq 2$ then $D(M_i) = 0$ implies that $D = (b_i\partial_x - a_i\partial_y)/M_i^s$. Then $b_i a_j - a_i b_j \neq 0$ for $j \neq i$ hence $D(M_j) = (b_i a_j - a_i b_j)(M_j/M_i^s)$ vanishes in $H^1(U, \mathscr{O}_U)$ iff $sa_i \leq a_j$ or $sb_i \leq b_j$. If $j < i$ then $b_j < b_i$, hence $sa_i \leq a_j$ must hold. Since the a_j form a decreasing sequence, we need $sa_i \leq a_{i-1}$. Similarly, $sb_j \leq b_{j+1}$. By (6.70.5.c), these are equivalent to $s \leq c_i - 1$. We have thus proved the following result of Riemenschneider (1974) and Pinkham (1977).

Proposition 6.73 *Let $M_i = x^{a_i} y^{b_i}$ for $i = 0, \ldots, r$ be the generators of R_{nq} as in (6.70.1). Then $T^1(S_{nq}) \subset H^1(U, T_U)$ has a basis consisting of*

$$\left\{ \tfrac{\partial_x}{M_1}, \tfrac{\partial_y}{M_{r-1}} \right\} \quad \textit{and} \quad \left\{ \tfrac{\partial_x}{M_i}, \tfrac{\partial_y}{M_i} : 2 \leq i \leq r-2 \right\}, \tag{6.73.1}$$

plus the possibly empty set

$$\left\{ \tfrac{b_i\partial_x - a_i\partial_y}{M_i^s} : 1 \leq i \leq r-1,\ 2 \leq s \leq c_i - 1 \right\} \tag{6.73.2}$$

where $c_i = \lceil \frac{a_{i-1}}{a_i} \rceil = \lceil \frac{b_{i+1}}{b_i} \rceil$ is defined in (6.70.2).

6.74 (Powers of ω) Fix any $m \in \mathbb{Z}$. Then $H^0(U, \omega_U^m)$ has a basis consisting of $M(dx \wedge dy)^m$ where M is any monomial. Thus $H^0(S_{nq}, \omega_{S_{nq}}^{[m]}) = H^0(U/G, \omega_{U/G}^m)$ has a basis consisting of

$$\{x^a y^b (dx \wedge dy)^m : a + qb \equiv -m(1+q) \mod n\}. \tag{6.74.1}$$

For $D \in T^1(S_{nq})$ let S_D denote the corresponding deformation. By (6.58) $x^a y^b (dx \wedge dy)^m f$ lifts to a section of $\omega_{S_D}^{[m]}$ iff

$$D(x^a y^b) + m x^a y^b \nabla D = 0 \in H^1(U, \mathscr{O}_U). \tag{6.74.2}$$

It is enough to check (6.74.2) for a minimal generating set of $H^0(S_{nq}, \omega_{S_{nq}}^{[m]})$ as an R_{nq}-module. In any given case, this can be worked out by hand, but there are two instances where the answer is simple.

(6.74.3) If $n \mid (q+1)m$ then $H^0(S_{nq}, \omega_{S_{nq}}^m)$ is cyclic with generator $(dx \wedge dy)^m$.

(6.74.4) If $m = -1$ then $xy(dx \wedge dy)^{-1}$ is G-invariant and every $x^a y^b (dx \wedge dy)^{-1}$ is a multiple of it, save for powers of x or y. Thus $\omega_{S_{nq}}^{-1}$ has 3 generating sections:

$$\frac{xy}{dx \wedge dy}, \quad \frac{x^{q+1}}{dx \wedge dy}, \quad \frac{y^{q'+1}}{dx \wedge dy}.$$

6.75 (V-deformations) If $n \mid (q+1)m$, then $(dx \wedge dy)^m$ is a generator by (6.74.3), thus the condition (6.74.2) is equivalent to $\nabla D = 0$.

Therefore $T_V^1(S_{nq})$ equals the intersection of $T^1(S_{nq})$ with the kernel of ∇. The former was computed in (6.73), the latter in (6.59.2). Thus we see that a basis of $T_V^1(S_{nq})$ is

$$\left\{\frac{(b_i - 1)\partial_x - (a_i - 1)\partial_y}{M_i} : 2 \leq i \leq r - 2\right\} \tag{6.75.1.a}$$

and, if M_i is a power of xy for some i, then we have to add

$$\left\{\frac{\partial_x - \partial_y}{M_i^s} : 2 \leq s \leq c_i - 1\right\}. \tag{6.75.1.b}$$

6.76 (W-deformations) By (6.74.4), $\omega_{X/G}^{-1}$ has three generating sections. Thus, by (6.74.2), D corresponds to a W-deformation iff $D(xy) - xy\nabla D = 0$, $D(x^{q+1}) - x^{q+1}\nabla D = 0$, and $D(y^{q'+1}) - y^{q'+1}\nabla D = 0$.

The first of these conditions is especially strong. We do not compute it here, rather go directly to the next case where the answer is simpler.

6.77 (VW-deformations) Combining (6.75) and (6.76) we get the description of VW-deformations. These satisfy the conditions

(6.77.1) $D(xy) = 0$, $D(x^{q+1}) = 0$ and $D(y^{q'+1}) = 0$.

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We computed the subspace K_{VW} where then first two hold in (6.60). It is spanned by the derivations $(\partial_x - \partial_y)(xy)^{-i}$ for $i \geq 2$. Comparing this with (6.73) we get the following.

Claim 6.77.2 If $T^1_{VW}(S_{nq}) \neq 0$, then R_{nq} has a minimal generator of the form $M_i = (xy)^a$. □

In order to put this into a cleaner form, assume that $(xy)^s$ is the smallest G-invariant power of xy. Note that $(xy)^n = M_0 M_r$ is G-invariant, but it is not one of the M_i. We have $s(q+1) \equiv 0 \mod n$, thus if $s < n$ then $b := (n, q+1) > 1$. We have thus shown the following.

Claim 6.77.3 Assume that $(n, q+1) = 1$. Then $T^1_{KSB}(S_{nq}) = T^1_{VW}(S_{nq}) = 0$ and $\dim T^1_V(S_{nq}) = r - 3$. □

Claim 6.77.4 Assume that $M_i = (xy)^a$ for some i (so $a_i = b_i = a$). Then the space of VW-deformations is spanned by

$$\left\{ \tfrac{\partial_x - \partial_y}{M_i^s} :\ 1 \leq s \leq \min\left\{ c_i - 1, \tfrac{q+1}{a}, \tfrac{q'+1}{a} \right\} \right\}.$$

Proof The first restriction on s we get from (6.73.2). Then $D(x^{q+1}) = 0$ is equivalent to $sa \leq q+1$ and $D(y^{q'+1}) = 0$ is equivalent to $sa \leq q' + 1$. These give the last 2 restrictions. □

We thus need to compare the two upper bounds occurring in (6.75.1.b) and (6.77.4). The key is the following general estimate.

Lemma 6.78 *Using the notation of (6.70) we have*

$$\tfrac{n}{a_i b_i} \leq \tfrac{a_{i-1}}{a_i}, \tfrac{b_{i+1}}{b_i} < \tfrac{n}{a_i b_i} + 1.$$

Proof Note that $n = a_i b_{i+1} - a_{i+1} b_i$ by (6.70.5.d). Dividing by $a_i b_i$ we get that

$$\tfrac{n}{a_i b_i} = \tfrac{b_{i+1}}{b_i} - \tfrac{a_{i+1}}{a_i}.$$

Since the a_i form a decreasing sequence, $\frac{a_{i+1}}{a_i} < 1$. □

The final estimate connecting (6.75.1.b) and (6.77.4) is easier to state using a different system of indexing the singularities.

Notation 6.79 Set $b = (n, q+1)$ and write $n = ab,\ q+1 = bc$ where $(a, c) = 1$. The inverse (modulo ab) of $bc - 1$ is written as $bc' - 1$. We thus have the singularity

$$S_{abc} := S_{nq} = \mathbb{A}^2 / \tfrac{1}{ab}(1, bc-1) \simeq \mathbb{A}^2 / \tfrac{1}{ab}(1, bc'-1). \tag{6.79.1}$$

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Note that $(xy)^a$ is the smallest G-invariant power of xy, but it need not be among the generators M_i; see (6.81).

Corollary 6.80 *Assume in addition that $M_i = (xy)^a$ for some i. Then*

$$\lfloor \tfrac{b}{a} \rfloor \le \min\left\{c_i - 1, \tfrac{q+1}{a}, \tfrac{q'+1}{a}\right\} \le c_i - 1 \le \lfloor \tfrac{b}{a} \rfloor + 1. \tag{6.80.1}$$

Proof First, we claim that

$$\tfrac{b}{a} \le \min\left\{\tfrac{a_{i-1}}{a_i}, \tfrac{b_{i+1}}{b_i}, \tfrac{q+1}{a}, \tfrac{q'+1}{a}\right\} \le \min\left\{\tfrac{a_{i-1}}{a_i}, \tfrac{b_{i+1}}{b_i}\right\} < \tfrac{b}{a} + 1. \tag{6.80.2}$$

To see this, note that $q = bc - 1, q' = bc' - 1$. Thus $b \le q + 1, q' + 1$, so it is enough to show that

$$\tfrac{b}{a} \le \min\left\{\tfrac{a_{i-1}}{a_i}, \tfrac{b_{i+1}}{b_i}\right\} < \tfrac{b}{a} + 1.$$

Since $n = ab$ and $a = a_i = b_i$, the latter is equivalent to (6.78). Taking the round-down gives (1) using (6.70.5.c). □

Example 6.81 Assume that $x^\alpha y^\beta$ is G-invariant. From $\alpha + \beta(bc - 1) \equiv 0 \mod ab$, we see that $\alpha \equiv \beta \mod b$. Thus if $0 < \alpha, \beta \le 2b$ then either $\alpha = \beta$ or $\alpha = \beta \pm b$.

It turns out that if $a \le b$ then we can write down these invariants explicitly. Corresponding to the first case we have $(xy)^a$ (and its square). In order to get the other cases, let $0 < e < a$ (resp. $0 < e' < a$) be the unique solution of $ec \equiv -1 \mod a$ (resp. $e'c' \equiv -1 \mod a$). Then $(b + e) + e(bc - 1) = b(ec + 1) \equiv 0 \mod ab$ and $e'(bc' - 1) + (b + e') = b(e'c' + 1) \equiv 0 \mod ab$. Thus we get the minimal generators

$$M_{i-1} = x^{b+e}y^e,\ M_i = x^a y^a,\ M_{i+1} = x^{e'} y^{b+e'}.$$

This gives that $c_i - 1 = \lfloor \frac{b+e}{a} \rfloor = \lfloor \frac{b+e'}{a} \rfloor$.

Fixing a, b we can choose any $0 < e < a$ such that $(a, e) = 1$ and then solve for c. Thus we see that if $b \equiv 0 \mod a$ then $\lfloor \frac{b}{a} \rfloor = c_i - 1$ for every e and if $b \equiv -1 \mod a$ then $\lfloor \frac{b}{a} \rfloor = c_i - 2$ for every e, but otherwise both are possible for suitable choice of e.

We see in (6.83) that the condition $a \le b$ holds iff S_{abc} has a nontrivial KSB-deformation, so this is a natural class to consider.

6.82 (Proof of 6.66) Comparing (6.75) and (6.77) we see that the derivations listed in (6.75.1) give V-deformations, but not W-deformations. The only possible exception occurs if $M_i = (xy)^a$ for some i. Thus we have two cases.

If $M_i = (xy)^a$ does not occur, then $\dim T^1_V(S_{nq}) = \dim T^1_{VW}(S_{nq}) + r - 3$.

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If $M_i = (xy)^a$ for some i, then (6.75.1) gives $r - 4$ basis vectors that give V-deformations, but not W-deformations. By (6.80), there is at most one derivation as in (6.75.2) that gives a V-deformation that is not a W-deformation. □

6.83 (KSB-deformations) From (6.58) and (6.70.2) we see that D corresponds to a KSB-deformation iff $D(x^iy^j) + mx^iy^j\nabla D = 0$ whenever $i + j(bc-1) \equiv -mbc \mod ab$.

First, we use this for $(dx \wedge dy)^{ab}$ to conclude that $\nabla D = 0$. Second, we note that since $(a, c) = 1$, the congruence $i + j(bc - 1) \equiv -mbc \mod ab$ holds for some m iff $i \equiv j \mod b$. The ring of such monomials is generated by x^b, xy, y^b. Thus D gives a first order KSB-deformation iff

(6.83.1) $\nabla D = 0$, $D(xy) = 0$, $D(x^b) = 0$ and $D(y^b) = 0$.

We thus get that $T^1_{KSB}(S_{abc})$ is spanned by the derivations

$$\left\{\tfrac{\partial_x - \partial_y}{(xy)^{as}}:\ 1 \le s \le \lfloor b/a \rfloor\right\}. \tag{6.83.2}$$

The corresponding deformations were written down in Wahl (1980, 2.7):

$$(uv - w^b - t_1w^{b-a} - \cdots - t_rw^{b-ra} = 0)/\tfrac{1}{a}(1, bc - 1, c). \tag{6.83.3}$$

To make this $\mathbb{G}_m^2$-equivariant, the $\mathbb{G}_m^2$-action on t_i should be the same as on $(xy)^{ai}$. Thus (6.83.5) describes a smooth subscheme T of $\mathrm{Def}_{KSB}(S_{abc})$ and $\dim T = \lfloor b/a \rfloor$. By (6.83.2), the tangent space of $\mathrm{Def}_{KSB}(S_{abc})$ has dimension $\lfloor b/a \rfloor$, so $T = \mathrm{Def}_{KSB}(S_{abc})$ and $\mathrm{Def}_{KSB}(S_{abc})$ is smooth.

In particular, there is a nontrivial 1-parameter KSB-deformation iff $a \le b$ and there is a KSB-smoothing iff $a|b$. Note that $a \le b$ is equivalent to $ab \le b^2$ and we have proved the following.

Claim 6.83.6 The singularity S_{nq} has

(a) a KSB-smoothing iff $n|(q + 1)^2$, and
(b) a nontrivial KSB-deformation iff $n \le (n, q + 1)^2$. Furthermore,
(c) $\dim T^1_{KSB}(S_{nq}) = \lfloor b/a \rfloor = \lfloor (n, q + 1)^2/n \rfloor$. □

If $a|b$ then write $b = ad$. We get the singularities

$$W_{adc} := \tfrac{1}{a^2d}(1, adc - 1) \simeq (uv - w^{ad} = 0)/\tfrac{1}{a}(1, -1, c). \tag{6.83.7}$$

In this case, $b/a = c_i - 1$ hence the arguments give the following.

Claim 6.83.8 For the singularities $W_{adc} = \mathbb{A}^2/\frac{1}{a^2d}(1, adc - 1)$ every VW-deformation is a KSB-deformation. □

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6.84 (Proof of 6.67) Note that (6.67.1) follows from (6.77.3) and (6.67.2) from (6.83.8) for first order deformations. Since $\mathrm{Def}_{KSB}(S_{n,q})$ is smooth by (2.29) or by the explicit description (6.83.5), equality of the tangent spaces $T^1_{KSB}(S_{n,q}) = T^1_{VW}(S_{n,q})$ implies that $\mathrm{Def}_{KSB}(S_{n,q}) = \mathrm{Def}_{VW}(S_{n,q})$.

In order to prove (6.67.3) we consider two cases. If R_{nq} does not have a minimal generator of the form $M_i = (xy)^a$, then $T^1_{VW}(S_{nq}) = T^1_{KSB}(S_{nq}) = \{0\}$ by (6.77.4).

Otherwise, we have proved in (6.83) that

$$\dim T^1_{KSB}(\mathbb{A}^2/\tfrac{1}{ab}(1, bc-1)) = \lfloor \tfrac{b}{a} \rfloor$$

and (6.80) shows that

$$\dim T^1_{VW}(\mathbb{A}^2/\tfrac{1}{ab}(1, bc-1)) = \min\Big\{c_i - 1, \tfrac{q+1}{a}, \tfrac{q'+1}{a}\Big\} \le \lfloor \tfrac{b}{a} \rfloor + 1. \quad \square$$

Examples 6.85 We work out (6.66.1), that is, list those cyclic quotients singularities for which every V-deformation is a KSB-deformation.

6.85.1 (*Double points*) These are the A_n singularities; every deformation is a KSB-deformation.

6.85.2 (*Triple points*) For cyclic quotient triple points the minimal generators of its coordinate ring are $x^n, x^{n-q}y, xy^{n-q'}, y^n$. Thus $\frac{n}{n-q}$ has a two-step continued fraction expansion involving c_1, c_2. Setting $c_1 = e, c_2 = d$ we have the singularities $\mathbb{A}^2/\frac{1}{ed-1}(1, ed-d-1)$, with invariants $x^{ed-1}, x^d y, xy^e, y^{ed-1}$. By (6.75) we have $T^1_V = T^1_{KSB} = 0$.

6.85.3 (*Quadruple points*) By (6.66) and (6.82), every cyclic quotient singularity of multiplicity 4 has a V-deformation that is not a KSB-deformation, unless M_2 (6.70.1) is a power of xy. Thus in this case the minimal generators of its coordinate ring are $x^n, x^{n-q}y, x^a y^a, xy^{n-q'}, y^n$.

The equation $M_2^{c_2} = M_1 M_3$ now implies that $q = q'$. Thus $\frac{n}{n-q}$ has a three-step continued fraction expansion involving $c_1, c_2, c_3 = c_1$. By expanding it we see that $c_1 = a$. Setting $c_2 = d$, the singularity is $\mathbb{A}^2/\frac{1}{a(ad-2)}(1, (ad-2)(a-1)-1)$, and the ring of invariants is $k[x^{a(ad-2)}, x^{ad-1}y, x^a y^a, xy^{ad-1}, y^{a(ad-2)}]$.

Thus $\lfloor (ad-2)/a \rfloor = d-1 = c_2 - 1$ and hence, by (6.75) and (6.83), $T^1_V = T^1_{KSB}$ is spanned by $\Big\{\frac{\partial_x - \partial_y}{(xy)^{as}} : \ 1 \le s \le d-1\Big\}$. These singularities admit a KSB-smoothing iff $a = 2$. Then, after replacing $d-1$ by d, the normal form becomes $\mathbb{A}^2/\frac{1}{4d}(1, 2d-1)$. Together with the A_n-series, these are the only cyclic quotient singularities with a KSB-smoothing for which every V-deformation is a KSB-deformation.

6.85.4 (*Higher multiplicity points*) By (6.66), every cyclic quotient singularity of multiplicity ≥ 5 has V-deformations that are not KSB-deformations.

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