J. Appl. Prob. **34**, 823–826 (1997)
Printed in Israel

Dear Editor,

On weak convergence within a family of life distributions

1. Introduction

It is well known that the weak convergence and moment convergence of distributions are not equivalent in general. However, for distributions within a suitable family, these two modes of convergence can be equivalent. For example, Basu and Bhattacharjee (1984) obtained the equivalence of weak convergence and moment convergence for life distributions within the harmonic new better than used in expectation family (HNBUE family) which is related to the exponential distribution, and Lin (1994, 1995) extended their result from the HNBUE family to some larger families. In this letter we further prove that the same equivalence property holds for the following larger family $\mathcal{G}_\alpha$ which is related to the gamma distribution.

Let X be a general non-negative random variable with distribution F, and denote $\mu_F = EX$. Also, let $Y_{\alpha,\beta}$ be a random variable obeying the gamma distribution $G_{\alpha,\beta}$ with density function

$$g_{\alpha,\beta}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \mathrm{e}^{-x/\beta} \qquad \text{for } x > 0,$$

where α and β are two positive constants. Recall that the mean of $G_{\alpha,\beta}$ is $\alpha\beta$. For each $\alpha > 0$, define the family $\mathcal{G}_\alpha$ to be the class of all life distributions F such that its moment generating function is less than or equal to that of $G_{\alpha,\mu_F/\alpha}$, namely

$$\mathcal{G}_\alpha = \bigcup_{\beta>0} \{F : \mu_F = \alpha\beta,\ E\mathrm{e}^{sX} \leqq E\mathrm{e}^{sY_{\alpha,\beta}} \text{ for } s \in (-\infty, 1/\beta)\} \equiv \bigcup_{\beta>0} \mathcal{G}_{\alpha,\beta}.$$

From Theorem 1.3.1 of Stoyan (1983) it follows that the HNBUE family is a subfamily of $\mathcal{G}_1$. And $\mathcal{G}_\alpha$ is probably the largest family ever studied for the equivalence of these two modes of convergence. If $F \in \mathcal{G}_{\alpha,\beta}$, we say that F is smaller than $G_{\alpha,\beta}$ in the moment generating function order, which is a stochastic order relation and stronger than the Laplace transform order (see Remark (c) below). We are ready to state the main result.

Theorem (cf. Theorem 2.1, Basu and Bhattacharjee (1984)). Let $\alpha > 0$ be a fixed constant, and let X_n be a random variable with distribution $F_n \in \mathcal{G}_\alpha$, where $n = 1, 2, 3, \cdots$.

(i) Suppose that $X_n \to X$ in distribution for some non-negative random variable X. Then

$$\lim_{n\to\infty} EX_n^r = EX^r < \infty \qquad \text{for all } r > 0. \tag{1}$$

Further, the distribution of X belongs to the family $\mathcal{G}_\alpha$ if $EX \neq 0$.

(ii) Conversely, suppose that the relation (1) holds for all positive integers r and for some non-negative random variable X with $EX \in (0, \infty)$. Then $X_n \to X$ in distribution.

2. Proofs

Our arguments are parallel to those of Basu and Bhattacharjee (1984), adapted to and modified for the family $\mathscr{G}_\alpha$. We start with a lemma.

Lemma. Let $\alpha > 0$ be a fixed constant.

(i) If $F \in \mathscr{G}_\alpha$, then the distribution F is characterized by the sequence of its moments.

(ii) If X_n is a random variable obeying distribution $F_n \in \mathscr{G}_\alpha$, $n = 1, 2, \cdots$, and if the relation (1) holds for all positive integers r and for some fixed non-negative random variable X with $EX \in (0, \infty)$, then the distribution of X is characterized by the sequence of its moments.

Proof. (i) Let X be a random variable with distribution F. Then by the definition of $\mathscr{G}_\alpha$ we have

$$Ee^{sX} \leqq Ee^{sY_{\alpha,\beta}} = (1 - \beta s)^{-\alpha} < \infty \qquad \text{for } s \in (-\infty, 1/\beta),$$

where $\beta = \mu_F / \alpha$. Therefore the distribution F is characterized by the sequence of its moments.

(ii) As in the proof of (i), it suffices to prove that $Ee^{sX} < \infty$ for each $s \in (-\infty, \alpha/EX)$. Since $Ee^{sX} < \infty$ for each $s \leqq 0$, it remains to prove that $Ee^{sX} < \infty$ for each $s \in (0, \alpha/EX)$. Hereafter, let s be a fixed point in the interval $(0, \alpha/EX)$ and define the counting measure μ by $\mu(\{m\}) = s^m/m!$ for $m = 0, 1, 2, \cdots$. Further, define the functions f, g on the support of μ by

$$f(m) = EX^m, \quad g(m) = EY^m_{\alpha,\beta}, \qquad \text{for } m = 0, 1, 2, \cdots,$$

where $\beta = EX/\alpha$. Similarly, for each $n = 1, 2, \cdots$, let us define the functions f_n, g_n by

$$f_n(m) = EX_n^m, \quad g_n(m) = EY^m_{\alpha,\beta_n}, \qquad \text{for } m = 0, 1, 2, \cdots,$$

where $\beta_n = EX_n/\alpha$. Then applying Fatou's lemma we have

$$Ee^{sX} = \int f d\mu = \int \liminf_{n \to \infty} f_n d\mu \leqq \liminf_{n \to \infty} \int f_n d\mu \leqq \liminf_{n \to \infty} \int g_n d\mu = Ee^{sY_{\alpha,\beta}} < \infty.$$

This completes the proof of the lemma.

Proof of Theorem. Let G_{α,β_n} be the gamma distribution with mean $\alpha\beta_n = EX_n$, and let Y_n be a random variable obeying G_{α,β_n}, where $n = 1, 2, 3, \cdots$.

(i) Suppose that $X_n \to X$ in distribution for some non-negative random variable X. Then we first prove that the conclusion (1) holds. From the assumption $F_n \in \mathscr{G}_\alpha$ it follows that for all $n \geqq 1$

$$(1 + \beta_n)^{-\alpha} = Ee^{-Y_n} \geqq Ee^{-X_n},$$

the last term tending to $Ee^{-X} \in (0, \infty)$ as n goes to infinity. Hence the sequence $\{\beta_n\}_{n=1}^{\infty}$ is bounded, namely, there exists a constant $B \in (0, \infty)$ such that

(2) $$0<\beta_n \leqq B \qquad \text{for all } n \geqq 1.$$

For each given $r>0$, take an integer $k>\max\{r, 1\}$. Then for $s_0=1/(2B)$ and for each $n=1, 2, 3,\cdots$, we have

$$(1-\beta_n s_0)^{-\alpha}=E\mathrm{e}^{s_0 Y_n} \geqq E\mathrm{e}^{s_0 X_n} \geqq \frac{EX_n^k}{k!} s_0^k,$$

and hence

$$EX_n^k \leqq \frac{k!}{s_0^k(1-\beta_n s_0)^\alpha} \leqq \frac{k!}{s_0^k(1-Bs_0)^\alpha},$$

in which the last inequality follows from (2). This means that the sequence $\{EX_n^k\}_{n=1}^\infty$ is bounded. Using the moment convergence theorem we conclude that $\lim_{n\to\infty} EX_n^r = EX^r<\infty$. This is the desired result (1). Suppose further $EX \neq 0$. Then we want to prove that the distribution of X, say F, belongs to $\mathscr{G}_\alpha$. Letting $r=1$, the equality in (1) reduces to $\lim_{n\to\infty} EX_n = EX$, or, equivalently, $\lim_{n\to\infty} \beta_n=(EX)/\alpha=\beta$ (say). For each $s<1/\beta$, we have

$$E\mathrm{e}^{sX}=\lim_{n\to\infty} E\mathrm{e}^{sX_n} \leqq \lim_{n\to\infty} E\mathrm{e}^{sY_n}=\lim_{n\to\infty} (1-\beta_n s)^{-\alpha}=(1-\beta s)^{-\alpha}=E\mathrm{e}^{sY_{\alpha,\beta}}.$$

Therefore, $F\in\mathscr{G}_\alpha$.

(ii) Suppose that the relation (1) holds for all positive integers r and for some fixed non-negative random variable X with $EX\in(0, \infty)$. Then we have $X_n \to X$ in distribution by using the lemma above and Theorem 8.48 of Breiman (1993) (or Theorem 30.2 of Billingsley (1995)). This completes the proof.

3. Remarks

(a) Denote by F_0 the degenerate distribution at the point zero. Then from the theorem above it is seen that the family $\mathscr{G}_\alpha \cup \{F_0\}$ is closed under weak convergence.

(b) Define the dual family $\mathscr{G}_\beta^*$ for $\mathscr{G}_\alpha$ by

$$\mathscr{G}_\beta^* = \bigcup_{\alpha>0} \{F: \mu_F=\alpha\beta,\ E\mathrm{e}^{sX} \leqq E\mathrm{e}^{sY_{\alpha,\beta}} \text{ for } s\in(-\infty, 1/\beta)\},$$

where $\beta>0$. Then the conclusions of the lemma and theorem still hold if the family $\mathscr{G}_\alpha$ is replaced by $\mathscr{G}_\beta^*$; the proofs are similar to the previous ones and are omitted. It is natural to pose the question: is it possible to extend the above theorem from $\mathscr{G}_\alpha$ to the larger family $\mathscr{G} \equiv \bigcup_{\alpha>0} \mathscr{G}_\alpha = \bigcup_{\beta>0} \mathscr{G}_\beta^*$?

(c) On the other hand, let the family $\mathscr{L}_\alpha$ consist of all life distributions F being smaller than $G_{\alpha,\mu_F/\alpha}$ in the Laplace transform order. (For applications of Laplace transform order, see, e.g., Shaked and Shanthikumar (1994) and Alzaid *et al.* (1991).) Namely, for each $\alpha>0$, define

$$\mathscr{L}_\alpha = \bigcup_{\beta>0} \{F: \mu_F=\alpha\beta,\ E\mathrm{e}^{-sX} \leqq E\mathrm{e}^{-sY_{\alpha,\beta}} \text{ for } s\geqq 0\} \equiv \bigcup_{\beta>0} \mathscr{L}_{\alpha,\beta}.$$

Then $\mathscr{G}_\alpha \subset \mathscr{L}_\alpha$ because $\mathscr{G}_{\alpha,\beta} \subset \mathscr{L}_{\alpha,\beta}$ for each $\beta>0$. The question whether the above theorem can be extended from $\mathscr{G}_\alpha$ to $\mathscr{L}_\alpha$ also remains open. A special case that the family $\mathscr{L}_1 \cup \{F_0\}$ is closed under weak convergence has been proved by Chaudhuri (1995).

https://doi.org/10.2307/3215110 Published online by Cambridge University Press

Acknowledgement

The author would like to thank the referee for helpful comments.

References

ALZAID, A., KIM, J. S. AND PROSCHAN, F. (1991) Laplace ordering and its applications. *J. Appl. Prob.* **28**, 116–130.

BASU, S. K. AND BHATTACHARJEE, M. C. (1984) On weak convergence within the HNBUE family of life distributions. *J. Appl. Prob.* **21**, 654–660.

BILLINGSLEY, P. (1995) *Probability and Measure.* 3rd edn. Wiley, New York.

BREIMAN, L. (1993) *Probability.* 2nd edn. SIAM, Philadelphia, PA.

CHAUDHURI, G. (1995) A note on the *L*-class of life distributions. *Sankhyā* A **57**, 158–160.

LIN, G. D. (1994) On equivalence of weak convergence and moment convergence of life distributions. *Ann. Inst. Statist. Math.* **46**, 587–592.

LIN, G. D. (1995) On weak convergence within a Lorenz ordering family of distributions. *J. Appl. Prob.* **32**, 1028–1031.

SHAKED, M. AND SHANTHIKUMAR, J. G. (1994) *Stochastic Orders and their Applications.* Academic Press, New York.

STOYAN, D. (1983) *Comparison Methods for Queues and Other Stochastic Models.* Wiley, New York.

Yours sincerely,

GWO DONG LIN

Institute of Statistical Science
Academia Sinica, Taipei 11529
Taiwan, Republic of China