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SOME INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS

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Abstract

We obtain various refinements and generalizations of a classical inequality of S. N. Bernstein on trigonometric polynomials. Some of the results take into account the size of one or more of the coefficients of the trigonometric polynomial in question. The results are obtained using interpolation formulas.

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1. Introduction and statement of results

Let $S(\theta) = \sum_{m=-n}^{n} b_m e^{im\theta}$ be a trigonometric polynomial of degree at most n. It is well known (see [1, page 39] or [4, page 85, Prob. no 82]) that

$$(1) \qquad |S'(\theta)| \leqslant n \max_{0 \leqslant X < 2\pi} |S(X)|, \qquad \theta \in \mathbb{R}.$$

In (1) the equality is possible only if

$$S(\theta) = ae^{in\theta} + be^{-in\theta}, \qquad a, b \in \mathbb{C}.$$

In this paper we obtain certain refinements of this inequality (known after S. N. Bernstein). As a very special case of one of our results it follows that (1) may be replaced by

$$(2) \qquad |S'(\theta)| \leqslant \left(n - \frac{1}{2}\right) \max_{0 \leqslant X < 2\pi} |S(X)| + \frac{|b_{-n}| + |b_n|}{2}, \qquad \theta \in \mathbf{R}.$$

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According to a result of Visser [8, page 85]

$$|b_{-n}| + |b_n| \leqslant \max_{0 \leqslant X < 2\pi} |S(X)|, \tag{3}$$

and so (2) is indeed a refinement of (1).

Here is another special case of our results.

THEOREM 1. *If* $S(\theta) = \sum_{m=-n}^{n} b_m e^{im\theta}$ *is a trigonometric polynomial of degree at most* n *then*

$$\left|\sum_{m=-n}^{-1} m b_m e^{im\theta}\right| + \left|\sum_{m=1}^{n} m b_m e^{im\theta}\right| \leqslant n \max_{0 \leqslant X < 2\pi} |S(X)|, \qquad \theta \in \mathbb{R}.$$

It is clear that Theorem 1 is stronger than Bernstein's inequality. But the following result is even stronger.

THEOREM 1′. *If* $S(\theta) = \sum_{m=-n}^{n} b_m e^{im\theta}$ *is a trigonometric polynomial of degree at most* n *then, for all* $R \geqslant 1$, *we have*

$$\left|\sum_{m=-n}^{-1} (R^{-m} - 1) b_m e^{im\theta}\right| + \left|\sum_{m=1}^{n} (R^{m} - 1) b_m e^{im\theta}\right| \leqslant (R^n - 1) \max_{0 \leqslant X < 2\pi} |S(X)|,$$
$$\theta \in \mathbb{R}.$$

If we divide both sides of this inequality by R^n and let $R \to \infty$, we see that Theorem 1′ immediately implies (3). Also, if the coefficients of the trigonometric polynomial satisfy $b_{-m} = \overline{b_m}$, $1 \leqslant m \leqslant n$ (which is certainly the case if S is a real trigonometric polynomial), then

$$\left|\sum_{m=1}^{n} (R^{m} - 1) b_m e^{im\theta}\right| \leqslant \frac{(R^n - 1)}{2} \max_{0 \leqslant X < 2\pi} |S(X)|,$$

for all real θ and $R \geqslant 1$.

Theorem 1 may also be refined in another way.

THEOREM 2. *If* $S(\theta) = \sum_{m=-n}^{n} b_m e^{im\theta}$ *is a trigonometric polynomial of degree at most* n *then*

$$\left|\sum_{m=-n}^{-1} m b_m e^{im\theta} + \frac{b_{-n}}{2} e^{-in\theta} - \frac{b_0}{4}\right| + \left|\sum_{m=1}^{n} m b_m e^{im\theta} - \frac{b_n}{2} e^{in\theta} + \frac{b_0}{4}\right|$$
$$\leqslant \left(n - \frac{1}{2}\right) \max_{0 \leqslant X < 2\pi} |S(X)|, \qquad \theta \in \mathbb{R}.$$

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This result implies that *the inequality*

$$(4)\quad \left|S'(\theta)+i\frac{b_{-n}}{2}e^{-in\theta}-i\frac{b_n}{2}e^{in\theta}\right|\leqslant\left(n-\frac{1}{2}\right)\max_{0\leqslant X<2\pi}|S(X)|,\qquad \theta\in\mathbb{R},$$

holds for all trigonometric polynomials $S(\theta)=\sum_{m=-n}^{n}b_m e^{im\theta}$ *of degree at most n.* The inequality (2) follows from (4). Also, if $P(Z)=\sum_{j=0}^{n}a_jZ^j$ is a polynomial of degree at most n then $S(\theta)=P(\cos\theta)$ is a trigonometric polynomial of degree at most n such that $b_{-n}=b_n=a_n/2^n$ and $\max_{0\leqslant X<2\pi}|S(X)|=\max_{-1\leqslant t\leqslant 1}|P(t)|$; in that case, the inequality (4) and the triangle inequality show that

$$(5)\quad |P'(X)|\leqslant\frac{(n-1/2)}{\sqrt{1-X^2}}\max_{-1\leqslant t\leqslant 1}|P(t)|+\frac{|a_n|}{n2^n}|T_n'(X)|,\qquad -1<X<1,$$

where $T_n(X)=\cos(n\arccos X)$ is the nth Chebyshev polynomial of the first kind. In view of Chebyshev's inequality [1, page 29]

$$|a_n|\leqslant 2^{n-1}\max_{-1\leqslant t\leqslant 1}|P(t)|,$$

we see that (5) is a refinement of another inequality of Bernstein, namely

$$|P'(X)|\leqslant\frac{n}{\sqrt{1-X^2}}\max_{-1\leqslant t\leqslant 1}|P(t)|,\qquad -1<X<1.$$

The two theorems stated above will be proved with the help of an interpolation formula. This latter will permit us to deduce

THEOREM 3. *If* $S(\theta)=\sum_{m=-n}^{n}b_m e^{im\theta}$ *is a trigonometric polynomial of degree at most n then, for all* $R\geqslant 1$, *we have*

$$\left|\sum_{m=-n}^{-1}(R^{-m}-1)b_m e^{i(m+n)\theta}+\sum_{m=1}^{n}(R^m-1)b_m e^{i(m-n)\theta}\right|$$
$$\leqslant(R^n-1)\max_{1\leqslant K\leqslant 2n}\left|S\left(\frac{K\pi}{n}\right)\right|,\qquad \theta\in\mathbb{R}.$$

If $S(\theta)=P(e^{i\theta})$, where P is a polynomial of degree at most n, then Theorem 3 implies that (see [2, Theorems 10 and 11])

$$\max_{|Z|=1}|P'(Z)|\leqslant n\max_{1\leqslant K\leqslant 2n}|P(e^{K\pi i/n})|.$$

THEOREM 3′. *If* $S(\theta)=\sum_{m=-n}^{n}b_m e^{im\theta}$ *is a trigonometric polynomial of degree at most n then, for all* $R\geqslant 1$, *we have*

$$\left|\sum_{m=-n}^{n}(R^{n-|m|}-1)b_m e^{im\theta}\right|\leqslant(R^n-1)\max_{1\leqslant K\leqslant 2n}|S(K\pi/n)|,\qquad \theta\in\mathbb{R}.$$

To illustrate this theorem suppose that f is a function of bounded variation, periodic with period 2π. Let $S_n(f,\theta) = \sum_{m=-n}^{n} C_m e^{im\theta}$ be the nth partial sum of the expansion of f in Fourier series and

$$\sigma_n(f,\theta) = \frac{1}{n}\sum_{m=0}^{n-1} S_m(f,\theta)$$

the associated Fejér mean. It is easy to verify that

$$\sigma_n(f,\theta) = \sum_{m=-n}^{n}\left(1 - \frac{|m|}{n}\right)C_m e^{im\theta}.$$

Hence, a corollary of Theorem 3′ (obtained by dividing both sides of the inequality by $(R-1)$, $R > 1$, and letting $R \to 1$) shows that

$$|\sigma_n(f,\theta)| \leqslant \max_{1 \leqslant K \leqslant 2n}\left|S_n\left(f, \frac{K\pi}{n}\right)\right|, \qquad \theta \in \mathbb{R}.$$

Finally, we shall prove

THEOREM 4. *If $S(\theta) = \sum_{m=-n}^{n} b_m e^{im\theta}$ is a trigonometric polynomial of degree at most n then, for all $R \geqslant 1$, we have*

$$\left(\left|\sum_{m=-n}^{-1}(R^{-m}-1)b_m e^{im\theta}\right| + \left|\sum_{m=1}^{n}(R^{m}-1)b_m e^{im\theta}\right|\right)^2 + \left|\sum_{m=-n}^{n}(R^{n-|m|}-1)b_m e^{im\theta}\right|^2 \leqslant \left((R^n-1)\max_{0 \leqslant X < 2\pi}|S(X)|\right)^2, \qquad \theta \in \mathbb{R}.$$

It is readily seen that Theorem 4 contains Theorem 1′ and a weaker version of Theorem 3′.

2. The interpolation formulas

Theorems 1′ and 3 are consequences of the following

LEMMA 1. *Let $S(\theta) = \sum_{m=-n}^{n} b_m e^{im\theta}$ be a trigonometric polynomial of degree at most n, $n \geqslant 2$, γ be any real number and $R \geqslant 1$. Then*

$$(6) \quad e^{-i\gamma}\sum_{m=-n}^{-1}(R^{-m}-1)b_m e^{im\theta} + e^{i\gamma}\sum_{m=1}^{n}(R^{m}-1)b_m e^{im\theta} = \frac{1}{2n}\sum_{K=1}^{2n}(-1)^K A_k(R,\gamma)S(\theta + (K\pi+\gamma)/n),$$

where

$$A_K(R,\gamma) = R^n - 1 + 2\sum_{j=1}^{n-1}(R^{n-j} - 1)\cos j(K\pi + \gamma)/n.$$

The coefficients $A_K(R,\gamma)$ are non-negative and

$$\frac{1}{2n}\sum_{K=1}^{2n} A_K(R,\gamma) = R^n - 1. \tag{7}$$

PROOF. Substituting the expressions for $A_K(R,\gamma)$ and $S(\theta + (K\pi + \gamma)/n)$, we obtain

$$\begin{aligned}
&\frac{1}{2n}\sum_{K=1}^{2n}(-1)^K A_K(R,\gamma) S(\theta + (K\pi + \gamma)/n)\\
&= \frac{(R^n - 1)}{2n}\sum_{K=1}^{2n}\sum_{m=-n}^{n}(-1)^K b_m e^{im(\theta + (K\pi+\gamma)/n)}\\
&\quad + \frac{1}{n}\sum_{K=1}^{2n}\sum_{j=1}^{n-1}\sum_{m=-n}^{n}(-1)^K b_m (R^{n-j} - 1)\cos(j(K\pi + \gamma)/n) e^{im(\theta + (K\pi+\gamma)/n)}\\
&= (R^n - 1)\left(b_{-n}e^{-i(n\theta+\gamma)} + b_n e^{i(n\theta+\gamma)}\right)\\
&\quad + \frac{1}{n}\sum_{K=1}^{2n}\sum_{j=1}^{n-1}\sum_{m=-n}^{n}(-1)^K b_m (R^{n-j} - 1)\cos(j(K\pi + \gamma)/n) e^{im(\theta + (K\pi+\gamma)/n)},
\end{aligned}$$

since $(-n \leqslant m \leqslant n)$

$$\frac{1}{2n}\sum_{K=1}^{2n}(-1)^K e^{mK\pi i/n} = \begin{cases} 1 & \text{if } m = \pm n,\\ 0 & \text{otherwise.}\end{cases}$$

Using now Euler's identity, $\cos z = e^{iz} + e^{-iz}/2$, we see that

$$\begin{aligned}
&\frac{1}{2n}\sum_{K=1}^{2n}(-1)^K A_K(R,\gamma) S(\theta + (K\pi + \gamma)/n)\\
&\qquad = (R^n - 1)\left(b_{-n}e^{-i(n\theta+\gamma)} + b_n e^{i(n\theta+\gamma)}\right)\\
&\qquad\quad + \frac{1}{2n}\sum_{j=1}^{n-1}\sum_{m=-n}^{n}\sum_{K=1}^{2n} b_m (R^{n-j} - 1) e^{im\theta + i(j+m)\gamma/n} e^{(n+j+m)K\pi i/n}\\
&\qquad\quad + \frac{1}{2n}\sum_{j=1}^{n-1}\sum_{m=-n}^{n}\sum_{K=1}^{2n} b_m (R^{n-j} - 1) e^{im\theta + i(m-j)\gamma/n} e^{(n+m-j)K\pi i/n}.
\end{aligned}$$

Since $1 \leqslant m+n+j \leqslant 3n-1$ and $-(n-1) \leqslant m+n-j \leqslant 2n-1$, we have

$$\frac{1}{2n}\sum_{K=1}^{2n} e^{(m+n+j)K\pi i/n} = \begin{cases} 1 & \text{if } m+n+j=2n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\frac{1}{2n}\sum_{K=1}^{2n} e^{(m+n-j)K\pi i/n} = \begin{cases} 1 & \text{if } m+n-j=0, \\ 0 & \text{otherwise } (1 \leqslant j < n,\ -n \leqslant m \leqslant n), \end{cases}$$

whence

$$\begin{aligned} &\frac{1}{2n}\sum_{K=1}^{2n} (-1)^K A_K(R,\gamma) S(\theta+(K\pi+\gamma)/n) \\ &\quad = (R^n-1)\left(b_{-n}e^{-i(n\theta+\gamma)} + b_n e^{i(n\theta+\gamma)}\right) \\ &\qquad + \sum_{m=1}^{n-1} b_m(R^m-1)e^{im\theta+i\gamma} + \sum_{m=-(n-1)}^{-1} b_m(R^{-m}-1)e^{im\theta-i\gamma} \\ &\quad = e^{i\gamma}\sum_{m=1}^{n}(R^m-1)b_m e^{im\theta} + e^{-i\gamma}\sum_{m=-n}^{-1}(R^{-m}-1)b_m e^{im\theta}. \end{aligned}$$

In order to verify that $A_K(R,\gamma) \geqslant 0$, we may use Lemma 2, below.
Finally, the relation (7) follows from (6) if we set $S(\theta) = e^{in\theta}$.

LEMMA 2 ([6, *page* 75]). *If* $\lambda_n \geqslant 0$, $\lambda_{n-1} - 2\lambda_n \geqslant 0$ *and* $\lambda_{j-1} - 2\lambda_j + \lambda_{j+1} \geqslant 0$ *for* $1 \leqslant j < n$, *then*

$$\lambda_0 + 2\sum_{j=1}^{n}\lambda_j \cos j\theta \geqslant 0$$

for all real θ.

For the proof of Theorem 2, we need

LEMMA 1′. *Let* $S(\theta) = \sum_{m=-n}^{n} b_m e^{im\theta}$ *be a trigonometric polynomial of degree at most* n. *Then, for all real* γ, *we have*

$$\begin{aligned} (8)\quad & e^{i\gamma}\sum_{m=1}^{n} m b_m e^{im\theta} - e^{-i\gamma}\sum_{m=-n}^{-1} m b_m e^{im\theta} \\ &\qquad = \frac{1}{2n}\sum_{K=1}^{2n}(-1)^K\left(\frac{\sin(K\pi+\gamma)/2}{\sin(K\pi+\gamma)/2n}\right)^2 S(\theta+(K\pi+\gamma)/n), \end{aligned}$$

and

$$\frac{1}{2n}\sum_{K=1}^{2n}\left(\frac{\sin(K\pi+\gamma)/2}{\sin(K\pi+\gamma)/2n}\right)^2 = n. \tag{9}$$

PROOF. If we divide both sides of (6) by $(R-1)$, $R>1$, and let $R\to 1$, we obtain

$$e^{i\gamma}\sum_{m=1}^{n} mb_m e^{im\theta} - e^{-i\gamma}\sum_{m=-n}^{-1} mb_m e^{im\theta}$$
$$=\frac{1}{2n}\sum_{K=1}^{2n}(-1)^K\left(n+2\sum_{j=1}^{n-1}(n-j)\cos\left(j(K\pi+\gamma)/n\right)\right)S(\theta+(K\pi+\gamma)/n).$$

It is then a matter of simple calculations to show that

$$n+2\sum_{j=1}^{n-1}(n-j)\cos\left(j(K\pi+\gamma)/n\right)=\left(\frac{\sin(K\pi+\gamma)/2}{\sin(K\pi+\gamma)/2n}\right)^2,$$

and we obtain immmediately (8).

The relation (9) follows from (8) if we set $S(\theta)=e^{in\theta}$.

The same method as used in the proof of Lemma 1 yields

LEMMA 3. *Let $S(\theta)=\sum_{m=-n}^{n} b_m e^{im\theta}$ be a trigonometric polynomial of degree at most n, $n\geqslant 2$, γ be any real number and $R\geqslant 1$. Then*

$$e^{-i\gamma}\sum_{m=-n}^{-1}(R^{-m}-1)b_m e^{im\theta}+\sum_{m=-n}^{n}(R^{n-|m|}-1)b_m e^{im\theta} \tag{10}$$
$$+e^{i\gamma}\sum_{m=1}^{n}(R^m-1)b_m e^{im\theta}$$
$$=\frac{1}{n}\sum_{K=1}^{n}A_{2K}(R,\gamma)S(\theta+(2K\pi+\gamma)/n).$$

Furthermore,

$$\frac{1}{n}\sum_{K=1}^{n}A_{2K}(R,\gamma)=R^n-1.$$

There is a similar formula where the interpolation points are $\theta + ((2K-1)\pi + \gamma)/n$ instead of $\theta + (2K\pi + \gamma)/n$. More precisely, we have

$$(10') \quad -e^{-i\gamma} \sum_{m=-n}^{-1} (R^{-m} - 1) b_m e^{im\theta} + \sum_{m=-n}^{n} (R^{n-|m|} - 1) b_m e^{im\theta} - e^{i\gamma} \sum_{m=1}^{n} (R^m - 1) b_m e^{im\theta} = \frac{1}{n} \sum_{K=1}^{n} A_{2K-1}(R,\gamma) S(\theta + ((2K-1)\pi + \gamma)/n),$$

with

$$\frac{1}{n} \sum_{K=1}^{n} A_{2K-1}(R,\gamma) = R^n - 1, \qquad n \geqslant 2.$$

If we add the corresponding members of (10) and (10′), we obtain

Lemma 3′. *Let $S(\theta) = \sum_{m=-n}^{n} b_m e^{im\theta}$ be a trigonometric polynomial of degree at most n, γ be any real number and $R \geqslant 1$. Then*

$$(11) \quad \sum_{m=-n}^{n} (R^{n-|m|} - 1) b_m e^{im\theta} = \frac{1}{2n} \sum_{K=1}^{2n} A_K(R,\gamma) S(\theta + (K\pi + \gamma)/n).$$

Remark 1. Lemma 1′ is due to Szegö [7, page 64]. The particular case $\gamma = \pi/2$ of that lemma is a formula known as Riesz's interpolation formula [5].

Remark 2. The interpolation formula obtained from (6) by taking $\gamma = 0$ and $b_{-m} = 0$, $1 \leqslant m \leqslant n$, has been used by Giroux and Rahman [3] in their work on polynomials having a prescribed zero on the unit circle $|z| = 1$.

3. Proofs of the theorems

Theorem 1 is deduced from Theorem 1′ by dividing both members of the inequality by $(R-1)$, $R > 1$, and letting $R \to 1$.

Proof of Theorem 1′. Using the formula (6), the triangle inequality and (7), we obtain

$$\left| e^{-i\gamma} \sum_{m=-n}^{-1} (R^{-m} - 1) b_m e^{im\theta} + e^{i\gamma} \sum_{m=1}^{n} (R^m - 1) b_m e^{im\theta} \right| \leqslant (R^n - 1) \max_{0 \leqslant X < 2\pi} |S(X)|,$$

and an appropriate choice of γ gives us the result.

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PROOF OF THEOREM 2. The formula (8) may be written

$$e^{i\gamma}\sum_{m=1}^{n} mb_m e^{im\theta} - e^{-i\gamma}\sum_{m=-n}^{-1} mb_m e^{im\theta}$$
$$= \frac{1}{2n}\sum_{K=1}^{2n}(-1)^K \sin^2(K\pi+\gamma)/2 \operatorname{cosec}^2(K\pi+\gamma)/2nS(\theta+(K\pi+\gamma)/n)$$
$$= \frac{1}{2n}\sum_{K=1}^{2n}(-1)^K \sin^2(K\pi+\gamma)/2 \cot^2(K\pi+\gamma)/2nS(\theta+(K\pi+\gamma)/n)$$
$$+\frac{\sin^2\gamma/2}{2n}\sum_{K=1}^{n} S(\theta+(2K\pi+\gamma)/n)$$
$$-\frac{\cos^2\gamma/2}{2n}\sum_{K=1}^{n} S(\theta+((2K-1)\pi+\gamma)/n).$$

Since

$$\frac{1}{n}\sum_{K=1}^{n} S(\theta+(2K\pi+\gamma)/n) = b_{-n}e^{-i(n\theta+\gamma)} + b_0 + b_n e^{i(n\theta+\gamma)}$$

and

$$\frac{1}{n}\sum_{K=1}^{n} S(\theta+((2K-1)\pi+\gamma)/n) = -\left(b_{-n}e^{-i(n\theta+\gamma)} - b_0 + b_n e^{i(n\theta+\gamma)}\right),$$

we obtain the interpolation formula

(12)

$$e^{i\gamma}\left[\sum_{m=1}^{n} mb_m e^{im\theta} - \frac{b_n}{2}e^{in\theta} + \frac{b_0}{4}\right] - e^{-i\gamma}\left[\sum_{m=-n}^{-1} mb_m e^{im\theta} + \frac{b_{-n}}{2}e^{-in\theta} - \frac{b_0}{4}\right]$$
$$= \frac{1}{2n}\sum_{K=1}^{2n}(-1)^K \sin^2((K\pi+\gamma)/2)\cot^2((K\pi+\gamma)/2n)S(\theta+(K\pi+\gamma)/n).$$

In particular, for $S(\theta) = e^{in\theta}$,

$$\frac{1}{2n}\sum_{K=1}^{2n} \sin^2((K\pi+\gamma)/2)\cot^2((K\pi+\gamma)/2n) = n - \frac{1}{2}. \tag{13}$$

It follows from (12) and (13) that

$$\left|e^{i\gamma}\left[\sum_{m=1}^{n} mb_m e^{im\theta} - \frac{b_n}{2}e^{in\theta} + \frac{b_0}{4}\right] - e^{-i\gamma}\left[\sum_{m=-n}^{-1} mb_m e^{im\theta} + \frac{b_{-n}}{2}e^{-in\theta} - \frac{b_0}{4}\right]\right|$$
$$\leqslant \frac{1}{2n}\sum_{K=1}^{2n} \sin^2((K\pi+\gamma)/2)\cot^2((K\pi+\gamma)/2n) \max_{0\leqslant X<2\pi}|S(X)|$$
$$= \left(n-\frac{1}{2}\right)\max_{0\leqslant X<2\pi}|S(X)|,$$

and an appropriate choice of γ gives us the result.

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PROOF OF THEOREM 3. Let θ be an arbitrary real number. Choosing $\gamma = -n\theta$ in (6) we obtain

$$\left| \sum_{m=-n}^{-1} (R^{-m} - 1) b_m e^{i(m+n)\theta} + \sum_{m=1}^{n} (R^m - 1) b_m e^{i(m-n)\theta} \right|$$
$$\leqslant \frac{1}{2n} \sum_{K=1}^{2n} A_K(R, -n\theta) \max_{1 \leqslant K \leqslant 2n} |S(K\pi/n)|,$$

which, in conjunction with (7) (applied with $\gamma = -n\theta$) gives the result.

PROOF OF THEOREM 3'. Choosing $\gamma = -n\theta$ in (11) we obtain

$$\left| \sum_{m=-n}^{n} (R^{n-|m|} - 1) b_m e^{im\theta} \right| \leqslant \frac{1}{2n} \sum_{K=1}^{2n} A_K(R, -n\theta) \max_{1 \leqslant K \leqslant 2n} |S(K\pi/n)|,$$

and the result follows again from (7).

PROOF OF THEOREM 4. Put $W = e^{i\gamma} (\gamma \in \mathbb{R})$, $a = \sum_{m=1}^{n} (R^m - 1) b_m e^{im\theta}$, $b = \sum_{m=-n}^{n} (R^{n-|m|} - 1) b_m e^{im\theta}$ and $c = \sum_{m=-n}^{-1} (R^{-m} - 1) b_m e^{im\theta}$, $(\theta \in \mathbb{R}, R \geqslant 1)$.

The formula (10) implies immediately that

$$|aW^2 + bW + c| \leqslant (R^n - 1) \max_{0 \leqslant X < 2\pi} |S(X)|. \tag{14}$$

It is known [9] that if $P(W) = C_0 + C_1 W + \cdots + C_m W^m$ is a polynomial of degree at most m then

$$2|C_0||C_m| + \sum_{K=0}^{m} |C_K|^2 \leqslant \left(\max_{|W|=1} |P(W)| \right)^2. \tag{15}$$

If we apply (15) to the polynomial of degree 2, $P(W) = aW^2 + bW + c$, which satisfies, in view of (14),

$$\max_{|W|=1} |P(W)| \leqslant (R^n - 1) \max_{0 \leqslant X < 2\pi} |S(X)|,$$

we obtain

$$2|a||c| + |a|^2 + |b|^2 + |c|^2 \leqslant \left((R^n - 1) \max_{0 \leqslant X < 2\pi} |S(X)| \right)^2,$$

which is equivalent to the desired result.

We wish to mention, to conclude, that the choice $\gamma = -n\theta$ in the interpolation formula (12) gives us a result which may be compared with Theorems 3 and 3'.

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