On the Neumann Problem for Monge–Ampère Type Equations

Feida Jiang, Neil S. Trudinger, and Ni Xiang

Abstract. In this paper, we study the global regularity for regular Monge–Ampère type equations associated with semilinear Neumann boundary conditions. By establishing a priori estimates for second order derivatives, the classical solvability of the Neumann boundary value problem is proved under natural conditions. The techniques build upon the delicate and intricate treatment of the standard Monge–Ampère case by Lions, Trudinger, and Urbas in 1986 and the recent barrier constructions and second derivative bounds by Jiang, Trudinger, and Yang for the Dirichlet problem. We also consider more general oblique boundary value problems in the strictly regular case.

1 Introduction

In this paper, we consider the following semilinear Neumann boundary value problem for the Monge–Ampère type equation

$$(1.1) \qquad \det\big[D^2 u - A(x, u, Du) \big] = B(x, u, Du), \qquad \text{in } \Omega,$$

$$(1.2) \qquad D_\nu u = \varphi(x, u), \qquad \text{on } \partial\Omega,$$

where Ω is a bounded domain in n dimensional Euclidean space $\mathbb{R}^n$ with smooth boundary, Du and $D^2 u$ denote the gradient vector and the Hessian matrix of the second order derivatives of the function $u \colon \Omega \to \mathbb{R}$, respectively, A is a given $n \times n$ symmetric matrix function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, B is a positive scalar valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, φ is a scalar valued function defined on $\partial\Omega \times \mathbb{R}$, and ν is the unit inner normal vector field on $\partial\Omega$. As usual, we use x, z, p, r to denote points in Ω, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n \times n}$ respectively. A solution $u \in C^2(\Omega)$ of equation (1.1) is elliptic when the augmented Hessian matrix $Mu = D^2 u - A(x, u, Du)$ is positive definite; that is $Mu > 0$, which implies $B > 0$. Also, a function u satisfying $Mu > 0$ is called an elliptic function of equation (1.1). Since the matrix A determines the augmented Hessian matrix Mu, we also call an elliptic solution (or function) an A-admissible solution (or function) or, by analogy with the case $A = 0$, an A-convex solution (or function).

We shall establish an existence theorem together with a priori estimates for elliptic solutions of the Neumann boundary value problem (1.1)–(1.2) in this paper, which extend the special case where A is independent of p in [17]. For this purpose, we

1334

need appropriate assumptions on A, B, φ and Ω. Assume that the matrix A is twice differentiable with respect to p and A, B, and φ are differentiable with respect to z. Following [24], we call the matrix A regular in Ω if A is co-dimension one convex with respect to p, in the sense that

$$(1.3) \qquad A_{ij,kl}(x,z,p)\xi_i\xi_j\eta_k\eta_l \geq 0$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$, $\xi \perp \eta$, where $A_{ij,kl} = D^2_{p_k p_l}A_{ij}$. If inequality (1.3) is strict, then the matrix A is called strictly regular. We also define the matrix A to be non-decreasing (strictly increasing) with respect to z if

$$D_z A_{ij}(x,z,p)\xi_i\xi_j \geq 0, (> 0)$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$. The inhomogeneous term B and boundary function φ are also non-decreasing, (strictly increasing), with respect to z if

$$B_z(x,z,p) \geq 0 \quad (> 0)$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and $\varphi_z(x,z) \geq 0, (> 0)$, for all $(x,z) \in \partial\Omega \times \mathbb{R}$. Note that if we write the boundary value problem (1.1)–(1.2) in the general form

$$(1.4) \qquad \mathcal{F}[u] := F(x,u,Du,D^2u) = 0, \qquad \text{in } \Omega,$$

$$(1.5) \qquad \mathcal{G}[u] := G(x,u,Du) = 0, \qquad \text{on } \partial\Omega,$$

where F and G are defined by

$$(1.6) \qquad F(x,z,p,r) = \det\left[r - A(x,z,p)\right] - B(x,z,p),$$

$$(1.7) \qquad G(x,z,p) = v \cdot p - \varphi(x,z),$$

then A, B, and φ non-decreasing (strictly increasing) in z, correspond to the standard monotonicity conditions $F_z \leq 0, G_z \leq 0, (F_z < 0, G_z < 0)$ for symmetric matrices r satisfying $r > A(x,z,p)$, that is, for points $(x,z,p,r) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$, where $\mathcal{F}$ is elliptic.

As with [17], we also need the domain Ω to satisfy an appropriate uniform convexity condition. Adapting [24], we define the domain Ω to be uniformly A-convex (A-convex) with respect to the boundary function φ and an interval valued function $\mathcal{I}$ on $\partial\Omega$ if $\Omega \in C^2$ and

$$(1.8) \qquad \left(D_i v_j(x) - D_{p_k}A_{ij}(x,z,p)v_k\right)\tau_i\tau_j < 0, (\leq 0),$$

for all $(x,z,p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$, satisfying $p \cdot v(x) \geq \varphi(x,z)$, $z \in \mathcal{I}(x)$ and vectors $\tau = \tau(x)$ tangent to $\partial\Omega$. For a given function u_0 on $\partial\Omega$, we define Ω to be uniformly A-convex, (A-convex), with respect to φ and u_0 if (1.8) holds for all $p \cdot v(x) \geq \varphi(x,u_0(x))$, that is, $\mathcal{I} = \{u_0\}$.

From the regularity of A (1.3), we can equivalently replace the boundary inequality $p \cdot v \geq \varphi(x,z)$ by the boundary equality $p \cdot v = \varphi(x,z)$, in the above definitions, as $D_{p_v}A_{ij}(x,z,p)\tau_i\tau_j$ is then non-decreasing with respect to p_v. This leads us to a further definition, which is independent of the boundary condition (1.2). Namely, Ω is uniformly A-convex with respect to $u \in C^1(\overline{\Omega})$ if

$$(1.9) \qquad \left(D_i v_j - D_{p_k}A_{ij}(\cdot,u,Du)v_k\right)\tau_i\tau_j \leq -\delta_0 \quad \text{on } \partial\Omega$$

for all vectors $\tau = \tau(x)$ tangent to $\partial\Omega$ and some positive constant δ_0. Accordingly, if A is regular, Ω is uniformly A-convex with respect to φ and u, and u satisfies (1.2), it follows that Ω is uniformly A-convex with respect to u.

In order to use the regularity of A in its most general form, we will need to assume the existence of a supersolution $\overline{u}$ to (1.1) satisfying

$$(1.10) \qquad \det[D^2\overline{u} - A(x, \overline{u}, D\overline{u})] \le B(x, \overline{u}, D\overline{u}) \quad \text{in } \Omega,$$

together with the same boundary condition,

$$(1.11) \qquad D_\nu\overline{u} = \varphi(x, \overline{u}) \quad \text{on } \partial\Omega.$$

We then have the following global second derivative estimate.

Theorem 1.1 *Let $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be an elliptic solution of the Neumann problem* (1.1)–(1.2) *in a $C^{3,1}$ domain $\Omega \subset \mathbb{R}^n$, which is uniformly A-convex with respect to u, where $A \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is regular and non-decreasing, $B > 0, \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is non-decreasing, and $\varphi \in C^{2,1}(\partial\Omega \times \mathbb{R})$ is non-decreasing. Suppose there exists an elliptic supersolution $\overline{u} \in C^2(\overline{\Omega})$ satisfying* (1.10)–(1.11). *Then we have the estimate*

$$(1.12) \qquad \sup_\Omega |D^2 u| \le C,$$

where C is a constant depending on $n, A, B, \Omega, \overline{u}, \varphi, \delta_0$, and $|u|_{1;\Omega}$.

Theorem 1.1 extends [17, Theorem 3.3] except for the supersolution hypothesis, as a supersolution is constructed in [17] in the course of the proof. We also point out that, as in [17], the restriction to the Neumann condition is critical for our proof, and, moreover, as shown by the Pogorelov example, (see [31, 34]), one cannot generally expect second derivative estimates and classical solutions of (1.1)–(1.2) for $A = 0$ when the geometric normal ν is replaced by an oblique vector β satisfying $\beta \cdot \nu > 0$; that is, in (1.7),

$$(1.13) \qquad G(x, z, p) = \beta \cdot p - \varphi(x, z),$$

no matter how smooth β, φ, B and $\partial\Omega$ are. However, if the matrix function A is strictly regular on $\overline{\Omega}$, so that we have a positive lower bound in (1.3) when z and p are bounded, then the proof is much simpler and also embraces oblique boundary conditions. Moreover, in this case the monotonicity and supersolution hypotheses in Theorem 1.1 can be dispensed with. Typically, second derivative behaviour for equation (1.1) in the strictly regular case is closer to that for uniformly elliptic equations while the challenge in the general case is to carry over the more intricate Monge–Ampère case, $A = 0$. Following [17], we can also relax the supersolution hypothesis for uniformly convex domains in the special case when $D_{px}A = 0$ and $D_{pz}A = 0$; that is,

$$(1.14) \qquad A(x, z, p) = A_0(x, z) + A_1(p),$$

where $A_0 \in C^2(\overline{\Omega} \times \mathbb{R})$ and $A_1 \in C^2(\mathbb{R}^n)$ is regular.

From Theorem 1.1, we obtain classical existence theorems for (1.1)–(1.2) under further hypotheses ensuring estimates for solutions and their gradients. For solution

estimates, by virtue of the comparison principle, we can simply assume the existence of bounded subsolutions and supersolutions.

However, more specific conditions for solution bounds will be treated in Section 3 of this paper, including an extension of the Bakel'man condition in [17, Theorem 2.1]. For the gradient estimate we adopt the same structure condition used for the Dirichlet problem in [9], namely,

$$(1.15) \qquad A(x, z, p) \geq -\mu_0(1 + |p|^2)I,$$

for all $x \in \Omega$, $|z| \leq M_0$, $p \in \mathbb{R}^n$ and some positive constant μ_0 depending on the constant M_0. Condition (1.15) provides a simple gradient bound for A-convex functions u in terms of a lower bound for $D_\nu u$ on the boundary. Combining the second derivative bounds with the lower order bounds and the global second derivative Hölder estimates as in [15–17, 23], we establish the following existence result by the method of continuity.

Theorem 1.2 *Suppose that A, B, φ, $\overline{u}$, and Ω satisfy the hypotheses of Theorem 1.1, with either A, B, or φ being strictly increasing. Assume also condition (1.15) and that there exists an elliptic subsolution $\underline{u} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of equation (1.1), with $D_\nu \underline{u} \geq \varphi(\,\cdot\,, \underline{u})$ on $\partial\Omega$ and that Ω is uniformly A-convex with respect to φ and $\mathcal{I} = [\underline{u}, \overline{u}]$, in the sense of (1.8). Then the Neumann boundary value problem (1.1)–(1.2) has a unique elliptic solution $u \in C^{3,\alpha}(\overline{\Omega})$ for any $\alpha < 1$.*

The uniqueness of solutions follows from the comparison principle for elliptic solutions of general oblique boundary value problems, (1.4)–(1.5); see Lemma 3.1. The regularity for the solution u in Theorem 1.2 can be improved by the linear elliptic theory [5] if the data are sufficiently smooth. For example, if A, B, φ, and $\partial\Omega$ are C^∞, then the solution $u \in C^\infty(\overline{\Omega})$. From the monotonicity of φ, it is also enough to assume (1.8) only holds for $p \cdot \nu \geq \varphi(\,\cdot\,, \underline{u})$ and $\underline{u} \leq z \leq \overline{u}$. Moreover, if A is independent of z, there is no need for the last inequality. Also taking account of our remarks after the statement of Theorem 1.1, we only need to assume the supersolution $\overline{u}$ satisfies (1.10) at points where it is elliptic and the boundary inequality $D_\nu \overline{u} \leq \varphi(\,\cdot\,, \overline{u})$, instead of (1.11), if either A satisfies (1.14) with Ω also uniformly convex or A is strictly regular in $\overline{\Omega}$.

The regular condition for A was originally introduced in [22] in its strict form for interior regularity of potential functions in optimal transportation, with the weak form (1.3) subsequently introduced in [30] for global regularity; see also [24]. It was subsequently shown to be sharp for C^1 regularity of potential functions in [21]. Optimal transportation equations are special cases of prescribed Jacobian equations, which have the general form,

$$(1.16) \qquad |\det DY(\,\cdot\,, u, Du)| = \psi(\,\cdot\,, u, Du),$$

where Y is a C^1 mapping from $\Omega \times \mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$, ψ is a non-negative scalar valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Assuming $\det Y_p \neq 0$, we see that for elliptic solutions, equation (1.16) can be written in the form (1.1) with

$$(1.17) \qquad A = -Y_p^{-1}(Y_x + Y_z \otimes p), \quad B = (\det Y_p)^{-1}\psi.$$

The natural boundary value problem for the prescribed Jacobian equation is the *second boundary value problem* to prescribe the image,

$$(1.18) \qquad Tu(\Omega) := Y(\,\cdot\,, u, Du)(\Omega) = \Omega^*,$$

where Ω^* is another given domain in $\mathbb{R}^n$. The global regularity of the second boundary value problem (1.16)–(1.18) has been studied in [1,25,27,30,32] for different forms of the mapping Y. As shown in [30] in the optimal transportation case and in [25] in the general case, condition (1.18) implies an oblique nonlinear boundary condition for elliptic functions u; that is, (1.5) holds for a function $G \in C^1(\partial\Omega \times \mathbb{R} \times \mathbb{R}^n)$ with

$$(1.19) \qquad G_p(\,\cdot\,, u, Du) \cdot v > 0 \quad \text{on } \partial\Omega.$$

The crucial estimate in these papers is the control on the obliqueness, that is, an estimate of the form, $G_p \cdot v \geq \delta$ for a positive constant δ and this is done in [30] in the optimal transportation case, and extended to the general case in [25], under appropriate uniform convexity conditions on the domain and target, with the latter equivalent to the uniform concavity of the function G with respect to the p variables. Because we are defining obliqueness with respect to the inner normal, in agreement with [17], our function G is the negative of that in [25, 30, 32]. Once the obliqueness is estimated, the boundary second derivative bounds follow in [7, 25, 30] from the same uniform convexity conditions, together with the regular condition (1.3), similarly to the Monge–Ampère case in [33]. Note that the uniform concavity of G excludes the Neumann condition treated here, and, moreover, the derivation of the boundary C^2 estimate is much simpler, being somewhat analogous to using the strict regular condition. We also point out a recent paper [2] considering optimal transportation on a hemisphere where the obliqueness is estimated without using any uniform convexity of domains, which still gives the boundary C^2 estimate in the two dimensional case. Prescribed Jacobian equations also arise in geometric optics where solutions correspond to reflectors or refractors transmitting light rays from a source to a target with prescribed intensities; see for example [7,12,19,26,28,35] and references therein.

On the geometric side, the Neumann boundary value problem in the more general context of augmented Hessian equations on manifolds arises in the study of the higher order Yamabe problem in conformal geometry; see [3, 4, 11, 13, 14]. To explain this we let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$ with nonempty smooth boundary $\partial\mathcal{M}$, let A_g denote the Schouten tensor of the metric g and let $\lambda(A_g) = (\lambda_1(A_g), \ldots, \lambda_n(A_g))$ denote the eigenvalues of A_g. Let $\Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with vertex at the origin and let f be a smooth symmetric function in Γ. The fully nonlinear Yamabe problem on manifolds with boundary is to find a metric $\widetilde{g}$ in the conformal class of the metric g with a prescribed function of eigenvalues of the Schouten tensor and prescribed mean curvature. For example, for a given constant $c \in \mathbb{R}$, we are interested in finding a metric $\widetilde{g}$ conformal to g such that

$$(1.20) \qquad \begin{aligned} F(A_{\widetilde{g}}) &:= f(\lambda(A_{\widetilde{g}})) = 1 \quad \text{for } \lambda(A_{\widetilde{g}}) \in \Gamma \text{ on } \mathcal{M}, \\ h_{\widetilde{g}} &= c \qquad\qquad\qquad \text{on } \partial\mathcal{M}, \end{aligned}$$

where $h_{\widetilde{g}}$ denotes the mean curvature of $\partial\mathcal{M}$ with respect to the inner normal. Writing $\widetilde{g} = e^{-2u}g$ for some smooth function u on $\mathcal{M}$, by the transformation laws for the

Schouten tensor and mean curvature, problem (1.20) is equivalent to the following semilinear Neumann boundary value problem:

(1.21)
$$f(\lambda_g(U)) = e^{-2u}, \qquad \lambda_g(U) \in \Gamma \ \text{on} \ \mathcal{M},$$
$$\frac{\partial u}{\partial \nu} = ce^{-u} - h_g, \quad \text{on} \ \partial\mathcal{M},$$

with

$$U = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g,$$

where $\lambda_g(U)$ denotes the eigenvalues of U with respect to g, ν is the unit inner normal vector field to $\partial\mathcal{M}$, and ∇ denotes the Levi-Civita connection with respect to g. If we choose $f = \det$ and $\Gamma = \Gamma_n := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \sum \lambda_i > 0\}$, then we have an example (1.21) of a semilinear Neumann boundary value problem (1.1)–(1.2) for a Monge–Ampère type equation. In conclusion, a prescribed mean curvature fully nonlinear Yamabe problem (1.20) is equivalent to a semilinear Neumann problem (1.21) for an augmented Hessian equation. The corresponding matrix functions in these cases will be strictly regular when expressed in terms of local coordinates so that in the Monge–Ampère case strong local estimates are available, with second order estimates being considerably simpler than the general regular case we treat here. In fact, the particular Neumann boundary value problem (1.21) with $f = \det$ was studied in [11]. In the special case of Euclidean space $\mathbb{R}^n$, the matrix A is given by

(1.22)
$$A = \frac{1}{2}|p|^2 I - p \otimes p,$$

in which case our A-convexity condition (1.8) reduces to simply $\kappa_1 + \varphi > 0, (\geq 0)$, where κ_1 denotes the minimum curvature of $\partial\Omega$.

The overall organisation of this paper follows that of the Dirichlet problem case [9], where again the main issue was to deal with the general case of regular A. Also, here the strictly regular case is considerably simpler in the case of smooth data but in the optimal transportation case, with only Hölder continuous densities, local and global second derivative estimates were obtained in [6,20], in agreement with the uniformly elliptic case. In Section 2 we prove Theorem 1.1, which constitutes the heart of the paper. In Section 3 we provide the gradient estimate to complete the proof of Theorem 1.2, along with alternative solution bounds for more general oblique boundary value problems. In the optimal transportation case we also prove a Bakel'man type estimate for solutions that extends the Monge–Ampère case in [17]. In Section 4 we switch to the strictly regular case and prove first and second derivative bounds for general oblique boundary value problems (1.5), where G is concave with respect to the p variables, which extend the semilinear conditions (1.13). For this purpose we extend our definition of A-convexity so that a C^2 domain Ω is uniformly A-convex, (A-convex), with respect to G and an interval $\mathcal{I}$ if (1.8) holds for all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$, satisfying $G(x, z, p) \geq 0, z \in \mathcal{I}$ and vectors τ tangent to $\partial\Omega$. When G is independent of z, this corresponds to the c-convexity conditions from optimal transportation [24, 30] and more generally to the Y-convexity conditions for prescribed Jacobian equations in [25]. Finally, we remark that a general theory of oblique boundary value problems for augmented Hessian equations, which embraces our results in Section 4, is presented in [8].

2 Second Derivative Estimates

In this section, we shall derive the second order derivative estimates and complete the proof of Theorem 1.1 by taking full advantage of the assumed C^2 supersolution $\overline{u}$. Note that we only need to get an upper bound for the second derivatives, since the lower bound can be derived from the ellipticity condition $D^2 u - A > 0$.

For the arguments below, we assume the functions φ, ν can be smoothly extended to $\overline{\Omega} \times \mathbb{R}$ and $\overline{\Omega}$, respectively. We also assume that near the boundary, ν is extended to be constant in the normal directions. From equation (1.1), we have

$$(2.1) \qquad \widetilde{F}[u] := \log \det[D^2 u - A(\,\cdot\,, u, Du)] = \widetilde{B}(\,\cdot\,, u, Du),$$

where $\widetilde{B} \triangleq \log B$. We have

$$\frac{\partial \widetilde{F}}{\partial w_{ij}} = w^{ij} \quad \text{and} \quad \frac{\partial^2 \widetilde{F}}{\partial w_{ij} \partial w_{kl}} = -w^{ik} w^{jl},$$

where $\{w_{ij}\} \triangleq \{u_{ij} - A_{ij}\}$ denotes the augmented Hessian matrix and $\{w^{ij}\}$ denotes the inverse of the matrix $\{w_{ij}\}$. We now introduce the following linearized operators of $\widetilde{F}$ and (2.1),

$$L \triangleq w^{ij}\big(D_{ij} - D_{p_l} A_{ij}(\,\cdot\,, u, Du)D_l\big), \quad \mathcal{L} \triangleq L - D_{p_l}\widetilde{B}(\,\cdot\,, u, Du)D_l.$$

For convenience in later discussion, we denote $D_{\xi\eta} u \triangleq D_{ij}u\,\xi_i\eta_j$, $w_{\xi\eta} \triangleq w_{ij}\xi_i\eta_j = D_{ij}u\,\xi_i\eta_j - A_{ij}\xi_i\eta_j$ for any vectors ξ and η. As usual, C denotes a constant depending on the known data and may change from line to line in the context.

Before we start to deal with the second derivative estimates, we recall a fundamental lemma in [7, 9], which is also crucial for constructing the global barrier function using the supersolution in our situation. We shall omit its proof, which is similar to those in [7, 9].

Lemma 2.1 Let $u \in C^2(\overline{\Omega})$ be an elliptic solution of (1.1), and let $\widetilde{u} \in C^2(\overline{\Omega})$ be an elliptic function of equation (1.1) in $\overline{\Omega}$ with $\widetilde{u} \geq u$ in $\overline{\Omega}$, where A is regular and non-decreasing. Then

$$(2.2) \qquad \mathcal{L}\big(e^{K(\widetilde{u}-u)}\big) \geq \epsilon_1 \sum_i w^{ii} - C$$

holds in Ω for sufficiently large positive constant K and uniform positive constants ϵ_1, C depending on $A, B, \Omega, |u|_{1;\Omega}$ and $\widetilde{u}$.

We assume that the domain Ω is uniformly A-convex, with respect to φ and u, and first consider the second derivative estimates on the boundary $\partial\Omega$ in nontangential directions. We introduce the tangential gradient operator $\delta = (\delta_1, \ldots, \delta_n)$, where $\delta_i = (\delta_{ij} - \nu_i\nu_j)D_j$. Applying this tangential operator to the boundary condition (1.2), we have

$$(D_k u)\delta_i\nu_k + \nu_k\delta_i D_k u = \delta_i\varphi, \quad \text{on } \partial\Omega,$$

hence we have

$$(2.3) \qquad |D_{\tau\nu}u| \leq C, \quad \text{on } \partial\Omega,$$

for any tangential vector field τ.

We next deduce the estimate for $D_{vv}u$ on $\partial\Omega$. By a direct calculation, we have

$$(2.4) \qquad Lu = w^{ij}\big(D_{ij}u - D_{p_l}A_{ij}(\cdot, u, Du)D_l u \big)$$
$$= n - w^{ij}\big(A_{ij} - D_{p_l}A_{ij}(\cdot, u, Du)D_l u \big).$$

Differentiating equation (2.1) with respect to x_k, we have, for $k = 1, \ldots, n$,

$$w^{ij}\big(D_{ij}u_k - D_{x_k}A_{ij} - D_z A_{ij}u_k - D_{p_l}A_{ij}D_l u_k \big) = D_{x_k}\widetilde{B} + D_z\widetilde{B}u_k + D_{p_l}\widetilde{B}D_l u_k,$$

which implies

$$(2.5)\quad Lu_k = D_{x_k}\widetilde{B} + D_z\widetilde{B}u_k + D_{p_l}\widetilde{B}D_{kl}u + w^{ij}\big(D_{x_k}A_{ij} + D_z A_{ij}u_k \big), \quad \text{for } k = 1, \ldots, n.$$

If we consider the function $h = v_k D_k u - \varphi(x, u)$, by (2.4) and (2.5), we immediately have

$$(2.6) \qquad |Lh| \le C\big(1 + \sum w^{ii} + |D^2 u|\big), \quad \text{in } \Omega.$$

From the positivity of B we can estimate

$$1 \le Cw^{ii} \quad \text{and} \quad (w_{ii})^{\frac{1}{n-1}} \le Cw^{ii}.$$

Thus, we obtain from (2.6) and the boundary condition (1.2),

$$(2.7) \qquad |Lh| \le C\big(1 + |D^2 u|^{\frac{n-2}{n-1}}\big) \sum w^{ii} \text{ in } \Omega \quad \text{and} \quad h = 0 \text{ on } \partial\Omega.$$

From the uniform A-convexity of Ω (1.9) and the regularity of A, there exists a defining function, $\phi \in C^2(\overline{\Omega})$, satisfying $\phi = 0$ on $\partial\Omega$, $D\phi \ne 0$ on $\partial\Omega$ and $\phi < 0$ in Ω, together with the inequality

$$(2.8) \qquad D_{ij}\phi - D_{p_k}A_{ij}(\cdot, u, Du)D_k\phi \ge \delta_1 I,$$

in a neighbourhood $\mathcal{N}$ of $\partial\Omega$, whenever $D_v u \ge \varphi(x, u)$, where δ_1 is a positive constant and I denotes the identity matrix, with $\mathcal{N}$ and δ_1 depending also on δ_0, A, and $|u|_{1;\Omega}$. We remark that (2.8) follows from (1.9), using the continuity of $D_p A$ with respect to x and z together with an appropriate extension of the distance function, as in for example [5, 30]. In particular, we can take $\phi = -d + td^2$ near $\partial\Omega$, for a large enough positive constant t, where $d(x) = \text{dist}(x, \partial\Omega)$ is the distance function of Ω. Accordingly,

$$(2.9) \qquad L\phi \ge \delta_1 \sum w^{ii},$$

for $h \ge 0$, $d < d_0$, for a positive constant d_0 also depending on δ_0, A and $|u|_{1;\Omega}$. By (2.7), (2.9), and choosing $-\phi$ as a barrier function, a standard barrier argument leads to

$$D_v h \le C\big(1 + M_2^{\frac{n-2}{n-1}}\big) \quad \text{on } \partial\Omega,$$

where $M_2 = \sup_\Omega |D^2 u|$, so that we have the estimate

$$(2.10) \qquad D_{vv}u \le C\big(1 + M_2\big)^{\frac{n-2}{n-1}} \quad \text{on } \partial\Omega.$$

We conclude from (2.3), (2.10) and the ellipticity of u that

$$(2.11) \qquad |D_{v\xi}u| \le C\big(1 + M_2\big)^{\frac{n-2}{n-1}} \quad \text{on } \partial\Omega,$$

for any direction ξ. We remark that if B is independent of p or $n = 2$, then the term M_2 is not present in (2.11).

We have now established the mixed tangential normal derivative bound and the double normal derivative bound on $\partial\Omega$ so that it remains to bound the double tangential second derivatives on $\partial\Omega$. We shall adapt the delicate method in [17], which is specific for the Neumann boundary value problem, to obtain the double tangential derivative bound on the boundary and consequently the global second derivative bound.

Proof of Theorem 1.1 First we note from the comparison principle, Lemma 3.1, that $\overline{u} \geq u$ in Ω or $u - \overline{u}$ is a constant. Discarding the second case, we modify the elliptic supersolution $\overline{u}$ by adding a perturbation function $-a\phi$, where a is a small positive constant and ϕ is the defining function of the domain Ω satisfying $\phi = 0$ on $\partial\Omega$, $\phi < 0$ in Ω and $D_\nu\phi = -1$ on $\partial\Omega$. Note that the new function $\widetilde{u} = \overline{u} - a\phi$ is still uniformly elliptic in Ω if a is sufficiently small. Also, by a direct computation, we have

$$D_\nu(\widetilde{u} - u) = D_\nu\overline{u} - D_\nu u - aD_\nu\phi = \varphi(\cdot, \overline{u}) - \varphi(\cdot, u) + a \geq a,$$

on $\partial\Omega$, where the non-decreasing of φ and $\overline{u} \geq u$ on $\partial\Omega$ are used. If we define a function with the form $\Phi = e^{K(\widetilde{u}-u)}$ with a positive constant K, we then have $D_\nu\Phi \geq Ka > 0$ on $\partial\Omega$. We now introduce an auxiliary function v, given by

$$(2.12) \qquad v = v(\cdot, \xi) = e^{\frac{\alpha}{2}|Du|^2 + \kappa\Phi}\left(w_{\xi\xi} - v'(\cdot, \xi)\right),$$

for $x \in \overline{\Omega}$, $|\xi| = 1$, where α, κ are positive constants to be determined, $\Phi = \frac{1}{\epsilon_1}e^{K(\widetilde{u}-u)}$ is the barrier function in Lemma 2.1 with the above constructed $\widetilde{u}$, and v' is defined by

$$(2.13) \qquad v'(\cdot, \xi) = 2(\xi \cdot \nu)\xi_i'\left(D_i\varphi(\cdot, u) - D_k u D_i\nu_k - A_{ij}\nu_j\right),$$

with $\xi' = \xi - (\xi \cdot \nu)\nu$. Here ν is a $C^{2,1}(\overline{\Omega})$ extension of the inner unit normal vector field on $\partial\Omega$. The strategy of our proof is to estimate v at a maximum point in $\overline{\Omega}$ and vector ξ, in the same form as (2.11). From this we conclude a corresponding global estimate for $D^2 u$ in Ω from which follows the desired estimate (1.12).

Case 1. We suppose that v takes its maximum at an interior point $x_0 \in \Omega$ and a unit vector ξ. Let

$$H = \log v = \log(w_{\xi\xi} - v') + \frac{\alpha}{2}|Du|^2 + \kappa\Phi,$$

then the function H also attains its maximum at the point $x_0 \in \Omega$ and the unit vector ξ. The following analysis follows the method of Pogorelov type estimates in [30], with some modification, adapted from [17], to handle the additional term v'. Accordingly we have, at the point x_0,

$$(2.14) \qquad 0 = D_i H = \frac{D_i(w_{\xi\xi} - v')}{w_{\xi\xi} - v'} + \alpha D_k u D_{ik} u + \kappa D_i\Phi, \quad \text{for } i = 1\cdots n,$$

$$0 \geq D_{ij} H = \frac{D_{ij}(w_{\xi\xi} - v')}{w_{\xi\xi} - v'} - \frac{D_i(w_{\xi\xi} - v')D_j(w_{\xi\xi} - v')}{(w_{\xi\xi} - v')^2}$$
$$+ \alpha(D_{ik}u D_{jk}u + D_k u D_{ijk}u) + \kappa D_{ij}\Phi,$$

and consequently, at x_0

$$(2.15) \quad 0 \geq \mathcal{L}H = \frac{1}{w_{\xi\xi} - v'}\mathcal{L}(w_{\xi\xi} - v') - \frac{1}{(w_{\xi\xi} - v')^2}w^{ij}D_i(w_{\xi\xi} - v')D_j(w_{\xi\xi} - v')$$
$$+ \alpha w^{ij}D_{ik}uD_{jk}u + \alpha D_k u \mathcal{L}u_k + \kappa \mathcal{L}\Phi.$$

Next, we shall estimate each term on the right-hand side of (2.15). We start with some identities. By differentiation of equation (2.1) in the direction ξ, we have in accordance with (2.5),

$$(2.16) \quad w^{ij}(D_{ij}u_\xi - D_\xi A_{ij} - D_z A_{ij}u_\xi - D_{p_l}A_{ij}D_l u_\xi) =$$
$$D_\xi \widetilde{B} + D_z \widetilde{B}u_\xi + D_{p_l}\widetilde{B}D_l u_\xi,$$

and a further differentiation in the direction of ξ yields,

$$(2.17) \quad w^{ij}[D_{ij}u_{\xi\xi} - D_{\xi\xi}A_{ij} - (D_{zz}A_{ij})(u_\xi)^2 - (D_{p_k p_l}A_{ij})D_k u_\xi D_l u_\xi$$
$$- (D_z A_{ij})u_{\xi\xi} - (D_{p_k}A_{ij})D_k u_{\xi\xi} - 2(D_{\xi z}A_{ij})u_\xi$$
$$- 2(D_{\xi p_k}A_{ij})D_k u_\xi - 2(D_{z p_k}A_{ij})(D_k u_\xi)u_\xi]$$
$$= w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} + D_{\xi\xi}\widetilde{B} + (D_{zz}\widetilde{B})(u_\xi)^2 + (D_{p_k p_l}B)D_k u_\xi D_l u_\xi$$
$$+ 2(D_{\xi z}\widetilde{B})u_\xi + 2(D_{\xi p_k}\widetilde{B})D_k u_\xi + 2(D_{z p_k}\widetilde{B})(D_k u_\xi)u_\xi$$
$$+ (D_z \widetilde{B})u_{\xi\xi} + (D_{p_k}\widetilde{B})D_k u_{\xi\xi}.$$

Using (2.17) and the regular condition (1.3) (see [30, (3.9)]), we have

$$(2.18) \quad \mathcal{L}u_{\xi\xi} \geq w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - C\big[(1 + w_{ii})\mathcal{T} + (w_{ii})^2\big],$$

where we denote $\mathcal{T} = w^{ii}$ to avoid any confusion with the usual summation convention. When calculating $\mathcal{L}A_{\xi\xi}$, there will occur third derivative terms of u, which are controlled using (2.16). We then obtain

$$(2.19) \quad |\mathcal{L}A_{\xi\xi}| \leq C\big[(1 + w_{ii})\mathcal{T} + w_{ii}\big]$$

and by a similar calculation, we have

$$(2.20) \quad |\mathcal{L}v'| \leq C\big[(1 + w_{ii})\mathcal{T} + w_{ii}\big].$$

Combining (2.18), (2.19), and (2.20), we have

$$(2.21) \quad \mathcal{L}(w_{\xi\xi} - v') \geq w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - C\big[(1 + w_{ii})\mathcal{T} + (w_{ii})^2\big].$$

By Cauchy's inequality, we have

$$(2.22) \quad w^{ij}D_i(w_{\xi\xi} - v')D_j(w_{\xi\xi} - v') \leq (1 + \theta)w^{ij}D_i w_{\xi\xi}D_j w_{\xi\xi} + C(\theta)w^{ij}D_i v'D_j v'$$

for any $\theta > 0$, where $C(\theta)$ is a positive constant depending on θ.

By (2.2), (2.5), (2.21), and (2.22), we obtain from (2.15)

$$(2.23) \qquad 0 \geq \frac{1}{w_{\xi\xi} - v'} w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} - \frac{1+\theta}{(w_{\xi\xi} - v')^2} w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi}$$
$$+ \alpha w_{ii} + \kappa \mathcal{T} - C \left\{ \frac{1}{w_{\xi\xi} - v'} \left[(1 + w_{ii}) \mathcal{T} + (w_{ii})^2 \right] + \alpha + \kappa \right\}$$
$$- \frac{C(\theta)}{(w_{\xi\xi} - v')^2} w^{ij} D_i v' D_j v'.$$

Without loss of generality, we assume that $\{w_{ij}\}$ is diagonal at x_0 with maximum eigenvalue w_{11}. We can always assume that $w_{11} > 1$ and is as large as we want; otherwise we are done. We proceed first to estimate the third derivative terms in (2.23). From the inequality [17, (3.48)], we have

$$(2.24) \qquad w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} - \frac{1}{w_{11}} w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi} \geq 0.$$

Moreover, since v' is bounded, w_{11} and $w_{\xi\xi}$ are comparable in the sense that for any $\theta > 0$, there exists a further constant $C(\theta)$ such that

$$(2.25) \qquad |w_{11} - w_{\xi\xi} + v'| < \theta w_{11},$$

if $w_{11} > C(\theta)$. From (2.24) and (2.25), we have

$$(2.26) \qquad w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} \geq \frac{1-\theta}{w_{\xi\xi} - v'} w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi}.$$

Next, we use $D_i H = 0$ in (2.14) to estimate

$$(2.27) \qquad w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi} \leq 2 w^{ii} \left[|D_i v'|^2 + (w_{\xi\xi} - v')^2 (\alpha D_k u D_{ik} u + \kappa D_i \Phi)^2 \right]$$
$$\leq 2 w^{ii} |D_i v'|^2 + C (w_{\xi\xi} - v')^2 (\alpha^2 w_{ii} + \kappa^2 \mathcal{T}).$$

Using (2.26) and (2.27) in (2.23), together with (2.25), we obtain the following for $w_{11} \geq C(\theta)$:

$$\alpha w_{ii} + \kappa \mathcal{T} \leq C \left[\alpha + \kappa + (1 + \alpha^2 \theta) w_{ii} + (1 + \kappa^2 \theta) \mathcal{T} \right].$$

By choosing α, κ large and fixing a small positive θ, we thus obtain an estimate $w_{ii}(x_0) \leq C$, which implies a corresponding estimate for $|D^2 u(x_0)|$.

Case 2. We consider the case $x_0 \in \partial\Omega$; namely, the function

$$v(x, \xi) = e^{\frac{\alpha}{2} |Du|^2 + \kappa \Phi} (w_{\xi\xi} - v')$$

attains its maximum over $\overline{\Omega}$ at $x_0 \in \partial\Omega$ and a unit vector ξ. We then consider the following three subcases of different directions of ξ. For this we employ the key trick from [17].

Subcase (i). $\xi = v$, where v is normal to $\partial\Omega$ at x_0. Since we already obtained the double normal derivative bound from (2.10), we have

$$v(x_0, v) \leq C (1 + M_2)^{\frac{n-2}{n-1}}, \quad \text{on } \partial\Omega.$$

Subcase (ii). ξ is neither normal nor tangential to $\partial\Omega$. The unit vector ξ can be written as $\xi = (\xi \cdot \tau)\tau + (\xi \cdot v)v$, where $\tau \in S^{n-1}$, with $\tau \cdot v = 0$, $(\xi \cdot \tau)^2 + (\xi \cdot v)^2 = 1$ and $\xi \cdot v \neq 0$. By the construction of v', we have at x_0,

$$w_{\xi\xi} = (\xi \cdot \tau)^2 w_{\tau\tau} + (\xi \cdot v)^2 w_{vv} + 2(\xi \cdot \tau)(\xi \cdot v)w_{\tau v}$$
$$= (\xi \cdot \tau)^2 w_{\tau\tau} + (\xi \cdot v)^2 w_{vv} + v'(x, \xi).$$

By the construction of v, we then have

$$v(x_0, \xi) = (\xi \cdot \tau)^2 v(x_0, \tau) + (\xi \cdot v)^2 v(x_0, v)$$
$$\leq (\xi \cdot \tau)^2 v(x_0, \xi) + (\xi \cdot v)^2 v(x_0, v),$$

which leads again to

$$v(x_0, \xi) \leq v(x_0, v) \leq C(1 + M_2)^{\frac{n-2}{n-1}}, \quad \text{on } \partial\Omega.$$

Subcase (iii). ξ is tangential to $\partial\Omega$ at x_0. From (2.13), we have $v'(x_0, \xi) = 0$. We then have, at x_0,

$$0 \geq D_v v = D_v \left[e^{\frac{\alpha}{2}|Du|^2 + \kappa\Phi}(w_{\xi\xi} - v') \right]$$
$$= e^{\frac{\alpha}{2}|Du|^2 + \kappa\Phi} \left[(w_{\xi\xi} - v')D_v\left(\frac{\alpha}{2}|Du|^2 + \kappa\Phi\right) + D_v(w_{\xi\xi} - v') \right]$$
$$= e^{\frac{\alpha}{2}|Du|^2 + \kappa\Phi} \left\{ \left[\alpha D_k u D_v(D_k u) + \kappa D_v\Phi \right] w_{\xi\xi} + D_v u_{\xi\xi} - D_v(A_{\xi\xi} + v') \right\}$$
$$= e^{\frac{\alpha}{2}|Du|^2 + \kappa\Phi} \left\{ \left[\kappa D_v\Phi + \alpha D_k u(\varphi_k + \varphi_z D_k u - D_i u D_k v_i) \right] w_{\xi\xi} \right.$$
$$\left. + D_v u_{\xi\xi} - D_v(A_{\xi\xi} + v') \right\}$$
$$\geq e^{\frac{\alpha}{2}|Du|^2 + \kappa\Phi} \left\{ (\kappa c_0 - \alpha M)w_{\xi\xi} + D_v u_{\xi\xi} - D_v(A_{\xi\xi} + v') \right\},$$

where $c_0 = \frac{Ka}{\epsilon_1}$,

$$M = \max_{x \in \partial\Omega} |D_k u(\varphi_k + \varphi_z D_k u - D_i u D_k v_i)|.$$

The above inequality gives a relationship between $w_{\xi\xi}(x_0)$ and $D_v u_{\xi\xi}(x_0)$, namely

$$(2.28) \qquad D_v u_{\xi\xi} \leq -(\kappa c_0 - \alpha M)w_{\xi\xi} + D_v(A_{\xi\xi} + v'), \quad \text{at } x_0.$$

On the other hand, by tangentially differentiating the boundary condition twice, we obtain

$$(D_k u)\delta_i\delta_j v_k + (\delta_i D_k u)\delta_j v_k + (\delta_j D_k u)\delta_i v_k + v_k\delta_i\delta_j D_k u = \delta_i\delta_j\varphi, \quad \text{on } \partial\Omega.$$

Hence for the tangential direction ξ at x_0, we have

$$(2.29) \qquad D_v u_{\xi\xi} \geq \varphi_z D_{ij}u\xi_i\xi_j - 2(\delta_i v_k)D_{jk}u\xi_i\xi_j + (\delta_i v_j)\xi_i\xi_j D_{vv}u - C$$
$$\geq \varphi_z D_{ij}u\xi_i\xi_j - 2(\delta_i v_k)D_{jk}u\xi_i\xi_j - C(1 + M_2)^{\frac{n-2}{n-1}}$$
$$\geq \varphi_z w_{\xi\xi} - 2(\delta_i v_k)D_{jk}u\xi_i\xi_j - C(1 + M_2)^{\frac{n-2}{n-1}},$$

where the double normal boundary estimate (2.10) is used in the second inequality. Inequality (2.29) clearly provides another relationship between $D_v u_{\xi\xi}(x_0)$ and $w_{\xi\xi}(x_0)$. Combining this with (2.28), we obtain

$$(2.30) \quad (\kappa c_0 - \alpha M + \varphi_z)w_{\xi\xi} \leq 2(\delta_i v_k)D_{jk}u\xi_i\xi_j + D_v(A_{\xi\xi} + v') + C(1 + M_2)^{\frac{n-2}{n-1}}, \quad \text{at } x_0.$$

Without loss of generality, we can assume the normal at x_0 to be $\nu = (0, \ldots, 0, 1)$, and correspondingly, we can assume $\{w_{ij}(x_0)\}_{i,j<n}$ is diagonal with maximum eigenvalue $w_{11}(x_0) > 1$, as in the interior case. Observing that the first term on the right-hand side of (2.30) only involves tangential second derivatives and using (2.11), we can then obtain the following estimate at x_0:

$$(\kappa c_0 - \alpha M + \varphi_z)w_{\xi\xi} \le C\big(w_{11} + |DD_\nu u|\big) + C(1 + M_2)^{\frac{n-2}{n-1}}$$
$$\le Cw_{\xi\xi} + C(1 + M_2)^{\frac{n-2}{n-1}}.$$

We now choose κ sufficiently large such that

$$\kappa \ge \frac{2}{c_0}\big[\alpha M - \inf \varphi_z - C\big],$$

and again we obtain

$$(2.31) \qquad\qquad \nu(x_0, \xi) \le C(1 + M_2)^{\frac{n-2}{n-1}}.$$

We now conclude from the above three subcases that if ν attains its maximum over $\overline{\Omega}$ at a point $x_0 \in \partial\Omega$, then $\nu(x_0, \xi)$ is bounded from above as in (2.31), which implies the second derivative $D_{\xi\xi}u(x_0)$ is similarly bounded from above. Combining the above two cases, and using the Cauchy inequality, we obtain the desired estimate (1.12) and complete the proof of Theorem 1.1. ∎

As was remarked in Section 1, we can relax the supersolution hypothesis when $D_{px}A = 0$ and $D_{pz}A = 0$; that is, A is of the form (1.14). Moreover the details are then much simpler, as we do not need to extend the Pogorelev argument to handle third derivatives. Here, we proceed in accordance with [17, Remark 1, Section 3], assuming initially that B is convex with respect to p, and replacing the auxiliary function ν in (2.12) by

$$\nu = \nu(x, \xi) = w_{\xi\xi} - \nu' + \frac{\alpha}{2}|Du|^2 + \kappa\Phi,$$

where now $\widetilde{u} \in C^2(\overline{\Omega})$ in $\Phi = \frac{1}{\epsilon_1}e^{K(\widetilde{u}-u)}$ is an elliptic function with $\widetilde{u} \ge u$ in Ω, as in Lemma 2.1. In place of (2.21), we now have the simpler inequality

$$L(w_{\xi\xi} - \nu') \ge -C(1 + \mathcal{T} + w_{ii}).$$

We obtain an estimate from above for $w_{\xi\xi}$ if the maximum of ν occurs at an interior point of Ω by again taking sufficiently large constants α and κ. If the maximum of ν occurs on the boundary $\partial\Omega$, then we proceed as in Case 2, except now the technical details are simpler, and we do not need $D_\nu\Phi \ge 0$ on $\partial\Omega$, but instead we need Ω uniformly convex, or more generally $\varphi_z + 2\kappa_1 > 0$, where κ_1 is the minimum curvature of $\partial\Omega$, to use (2.29). We then obtain the estimate (1.12) as before, except that the dependence on $\overline{u}$ is replaced by a dependence on an elliptic function $\widetilde{u}$. The removal of the condition that B is convex in p can then be addressed in the same way as in [17] by using Theorem 1.2 to construct a supersolution when B is replaced by its infimum and invoking the full strength of Theorem 1.1.

Remark on Lemma 2.1

The proof of Lemma 2.1 following [7,9] applies very generally. In fact, similarly to [24, Theorem 2.1], we can replace the function "log det" in (2.1) by any increasing concave C^1 function f on an open convex set Γ in the linear space of $n \times n$ symmetric matrices $\mathbb{S}^n$, which is closed under addition of the positive cone. Here the ellipticity conditions are replaced by the augmented Hessians $Mu(\Omega)$, $M\widetilde{u}(\overline{\Omega}) \subset \Gamma$, which imply that the operator $\widetilde{F}$ is elliptic with respect to u and $\widetilde{u}$ on Ω and $\overline{\Omega}$, respectively, and w^{ij} is replaced by $\widetilde{F}_{r_{ij}}$ in the definition of L. The general case is covered with a slightly different proof in part II of [8]; see also [10] for the k-Hessian case. However for the special case of (2.1), the proof of Lemma 2.1 from [7, 9] can also be simplified somewhat by avoiding the perturbation of $\widetilde{u}$ that is one of the key ingredients of the general argument used there. To see this, we can modify the calculations in the proof of [7, Lemma 2.2], with $\epsilon = 0$ and $v = \widetilde{u} - u$, (without using concavity!), to arrive at the inequality

$$Le^{Kv} \geq Ke^{Kv}\left\{ w^{ij}[D_{ij}\widetilde{u} - A_{ij}(\,\cdot\,,\widetilde{u}, D\widetilde{u}) - w_{ij}] - \eta w^{ii} - D_{p_l}\widetilde{B}(\,\cdot\,, u, Du)D_l v \right\},$$

for any positive constant η and sufficiently large constant K depending also on η. We then obtain (2.2) using the simple inequality

$$w^{ij}\left[D_{ij}\widetilde{u} - A_{ij}(\,\cdot\,,\widetilde{u}, D\widetilde{u})\right] \geq w^{ii}\lambda[M\widetilde{u}] > 0,$$

where $\lambda[M\widetilde{u}]$ denotes the minimum eigenvalue of $M\widetilde{u}$, and taking η sufficiently small.

3 Existence and Solution Estimates

In this section we complete the proof of Theorem 1.2 and provide alternative conditions for the maximum modulus for solutions of the Neumann problem (1.1)–(1.2). First, we formulate a comparison principle for general oblique boundary value problems (1.4)–(1.5) with F defined by (1.6), with A and B non-decreasing in z, and $G \in C^1(\partial\Omega \times \mathbb{R} \times \mathbb{R}^n)$, non-increasing in z.

Lemma 3.1 *Let $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with $\mathcal{F}$ elliptic, with respect to u, in Ω and $\mathcal{G}$ oblique with respect to $[u, v]$ on $\partial\Omega$, where $[u, v] = \{\theta u + (1-\theta)v : 0 \leq \theta \leq 1\}$. Assume also that either G is strictly decreasing in z or A or B are strictly increasing in z. Then if $\mathcal{F}[u] \geq \mathcal{F}[v]$ on the subset of Ω, where $\mathcal{F}$ is elliptic with respect to v and $\mathcal{G}[u] \geq \mathcal{G}[v]$ on $\partial\Omega$, we have*

$$(3.1) \qquad\qquad\qquad u \leq v, \quad in \ \Omega.$$

Moreover, if we assume that $\mathcal{F}$ is elliptic with respect to $[u, v]$ on all of Ω, we can relax the strict monotonicity condition on A, B, or G, provided $u - v$ is not a constant.

The proof of Lemma 3.1 is standard. By approximating Ω by a subdomain and approximating u by a smaller elliptic function $\underline{u}$ satisfying $\mathcal{F}[\underline{u}] > \mathcal{F}[u]$, we infer that the function $u - v$ can only take a positive maximum on the boundary $\partial\Omega$, and (3.1) follows from the obliqueness and the strict monotonicity of G. When G is only non-increasing in z, then we can take $\underline{u} = u - \epsilon(\phi - \min\phi)$ for a defining function $\phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $\phi = 0$ on $\partial\Omega$, $\phi < 0$ in Ω and sufficiently small $\epsilon > 0$,

to ensure $\mathcal{G}[\underline{u}] > \mathcal{G}[u]$ on $\partial\Omega$, whence a positive maximum of $\underline{u} - v$ must be taken on in Ω, and we conclude (3.1) from the strict monotonicity of F with respect to z. Note that when G is strictly decreasing, we need only assume that $\mathcal{G}$ is weakly oblique; that is, $G_p \cdot v \geq 0$ on $\partial\Omega$, while when F is strictly decreasing we need only assume $\mathcal{F}$ is degenerate elliptic. In the case when there is no strict monotonicity, the difference $w = u - v$ will satisfy a linear uniformly elliptic differential inequality of the form

$$\mathcal{L}w := a^{ij}D_{ij}w + b_i D_i w + cw \geq 0,$$

together with an oblique boundary inequality, $\beta \cdot Dw \geq \gamma w$, with coefficients $c \leq 0$ and $\gamma \geq 0$, and the result follows from the strong maximum principle and Hopf boundary point lemma; see [5].

We remark that $\mathcal{F}$ is automatically elliptic with respect to v at an interior positive maximum of $u - v$, provided $\mathcal{F}$ is elliptic with respect to u. In fact, assuming that the positive maximum of $u - v$ is attained at a point $x_0 \in \Omega$, we have $Mv(x_0) \geq Mu(x_0)$ from the monotonicity of A. Therefore, we only require $\mathcal{F}[u] \geq \mathcal{F}[v]$ on the subset of Ω, where $\mathcal{F}$ is elliptic with respect to v, but not all of the domain Ω.

From Lemma 3.1 we have immediately the uniqueness in Theorem 1.2 and the inequality $\underline{u} \leq u \leq \overline{u}$, where $\overline{u}$ and $\underline{u}$ are the assumed elliptic supersolution (1.10)–(1.11) and subsolution.

Next, we obtain a gradient bound for A-convex functions for Neumann problem (1.1)–(1.2), where A satisfies a quadratic bound from below, (1.15), by a modification of our argument for the Dirichlet problem in [9]. For this purpose, we formulate the following gradient estimate as a lemma.

Lemma 3.2   *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy*

$$(3.2) \qquad\qquad D^2 u \geq -\mu_0\left(1 + |Du|^2\right)I,$$

in a C^2 domain $\Omega \subset \mathbb{R}^n$, with

$$(3.3) \qquad\qquad D_v u \geq -\sigma,$$

on $\partial\Omega$, where μ_0 and σ are non-negative constants. Then we have the estimate

$$(3.4) \qquad\qquad |Du| \leq C,$$

where C depends on μ_0, σ, Ω and $\sup|u|$.

Proof   Defining $\widetilde{u} = u - \sigma\phi$, whereas in Section 2, $\phi \in C^2(\overline{\Omega})$ is a negative defining function for Ω satisfying $D_v\phi = -1$ on $\partial\Omega$, we see that $v \cdot D\widetilde{u} \geq 0$ on $\partial\Omega$. Consequently, at a maximum point $x_0 \in \overline{\Omega}$ of the function

$$(3.5) \qquad\qquad w = e^{\kappa\widetilde{u}}|D\widetilde{u}|^2,$$

we have

$$(3.6) \qquad\qquad D\widetilde{u} \cdot Dw \leq 0.$$

From (3.2), we have

$$(3.7) \quad D^2\widetilde{u} = D^2 u - \sigma D^2\phi \geq -\mu_0\left(1 + |Du|^2\right)I - \sigma\Lambda_\phi I$$
$$\geq -\mu_0\left(1 + 2|D\widetilde{u}|^2 + 2\sigma^2|D\phi|^2\right)I - \sigma\Lambda_\phi I \geq -\mu_1\left(1 + |D\widetilde{u}|^2\right)I,$$

for some positive constant μ_1 depending on μ_0, σ, $D\phi$ and Λ_ϕ, where Λ_ϕ denotes the maximum eigenvalue of the Hessian matrix of ϕ and depends on the domain Ω. With the lower quadratic bound (3.7) for the Hessian matrix $D^2\widetilde{u}$ in hand, by choosing the constant κ sufficiently large as in [9, Section 4], we can obtain

$$(3.8) \qquad\qquad |D\widetilde{u}| \le C,$$

from (3.6), at x_0, where the constant C depends on μ_0, σ and Ω. We then conclude a global gradient estimate from (3.8) and the construction of $\widetilde{u}$, $|Du| \le C$, where C depends on μ_0, Ω, σ, and $\sup |u|$. ∎

We remark that by taking more careful account of the constant dependence in the proof of Lemma 3.2, we infer a sharper estimate

$$(3.9) \qquad\qquad |Du| \le C(1 + \sigma),$$

where C depends on μ_0, Ω and $\sup |u|$.

Note that the gradient estimate (3.4) in Lemma 3.2 and the sharper gradient estimate (3.9) hold for any solution u satisfying the weak convexity condition (3.2) and the lower bound condition (3.3) for normal derivative on the boundary. We now apply Lemma 3.2 to obtain the gradient estimate for A-convex solutions of the Neumann problem (1.1)–(1.2) with A satisfying the lower quadratic bound (1.15). From the A-convexity of the solution u and the quadratic structure condition (1.15), the solution u satisfies the weak convexity condition (3.2). The Neumann boundary condition (1.2) provides us a lower bound $D_\nu u \ge \inf_{\partial\Omega} \varphi(x, u)$. Applying Lemma 3.2, we then obtain the global gradient estimate for Neumann problem (1.1)–(1.2); that is, $|Du| \le C$ for C depending on μ_0, Ω, φ, and $\sup |u|$.

Since we now have obtained the derivative estimates up to second order, we can use the continuity method to prove our existence theorem.

Proof of Theorem 1.2. From the second derivative estimate, Theorem 1.1, and the preceding solution and gradient estimates, we can derive a global second derivative Hölder estimate

$$(3.10) \qquad\qquad |u|_{2,\alpha;\Omega} \le C,$$

for elliptic solutions $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ of the semilinear Neumann boundary value problem (1.1)–(1.2) for $0 < \alpha < 1$. The estimate (3.10) is obtained in [16, Theorem 3.2] (see also [15, 23]). With this $C^{2,\alpha}$ estimate, we can use the method of continuity, [5, Theorems 17.22 and 17.28], to derive the existence of a solution $u \in C^{2,\alpha}(\overline{\Omega})$, using the supersolution $\overline{u}$ as an initial solution. To be rigorous, we should assume that A and B are $C^{2,\alpha}$ smooth, φ is $C^{3,\alpha}$ smooth, and $\Omega \in C^{4,\alpha}$ for some $\alpha > 0$ to get a solution $u \in C^{4,\alpha}(\overline{\Omega})$ by the Schauder theory, (see [5, Section 6.7]), and then by approximation get a solution $u \in C^{3,\alpha}(\overline{\Omega})$. Alternatively we can use the Aleksandrov–Bakel'man maximum principles (see [5, Theorems 9.1 and 9.6]) to carry over the proof of Theorem 1.1 to solutions $u \in W^{4,n}(\Omega) \cap C^3(\overline{\Omega})$ and use L^p regularity as well ([5, Section 9.5]) to improve $C^{2,\alpha}(\overline{\Omega})$ solutions with $0 < \alpha < 1$ to be in the Sobolev spaces $W^{4,p}(\Omega) \cap C^{3,\delta}(\overline{\Omega})$ for all $p < \infty$, $0 < \delta < 1$. ∎

In the rest of this section we will consider more explicit conditions for solution bounds. Here we consider the oblique boundary value problems (1.4)–(1.5) with F

defined by (1.6) and G defined by (1.13), that is, the Monge–Ampère type equation (1.1) together with the oblique boundary condition

$$(3.11) \qquad D_\beta u = \varphi(x, u), \quad \text{on } \partial\Omega.$$

First, we note that we also obtain bounds for solutions u of (1.1)–(1.2) if $\overline{u}$ and $\underline{u}$ are only assumed to be supersolutions and subsolutions, without any assumed boundary conditions, provided we strengthen the monotonicity of φ. In particular, we can assume, as in [17], that there exists a positive constant γ_0 such that

$$(3.12) \qquad \varphi_z(x, z) \geq \gamma_0$$

for all $(x, z) \in \partial\Omega \times \mathbb{R}$. In the light of Lemma 3.1, we can interpret a supersolution as satisfying (1.10) only at points of ellipticity. Since A and B are non-decreasing, supersolutions and elliptic subsolutions are preserved under addition and subtraction respectively of positive constants. Accordingly, by subtracting a positive constant from $\underline{u}$ and using (3.12), we can assume $D_\beta \underline{u} \geq \varphi(x, \underline{u})$ on $\partial\Omega$, whence $u \geq \underline{u}$ in Ω. Similarly, by adding a positive constant to $\overline{u}$ we obtain $D_\beta \overline{u} \leq \varphi(x, \overline{u})$ on $\partial\Omega$, so that $u \leq \overline{u}$ in Ω. Note that for this argument we can replace (3.12) by the weaker conditions

$$(3.13) \qquad (\operatorname{sign} z)\varphi(\,\cdot\,, z) \to \infty, \text{ as } |z| \to \infty.$$

The conditions (3.12), (3.13) can be further weakened when constants are subsolutions or supersolutions. We first consider the bound from below, under the following conditions:

$$(3.14) \quad A(x, z, 0) \leq 0, \ \det[-A(x, z, 0)] > B(x, z, 0), \quad \text{for all } x \in \Omega, \ z < -K,$$

$$(3.15) \qquad \varphi(x, z) < 0, \quad \text{for all } x \in \partial\Omega, \ z < -K,$$

where K is a positive constant. Under assumptions (3.14) and (3.15), we can readily obtain the solution bound as follows. Suppose u attains its minimum over $\overline{\Omega}$ at a point x_0 and $u(x_0) < -K$. If $x_0 \in \Omega$, we have $Du(x_0) = 0$, $D^2 u(x_0) \geq 0$. From equation (1.1), we have $\det[-A(x_0, u(x_0), 0)] - B(x_0, u(x_0), 0) \leq 0$ so that by (3.14), we must have $u(x_0) \geq -K$. If $x_0 \in \partial\Omega$, we have $D_\beta u(x_0) \geq 0$. From the oblique boundary condition, we have $\varphi(x_0, u(x_0)) \geq 0$. By (3.15), we again have $u(x_0) \geq -K$. Note that condition (3.14) implies sufficiently small constants are subsolutions of the oblique boundary value problem (1.1) with (3.11), thereby providing lower solution bounds, by the comparison principle, Lemma 3.1. Therefore, the subsolution assumption in Theorem 1.2 can be replaced by the structure conditions (3.14) and (3.15), with $\min \underline{u}$ replaced by $-K$ in $\mathcal{I}$. We also remark that condition (3.14) follows from a uniform monotonicity condition on A, namely $D_z A_{ij}(x, z, p)\xi_i \xi_j \geq \gamma_1 |\xi|^2$, for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and some $\gamma_1 > 0$, which is a stronger form of the A4w condition used for generated prescribed Jacobian equations in geometric optics in [7, 28], together with B being non-decreasing in z.

In this sense, condition (3.14) is a weakening of the uniform monotonicity of A, while condition (3.15) is a weakening of the uniform monotonicity of φ. On the other hand, condition (3.14) is restrictive in that it excludes the case when A is independent of z, which occurs in optimal transportation.

Corresponding conditions also provide bounds from above. Here though, the analogue of (3.14) is more general; namely,

$$(3.16) \qquad \det[-A(x,z,0)] < B(x,z,0), \quad \text{for all } x \in \Omega, \ z > K, \ A(x,z,0) < 0,$$

while instead of (3.15), we have

$$\varphi(x,z) > 0, \quad \text{for all } x \in \partial\Omega, \ z > K,$$

where K is a positive constant. Note that condition (3.16) extends the condition in [9, Section 4], namely that the maximum eigenvalue of $A(x,z,0)$ is non-negative for all $x \in \Omega, \ z > K$ for some positive constant K and implies that constants larger than K will be supersolutions, where they are elliptic.

To complete this section, we derive a lower bound for optimal transportation equations and present the corresponding existence result.

Optimal Transportation Equations

In the optimal transportation case, we can replace the existence of a subsolution in Theorem 1.2 by an extension of the sharp conditions [17, (1.4), (1.5)] through an extension of the Aleksandrov–Bakel'man estimate in [17, Theorem 2.1]. Optimal transportation equations are special cases of prescribed Jacobian equations where the mapping Y is generated by a cost function c defined on a domain $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^n$. We assume $\overline{\Omega} \times \overline{\Lambda} \subset \mathcal{D}$ for some domain $\Lambda \subset \mathbb{R}^n$, and $c \in C^2(\mathcal{D})$ satisfies the following conditions (from [22]):

A1 For each $x \in \Omega$, the mapping $c_x(x, \cdot)$ is one-to-one in

$$y \in \mathcal{D}_x^* = \{y \in \mathbb{R}^n \mid (x,y) \in \mathcal{D}\};$$

A2 $\det c_{x,y} \neq 0$ on $\mathcal{D}$.

Then the mapping Y is given by

$$(3.17) \qquad\qquad Y(x,p) = c_x^{-1}(x, \cdot)(p)$$

and is well defined for $p \in \mathcal{U}_x = \{p \in \mathbb{R}^n \mid p = c_x(x,y) \text{ for some } y \in \mathcal{D}_x^*\}$. In the resultant Monge–Ampère type equation, we then have from (1.17),

$$A(x,z,p) = A(x,p) = c_{xx}(x, Y(x,p)), \quad B = |\det c_{x,y}|\psi,$$

and equation (1.1) is well defined for solutions u that are A-convex and satisfy $Du(x) \in \mathcal{U}_x$ for each $x \in \Omega$. We call such solutions admissible. In the optimal transportation case, c-affine functions, that is, functions of the form $\overline{u} = c(x,y) + c_0$, for constant c_0 and $(\Omega, \{y\}) \subset \mathcal{D}$ are automatically supersolutions, as they satisfy the homogeneous equation

$$\det(D^2\overline{u} - A(x,D\overline{u})) = 0,$$

and hence provide upper bounds for solutions of (weakly) oblique boundary value problems,

$$D_\beta u = \varphi(x,u), \quad \text{on } \partial\Omega,$$

where $\beta \cdot v \geq 0$ on $\partial\Omega$, under a uniform monotonicity condition (3.12). For lower bounds we impose a structure condition

$$(3.18) \qquad\qquad \psi(x, z, p) \leq \frac{f(x)}{f^* \circ Y(x, p)}$$

for all $x \in \Omega$, $z \leq m_0$, $Y(x, p) \in \Lambda$, where $f \geq 0, \in L^1(\Omega)$, $f^* > 0, \in L^1_{loc}(\Lambda)$ satisfy

$$(3.19) \qquad\qquad \int_\Omega f < \int_\Lambda f^*$$

and m_0 is a constant.

We now have the lower solution bound in the optimal transportation case.

Lemma 3.3 *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be an admissible solution of equation (1.16), in the optimal transportation case (3.17), with cost function c satisfying A1 and A2. Suppose that ψ satisfies (3.18) and*

$$(3.20) \qquad\qquad D_\beta u \leq \gamma_0 u + \varphi_0 \quad\text{ on } \partial\Omega,$$

for $u \leq m_0$, where $\beta \in L^\infty(\partial\Omega)$, $\beta \cdot v \geq 0$ on $\partial\Omega$ and $\gamma_0 > 0$ and $\varphi_0 \geq 0$ are constants. Then we have the lower bound $u \geq -C$, in Ω, where C is a positive constant depending on $\Omega, f, f^*, \beta, \gamma_0, \varphi_0$ and c.

Proof Our proof is adapted from the second author's 2004 Singapore Institute of Mathematical Sciences lectures and the case where $c(x, y) = x \cdot y$, that is, $Y = p$ and $A = 0$, in [17]. First, we note that if we have a global support from below at a point $x_0 \in \Omega$, that is,

$$(3.21) \qquad\qquad u(x) \geq u(x_0) + c(x, y_0) - c(x_0, y_0)$$

for all $x \in \Omega$, then we must have $y_0 = Y(x_0, Du(x_0))$. Defining $T = Y(\cdot, Du)$, we have by (1.16), (3.18), and the change of variable formula

$$\int_\Omega f \geq \int_\Omega |\det DT| f^* \circ T \geq \int_{T(\Omega_0)} f^*$$

where $\Omega_0 = \{x \in \Omega \mid u(x) < m_0\}$. Hence, by our condition (3.19) on f and f^*, there exists a point $y_0 \in \Lambda - T(\Omega_0)$. It then follows by upward vertical translation of a c-affine lower bound, that there exists a point $x_0 \in \partial\Omega_0$ such that

$$u(x) \geq u(x_0) + c(x, y_0) - c(x_0, y_0)$$

for all $x \in \Omega$. If $x_0 \in \partial\Omega$, we must also have

$$D_\beta u(x_0) \geq D_\beta c(x_0, y_0)$$

whence by the boundary inequality (3.20), we obtain

$$u(x_0) \geq \frac{1}{\gamma_0}[D_\beta c(x_0, y_0) - \varphi_0].$$

If $x_0 \notin \partial\Omega$, then we must have $u(x_0) = m_0$. Hence by (3.21), we obtain the following for $x_0 \in \partial\Omega$:

$$(3.22) \qquad u(x) \geq u(x_0) + c(x, y_0) - c(x_0, y_0)$$

$$\geq \frac{1}{\gamma_0}\big[D_\beta c(x_0, y_0) - \varphi_0 \big] + c(x, y_0) - c(x_0, y_0)$$

$$\geq -\frac{\varphi_0}{\gamma_0} - \Big(\frac{|\beta|}{\gamma_0} + \operatorname{diam}\Omega \Big) \sup_\Omega |c_x(\cdot, y_0)|,$$

while for $x_0 \notin \partial\Omega$, we obtain

$$(3.23) \qquad u(x) \geq m_0 - (\operatorname{diam}\Omega) \sup_\Omega |c_x(\cdot, y_0)|.$$

To remove the dependence on y_0 in (3.22) and (3.23), we can consider an exhaustion of Λ, say by defining subdomains

$$\Lambda_R = \Big\{ y \in \Lambda \mid |y| < R, \operatorname{dist}(y, \partial\Lambda) > \tfrac{1}{R} \Big\}$$

for $R \geq 1$. Then by (3.19), we have

$$\int_\Omega f < \int_{\Lambda_R} f^*$$

for some sufficiently large R, and we obtain from (3.22) and (3.23), the estimate,

$$u(x) \geq \min\Big\{ m_0, -\frac{\varphi_0}{\gamma_0} \Big\} - \Big(\frac{|\beta|}{\gamma_0} + \operatorname{diam}\Omega \Big) \sup_{\Omega \times \Lambda_R} |Dc|.$$

This completes the proof of Lemma 3.3. ∎

As a corollary of Lemma 3.3 and the proof of Theorem 1.2, we then have the following variant of Theorem 1.2 in the optimal transportation case. For this purpose, we note that the boundary condition (1.2) and the monotonicity condition (3.12) imply (3.20) with $\beta = \nu$ and

$$\varphi_0 = -\gamma_0 m_0 + \sup_{\partial\Omega} \varphi(\cdot, m_0).$$

Corollary 3.1 *Suppose that equation (1.1) is a prescribed Jacobian equation of the form (1.16) generated by a cost function $c \in C^2(\mathcal{D})$ satisfying conditions A1 and A2 and $\mathcal{U}_x = \mathbb{R}^n$ for all $x \in \Omega$, with ψ satisfying the structure conditions (3.18) and (3.19). Let A, B, φ, and Ω satisfy the hypotheses of Theorem 1.2 except for the existence of an elliptic subsolution, with φ satisfying (3.12) and Ω assumed to be uniformly A-convex with respect to φ and $-C$; that is (1.8) holds for $p \cdot \nu \geq \varphi(\cdot, -C)$ on $\partial\Omega$, where C is the constant in Lemma 3.3. Then the Neumann boundary value problem (1.1)–(1.2) has a unique elliptic solution $u \in C^{3,\alpha}(\overline{\Omega})$ for any $\alpha < 1$.*

We remark that as in [17], condition (3.19) is necessary for an elliptic solution $u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ of (1.16), with $Du(x) \in \mathcal{U}_x$ for all $x \in \overline{\Omega}$.

In accordance with our remarks following the statement of Theorem 1.2, pertaining to the special case (1.14), and using the argument at the end of Section 2, we can remove the supersolution condition in Corollary 3.1 for convex domains. To apply the argument at the end of Section 2 (before Remark on Lemma 2.1), we also need to

use the existence of an elliptic function, as provided by [7, Lemma 2.1]. In this way, we obtain an extension of [17, Theorem 1.1], which corresponds to the special case $c(x, y) = x \cdot y$, (or equivalently, the case $c(x, y) = -|x - y|^2/2$). Note that the matrix $A = A(p)$ satisfies (1.14) when the cost function $c = c(x - y)$. Examples of regular and strictly regular cost functions are given in [18, 30]. However, most of these examples do not satisfy $\mathcal{U}_x = \mathbb{R}^n$, and in general we need additional controls on gradients to prove classical existence theorems.

We also remark that Lemma 3.3 and Corollary 3.1 are readily extended to generated prescribed Jacobian equations [28].

4 Oblique Boundary Value Problems

In this section we consider more general oblique boundary value problems for Monge–Ampère type equations under the hypothesis that the matrix function A is strictly regular. As remarked in Section 1, this condition also leads to a much simpler proof in the Neumann case. Also, we do not need to restrict to semilinear problems of the form (1.13) but can consider nonlinear boundary conditions of the general form (1.5), where G is also concave with respect to p. Our approach is already indicated in [30, Section 4], and we will carry over some of the basic details from there. Moreover, our results can also be seen as special cases of those for general augmented Hessian equations in [8]. For second derivative estimates, we will assume that the function $G \in C^2(\partial\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is oblique with respect to a solution u; that is, from (1.19),

$$(4.1) \qquad G_p(\cdot, u, Du) \cdot \nu \geq \beta_0, \quad \text{on } \partial\Omega,$$

for a positive constant β_0, and is concave in p, with respect to u, in the sense that

$$(4.2) \qquad G_{pp}(\cdot, u, Du) \leq 0, \quad \text{on } \partial\Omega.$$

We now have the following extension and improvement of Theorem 1.1 in the strictly regular case.

Theorem 4.1 *Let $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be an elliptic solution of the boundary value problem (1.1)–(1.5) in a $C^{3,1}$ domain $\Omega \subset \mathbb{R}^n$, which is uniformly A-convex with respect to G and u, where $A \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is strictly regular in $\overline{\Omega}$, $B > 0, \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, and $G \in C^{2,1}(\partial\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfies (4.1) and (4.2). Then we have the estimate*

$$(4.3) \qquad \sup_{\Omega} |D^2 u| \leq C,$$

where C is a constant depending on $n, A, B, G, \Omega, \beta_0$, and $|u|_{1;\Omega}$.

Proof As in the proof of Theorem 1.1, we first consider the estimation of the nontangential second derivatives. In the semilinear case (1.13), we can simply replace ν by β there and deduce in place of (2.11), the estimate

$$(4.4) \qquad |D_{\beta\xi} u| \leq C(1 + M_2)^{\frac{n-2}{n-1}}, \quad \text{on } \partial\Omega,$$

for any direction ξ, where as in Section 2, $M_2 = \sup_{\Omega} |D^2 u|$. In the general case, we have the same estimate (4.4) from [30, estimate (4.4)], where now $\beta = G_p(\cdot, u, Du)$.

Now, differentiating the boundary condition (1.5) twice with respect to a tangential C^2 vector field τ we obtain as in the estimate (4.10) in [30],

$$(4.5) \qquad u_{\tau\tau\beta} \geq -D_{p_k p_l} G u_{k\tau} u_{l\tau} - C(1 + M_2) \geq -C(1 + M_2), \qquad \text{on } \partial\Omega,$$

by virtue of the concavity of G with respect to p. For convenience, we write $u_{i\tau} = u_{ij}\tau_j$, $u_{\tau\tau} = u_{ij}\tau_i\tau_j$, $u_{\tau\tau\beta} = u_{ijk}\tau_i\tau_j\beta_k$. To handle the pure tangential derivatives we extend the C^2 vector field τ to all of $\overline{\Omega}$ and set

$$v = w_{\tau\tau} - K(1 + M_2)\phi,$$

whereas in the proof of Theorem 1.1, $\phi \in C^2(\overline{\Omega})$ is a negative defining function for Ω satisfying $D_\nu\phi = -1$ on $\partial\Omega$ and K is a constant such that

$$D_\beta(w_{ij}\tau_i\tau_j) > -K(1 + M_2)\beta_0, \qquad \text{on } \partial\Omega.$$

In particular we can fix τ with $\tau_i = x_i - (x \cdot \nu)\nu_i$, $i = 1, \ldots, n$, where as in Section 2, ν is a smooth extension of the inner normal ν to $\overline{\Omega}$. It then follows that $D_\beta v > 0$ on $\partial\Omega$ so that v must take its maximum on $\overline{\Omega}$ at an interior point $x_0 \in \Omega$, with $\mathcal{L}v(x_0) \leq 0$. Now we can adapt the proof of the interior second derivative estimate in [22, 29], differentiating the equation (1.1), in the form (2.1), twice with respect to τ and using also the concavity of the function "log det", together with (4.5) to control K, to estimate at x_0,

$$(4.6) \qquad w^{ij}A_{ij,kl}u_{k\tau}u_{l\tau} \leq C\big[(1 + M_2)w^{ii} + |Du_\tau|^2\big],$$

where the last term $|Du_\tau|^2$ is from the twice differentiation of B. We remark that if B is convex with respect to p, then the term $|Du_\tau|^2$ is not present in (4.6). We note that when we twice differentiate (1.1) with respect to a variable vector field τ, to calculate $\mathcal{L}v$, we encounter terms arising from derivatives of τ that are not present in the constant case (2.17). Apart from the terms in third derivatives these can be directly estimated by $C(1 + M_2)w^{ii}$. Retaining the third derivative terms, we would supplement the right-hand side of (4.6), by

$$(4.7) \qquad \begin{aligned} &-w^{ik}w^{jl}D_\tau w_{ij}D_\tau w_{kl} - 4w^{ij}D_i\tau_k D_\tau w_{jk} \\ &= -w^{ik}w^{jl}D_\tau w_{ij}D_\tau w_{kl} - 4w^{ik}w^{jl}w_{js}D_i\tau_s D_\tau w_{kl} \\ &\leq -w^{ik}w^{jl}\big(D_\tau w_{ij} + 2w_{js}D_i\tau_s\big)\big(D_\tau w_{kl} + 2w_{lt}D_k\tau_t\big) \\ &\quad + 4w^{ik}w^{jl}w_{js}w_{lt}D_i\tau_s D_k\tau_t \\ &\leq 4w^{ij}w_{kl}D_i\tau_k D_j\tau_l \leq C(1 + M_2)w^{ii}, \end{aligned}$$

so that estimate (4.6) is unaffected. To use the strictly regular condition,

$$A_{ij,kl}\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2,$$

for all $\xi, \eta \in \mathbb{R}^n$ satisfying the orthogonality $\xi \perp \eta$, where c_0 is a positive constant depending on A and $|u|_{1;\Omega}$, we choose coordinates so that w is diagonalised at x_0, so that

$$(4.8) \quad \begin{aligned} w^{ij}A_{ij,kl}w_{k\tau}w_{l\tau} &= w^{ii}A_{ii,kl}w_{kk}w_{ll}\tau_k\tau_l \geq \sum_{k,l \neq i} w^{ii}A_{ii,kl}w_{kk}w_{ll}\tau_k\tau_l - CM_2 \\ &\geq c_0 w^{ii}\sum(w_{kk}\tau_k)^2 - CM_2. \end{aligned}$$

Hence, we obtain the following from (4.6), (4.8), and (2.7):

$$(4.9) \qquad D_{\tau\tau}u(x_0) \leq C(1 + M_2)^{\frac{1}{2}}.$$

At this point we need to return to our choice of ϕ to ensure that $\inf \phi \geq -\epsilon$ for some small positive constant ϵ. This can be done, for example, by mollification of the function $- \inf\{d, \epsilon\}$ for sufficiently small ϵ, where the constant $C = C_\epsilon$ in (4.9) will depend also on ϵ. Alternatively, we may simply restrict to a boundary strip $\Omega_\epsilon = \{\phi > -\epsilon\}$ and use the interior second derivative estimates [22, 30] to estimate v on the inner boundary $\{\phi = -\epsilon\}$. Accordingly, we obtain the following from (4.9):

$$v(x_0) \leq C_\epsilon(1 + M_2)^{\frac{1}{2}} + \epsilon M_2,$$

and hence we get an estimate

$$(4.10) \qquad D_{\tau\tau}u \leq C_\epsilon(1 + M_2)^{\frac{1}{2}} + \epsilon M_2 \quad \text{on } \partial\Omega.$$

Since for any direction ξ, we have,

$$u_{\xi\xi} = u_{\tau\tau} + b(u_{\tau\beta} + u_{\beta\tau}) + b^2 u_{\beta\beta},$$

where

$$b = \frac{\xi \cdot v}{\beta \cdot v}, \quad \tau = \xi - b\beta,$$

we then obtain a boundary estimate in the form,

$$\sup_{\partial\Omega} |D^2 u| \leq \epsilon M_2 + C_\epsilon,$$

for any sufficiently small $\epsilon > 0$, by combining (4.4) and (4.10). The global second derivative estimate (4.3) now follows from the global second derivative estimates in [29, 30] by choosing ϵ sufficiently small. ∎

The details in the proof of Theorem 4.1 can be further varied. For example, we can replace v by

$$(4.11) \qquad v = (1 - K\phi)w_{\tau\tau}$$

for a sufficiently large constant K, where ϕ is the same negative defining function as in the proof of Theorem 4.1. As remarked in Section 1, we also obtain a much simpler proof of Theorem 1.1 in the strictly regular case without need for the supersolution and monotonicity hypotheses. Moreover, by flattening the boundary $\partial\Omega$ in a neighbourhood $\mathcal{N}$ of a fixed point $x_1 \in \partial\Omega$, we can localise the second derivative estimate by modifying (4.11):

$$v = \eta(1 - K\phi)w_{\tau\tau},$$

where η is a suitable cut-off function satisfying $D_v \eta = 0$ on $\mathcal{N} \cap \partial\Omega$. Accordingly, we obtain for any ball $B = B_R(x_0)$ of radius $R > 0$ and centre x_0, the local estimate

$$(4.12) \qquad |D^2 u(x_0)| \leq \frac{C}{R^2},$$

for elliptic solutions $u \in C^4(B \cap \Omega) \cap C^3(B \cap \overline{\Omega})$ of (1.1) satisfying (1.2) on $B \cap \partial\Omega$, where $B \cap \partial\Omega$ is uniformly A-convex with respect to G and u in the sense that

$$(D_i v_j - D_{p_k} A_{ij}(\cdot, u, Du)v_k)\tau_i \tau_j \leq -\delta_0$$

on $B \cap \partial\Omega$ for $G(x, u, Du) \geq 0$ and any unit tangential vector τ and a positive constant δ_0. The constant C in (4.12) depends on $n, A, B, \Omega, \delta_0, \phi$, and $|u|_{1;\Omega}$. We also point out that comparability of differentiation with respect to a general vector field and a constant vector field in the proof of Theorem 4.1, which follows from the identity (4.7), is special to the Monge–Ampère case. A different and more detailed proof of the critical tangential estimate (4.10) is provided for more general augmented Hessian equations in [8, Lemma 2.3].

Returning to the example from conformal geometry in Section 1, namely (1.21), (1.22) with $\mathcal{M} = \Omega \subset \mathbb{R}^n$, the A-convexity condition also simplifies in that Ω is A-convex (uniformly A-convex) with respect to G and u if and only if

$$\kappa_1 \geq, (>), -ce^{-u} + h_{\partial\Omega} \quad \text{on } \partial\Omega,$$

where κ_1 denotes the minimum curvature of $\partial\Omega$, and Theorem 4.1 extends the second derivative estimates in [11] for this special case with $c > 0$. We remark though that the strictly regular case in Theorem 4.1 also extends to general augmented Hessian equations, and corresponding second derivative estimates for (1.21) for general f are proved in [8].

From Theorem 4.1, we can obtain existence theorems, which also extend Theorem 1.2 and Corollary 3.1 in the strictly regular case. First, we prove an appropriate extension of the gradient bound Lemma 3.2.

Lemma 4.1 *Let* $u \in C^2(\overline{\Omega})$ *satisfy* (3.2) *in a* C^2 *domain* $\Omega \subset \mathbb{R}^n$ *and let*

$$\tag{4.13} |D_\beta u| \leq \sigma_0, \quad \beta \cdot \nu \geq \beta_0$$

on $\partial\Omega$, *where* $\beta \in L^\infty(\partial\Omega)$, $|\beta| = 1$ *and* σ_0 *and* β_0 *are positive constants. Then we have the estimate*

$$\tag{4.14} |Du| \leq C,$$

where C *depends on* $\mu_0, \sigma_0, \beta_0, \Omega$, *and* $\sup |u|$.

Proof Invoking the tangential gradient δu, we have the formula

$$\tag{4.15} D_\nu u = \frac{1}{\beta \cdot \nu}(D_\beta u - \beta \cdot \delta u)$$

so that we can estimate

$$|Du| \leq \frac{1}{\beta_0}(|\delta u| + \sigma_0) + |\delta u|$$

on $\partial\Omega$, whence from (3.2), we obtain

$$\tag{4.16} D^2 u \geq -\mu_1(1 + |\delta u|^2)I$$

on $\partial\Omega$, for a further constant μ_1, depending on μ_0, β_0, and σ_0. Now we consider in place of (3.5), the function

$$\tag{4.17} w = e^{\kappa u}|\delta u|^2,$$

so that at a point $x_0 \in \partial\Omega$ where w is maximised we have

$$
\begin{aligned}
0 = \delta u \cdot \delta w &= e^{\kappa u} \big(\kappa |\delta u|^4 + 2\delta_i u \delta_j u \delta_i \delta_j u \big) \\
&= e^{\kappa u} \big[\kappa |\delta u|^4 + 2\delta_i u \delta_j u \big(D_{ij} u - D_\nu u \delta_i \nu_j \big) \big] \\
&\geq e^{\kappa u} \big[\kappa |\delta u|^4 - 2\mu_1 |\delta u|^2 (1 + |\delta u|^2) - C|\delta u|^2 \big],
\end{aligned}
$$

from (4.15) and (4.16), where C is a constant depending on β_0, σ_0, and $\partial\Omega$. By choosing κ sufficiently large we conclude the estimate (4.14) on $\partial\Omega$ and the estimate in all of Ω then follows from [9] or Lemma 3.2. ∎

Lemma 4.1 provides an extension of [17, Theorem 2.2] to the weaker convexity condition (3.2). If we assume a stronger quadratic control from below on the Hessian, namely

$$
(4.18) \qquad D_{ij} u \xi_i \xi_j \geq -\mu_0 (1 + |D_\xi u|^2)
$$

for some constant μ_0 and any unit vector ξ, we can reduce to Theorem 2.2 and the corresponding remark in [17] as condition (4.18) implies that the function $e^{\kappa u}$ is semi-convex for large κ. We also remark that the gradient estimates in Lemmas 3.2 and 4.1 have local versions. In particular, if we fix any ball $B = B_R(x_0)$ of radius R and centre $x_0 \in \overline{\Omega}$, and suppose $u \in C^2(\Omega \cap B) \cap C^1(\overline{\Omega} \cap B)$ satisfies (3.2) in $\Omega \cap B$ and (4.13) in $\partial\Omega \cap B$. Then we have an estimate

$$
(4.19) \qquad |Du(x_0)| \leq \frac{C}{R},
$$

where C depends on $\mu_0, \sigma_0, \beta_0, \Omega$, and $\sup |u|$. To prove (4.19) we modify our proof of the global estimate Lemma 4.1 by maximizing in place of the auxiliary functions in [9] and (4.17) above, the functions

$$
w_1 = \eta^2 e^{\kappa u} |Du|^2, \quad w_2 = \eta^2 e^{\kappa u} |\delta u|^2
$$

over $\Omega \cap B, \partial\Omega \cap B$ respectively, where $\eta \in C_0^1(B)$ is a cut-off function chosen so that $0 \leq \eta \leq 1, \eta(x_0) = 1$ and $|D\eta| \leq 2/R$.

Note that (4.18) is satisfied in the special case (1.22), so we obtain, for solutions of (1.21), (1.22), both local and global, gradient and second derivative estimates in terms of $\Omega, h_{\partial\Omega}$, and $\sup |u|$.

In order to apply Lemma 4.1, we also need to assume that G is uniformly oblique in the sense that

$$
(4.20) \qquad G_p(x, z, p) \cdot \nu \geq \beta_0, \quad |G_p(x, z, p)| \leq \sigma_0 \quad \text{on } \partial\Omega,
$$

for all $x \in \Omega, |z| \leq M_0, p \in \mathbb{R}^n$ and positive constants β_0 and σ_0, depending on the constant M_0. Using the mean value theorem, we can thus write G in the semilinear form (1.13) so that Lemma 4.1, as well as the solution estimates in Section 3, are applicable.

We then have the following analogue of Theorem 1.2 with a much weaker supersolution condition.

Theorem 4.2 *Suppose that $A, B, G,$ and Ω satisfy the hypotheses of Theorem 4.1 with G uniformly oblique satisfying (4.20), and concave in p for all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$. Assume also that A and B are non-decreasing in z, G is strictly decreasing in z, A satisfies*

(1.15) *and that there exists a supersolution $\overline{u}$ and an elliptic subsolution $\underline{u}$ of equation* (1.1) *in $C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfying $\mathcal{G}[\overline{u}] \leq 0$ and $\mathcal{G}[\underline{u}] \geq 0$ respectively on $\partial\Omega$ with Ω uniformly A-convex with respect to G and $\mathcal{I} = [\underline{u}, \overline{u}]$. Then the boundary value problem* (1.1)-(1.5) *has a unique elliptic solution $u \in C^{3,\alpha}(\overline{\Omega})$ for any $\alpha < 1$.*

Analogously to Corollary 3.1, we also have from Lemma 3.3 an existence theorem in the optimal transportation case. Here we may also extend the condition (3.12) by assuming there exists a positive constant γ_0 such that

$$(4.21) \qquad\qquad G_z(x, z, p) \leq -\gamma_0$$

for all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$.

Corollary 4.1 *Suppose that equation* (1.1) *is a prescribed Jacobian equation of the form* (1.16) *generated by a cost function $c \in C^2(\mathcal{D})$ satisfying conditions A1 and A2 and $\mathcal{U}_x = \mathbb{R}^n$ for all $x \in \Omega$, with ψ satisfying the structure conditions* (3.18) *and* (3.19). *Suppose also that A, B, G, and Ω satisfy the hypotheses of Theorem* 4.1 *with G uniformly oblique satisfying* (4.20), *uniformly monotone satisfying* (4.21) *and concave in p for all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$, A satisfying* (1.15), *B non-decreasing and Ω uniformly A-convex with respect to G and $-C$, where C is the constant in Lemma* 3.3. *Then the boundary value problem* (1.1)–(1.5) *has a unique elliptic solution $u \in C^{3,\alpha}(\overline{\Omega})$ for any $\alpha < 1$.*

As with Corollary 3.1, the applicability of Corollary 4.1 is limited by the admissibility condition $\mathcal{U}_x = \mathbb{R}^n$, as well as the boundary A-convexity conditions. Since the function $G(\cdot, u, Du)$ itself satisfies an elliptic Dirichlet problem with vanishing boundary values, geometric conditions of the latter type will be necessary for classical solvability; see [5]. This situation is explored further in the context of regularity of generalized solutions in conjunction with [8].

Finally, we remark that when G is assumed nonnegative and uniformly concave in p with respect to u in some boundary neighbourhood $\mathcal{N}$, we only need A to be regular in Theorems 4.1, 4.2, and Corollary 4.1, and the global second derivative estimates follow exactly as in [30, Section 4]; see also [33]. Also the proof of Theorem 4.1 would carry over to the cases when G is non-increasing and A is non-decreasing, with either G_z sufficiently small or $D_z A$ sufficiently large and A again only assumed regular (using in the first case the existence of an elliptic function and Lemma 2.1). These aspects are treated in part II of [8] for general augmented Hessian equations.

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Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, P.R. China
e-mail: jiangfeida@math.tsinghua.edu.cn

*Centre for Mathematics and Its Applications, The Australian National University, Canberra ACT 0200,
Australia*
e-mail: neil.trudinger@anu.edu.au

*Faculty of Mathematics and Statistics, Hubei Key Laboratory of Applied Mathematics, Hubei University,
Wuhan 430062, P.R. China*
e-mail: nixiang@hubu.edu.cn