Can. J. Math., Vol. XXXIII, No. 6, 1981, pp. 1439–1447

TRANSITIVITY AND ORTHO-BASES

JACOB KOFNER

Throughout this paper "space" means "T_1 topological space."

1. The concept of an ortho-base was introduced by W. F. Lindgren and P. J. Nyikos.

Definition 1. A base $\mathscr{B}$ of a space X is called an *ortho-base* provided that for each subcollection $\mathscr{B}_0 \subset \mathscr{B}$ either $\bigcap \mathscr{B}_0$ is open or $\mathscr{B}_0$ is a local base of a point $x \in X$ [**17**].

Ortho-bases are related to interior-preserving collections which have been known for some time.

Definition 2. A collection of open sets of a space X is called *interior-preserving* provided that the intersection of any subcollection is open. A space X is called *orthocompact* provided that each open cover has an open interior-preserving refinement.

It was proved in [**17**], in particular, that each space with an ortho-base is orthocompact, and each orthocompact developable space (which is the same as a non-archimedean quasi-metrizable developable space [**4**]) has an ortho-base. This paper is primarily devoted to the solution of Problem 6.9 of [**17**]: whether, in spaces with ortho-bases, being a γ-space implies (non-archimedean) quasi-metrizability.

Definition 3. A space X is *quasi-metrizable* provided that it admits a quasi-metric d, i.e., a generalized metric satisfying the triangle inequality,

$$d(x, z) \leqq d(x, y) + d(y, x).$$

("A space *admits* a generalized metric d" means that for each $x \in X$ the spheres $S^d(x, \epsilon) = \{y | d(x, y) < \epsilon\}$, $\epsilon > 0$, form a local base at x.) If the triangle inequality is strengthened to

$$d(x, z) \leqq \max \{d(x, y), d(y, z)\},$$

then d is a *non-archimedean quasi-metric* and X is *non-archimedean quasi-metrizable*. If the triangle inequality is relaxed to $d(x, z_n) \to 0$ whenever $d(x, y_n) \to 0$ and $d(y_n, z_n) \to 0$, then X is a *γ-space*.

Obviously a non-archimedean quasi-metrizable space is a γ-space. However, a quasi-metrizable space need not be non-archimedean quasi-

Received July 5, 1980 and in revised form January 20, 1981.

metrizable [**14**]. The γ-space conjecture states that every γ-space is quasi-metrizable. The problem, whether the conjecture is true, listed as Classic Problem VIII in [**18**], is open and only partial solutions have been obtained [**11**], [**12**], [**15**] (cf. also [**1**], [**7**]). G. Gruenhage has shown in [**11**] that each paracompact γ-space with an ortho-base is non-archimedean quasi-metrizable (G. Gruenhage has also proved in [**11**] that a first countable paracompact linearly ordered space with an ortho-base, due to P. J. Nyikos, fails to be a γ-space.) We will prove here a general result concerning the transitivity of spaces with ortho-bases, which will imply that all γ-spaces with an ortho-base are non-archimedean quasi-metrizable. This will prove the γ-space conjecture for the spaces with ortho-bases and will provide a positive solution to Problem 6.9 of [**17**].

Notice that a space is non-archimedean quasi-metrizable if and only if it has a σ-interior preserving base, i.e., a base which is a countable union of interior preserving collections [**14**], [**27**]. An analogous characterization of γ-spaces requires the following definitions.

Definition 4. A collection $\mathscr{P}$ of pairs $\langle G', G''\rangle$ of open sets of a space X, $G' \subset G''$, is called a *local pair base* of $x \in X$, provided that for each neighbourhood G of x there exists a pair $\langle G', G''\rangle \in \mathscr{P}$ such that $x \in G'$ and $G'' \subset G$; $\mathscr{P}$ is called a *pair-base of* X provided that it is a local pair base of each $x \in X$.

Definition 5. A collection Q of pairs $\langle G', G''\rangle$ of open sets of a space X, $G' \subset G''$, is called *interior preserving*, provided that for each subcollection $Q_0 \subset Q$,

$$\bigcap \{G' |\ \langle G', G''\rangle \in Q_0\} \subset \operatorname{int} \bigcap \{G'' |\ \langle G', G''\rangle \in Q_0\}.$$

A space is called *preorthocompact* provided that for each open cover there is an interior-preserving collection Q of pairs of open sets $\langle G', G''\rangle$, $G' \subset G''$, such that $\{G'' |\ \langle G', G''\rangle \in Q\}$ refines the cover while $\{G' |\ \langle G', G''\rangle \in Q\}$ covers the space.

We can now state that a space is a γ-space if and only if it has a σ-interior preserving pair-base, i.e., a pair-base which is a countable union of interior-preserving collections, cf. [**6**].

The analogy between Definitions 2, 4 and 1 suggests the following

Definition 6. A pair-base $\mathscr{P}$ of a space X is called an *ortho-pair-base* provided that for each subcollection $\mathscr{P}_0 \subset \mathscr{P}$ either

$$\bigcap \{G' |\ \langle G', G''\rangle \in \mathscr{P}_0\} \subset \operatorname{int} \bigcap \{G'' |\ \langle G', G''\rangle \in \mathscr{P}_0\}$$

or $\mathscr{P}_0$ is a local pair-base of a point $x \in X$.

Each space with an ortho-pair-base is preorthocompact, and each preorthocompact developable space (which is the same as a developable

https://doi.org/10.4153/CJM-1981-110-3 Published online by Cambridge University Press

γ-space [**12**] has an ortho-pair-base. The proof of this is quite similar to the proof in [**17**] of the analogous result on ortho-bases and ortho-compactness, mentioned above.

H. Junnila has proved in [**12**] that each developable γ-space is quasi-metrizable. We will generalize here H. Junnila's result to all γ-spaces with ortho-pair-bases.

2. Given a space X, a *binary relation* U on X, i.e., $U \subset X \times X$, is called a *neighbournet* in X, provided that for each $x \in X$,

$$U\{x\} = \{y| \langle x, y\rangle \in U\}$$

is a neighbourhood of x. If U is a neighbourhood of the diagonal in $X \times X$, then it is a neighbournet in X; but the converse need not be true.

Given a neighbournet U in X and a set $G \subset X$, we define

$$UG = U(G) = \cup \{U\{x\}| x \in G\}.$$

Given two neighbournets U and V we define a new neighbournet $UV = U \circ V$ such that

$$(U \circ V)\{x\} = U(V\{x\}) \text{ for each } x \in X,$$

and $U^k = U \circ U \circ \ldots \circ U$ (k times). A neighbournet is *transitive* provided that $U^2 \subset U$, i.e., $\langle x, z\rangle \in U$ whenever $\langle x, y\rangle$, $\langle y, z\rangle \in U$. A neighbournet is *normal* provided that there exists a sequence of neighbournets U_n, $n = 1, 2, \ldots, U_{n+1}^2$ and $U_1 = U$. A sequence of neighbournets $\langle U_n\rangle$ is called *basic*, provided that for each $x \in X$, $\{U_n\{x\}| n = 1, 2, \ldots\}$ is a local base of x [**13**].

PROPOSITION 1. (i) *A space X is quasi-metrizable if and only if there is a basic sequence of normal neighbournets in X.*

(ii) *A space X is non-archimedean quasi-metrizable if and only if there is a basic sequence of transitive neighbournets in X.*

(iii) *A space X is a γ-space if and only if there is a sequence of neighbournets $\langle U_n\rangle$ in X such that $\langle U_n{}^2\rangle$ is basic* [**13**].

Concerning (iii) we remark that if $\langle U_n{}^2\rangle$ is basic then so is $\langle U_n{}^k\rangle$ for each $k \geqq 1$.

It follows immediately from Proposition 1 that, in order to show that a γ-space X is (non-archimedean) quasi-metrizable, it is enough to prove, for example, that for each neighbournet U in X there exists a normal (transitive) neighbournet $V \subset U^k$ for any fixed $k \geqq 1$. This suggests the following definition and proposition due to P. Fletcher and W. F. Lindgren.

Definition 7. A space X is called *k-pretransitive* (*k-transitive*), $k \geqq 1$, provided that for each neighbournet U in X there is a normal (transitive) neighbournet $V \subset U^k$ (cf. [**3**]).

Obviously, each k-(pre)transitive space is m-(pre)transitive for each $m \geqq k$.

PROPOSITION 2. *Each k-pretransitive (k-transitive) γ-space is (non-archimedean) quasi-metrizable.*

We will show that each space with an ortho-(pair-)base is 2-(pre)-transitive; hence, each γ-space with an ortho-(pair-)base is non-archimedean quasi-metrizable (quasi-metrizable).

It is worth noting that k-transitivity and k-pretransitivity, paracompactness-like properties of topological spaces with neighbournets in the role of covers, seem to be of certain intrinsic interest. In many cases it is not easy to show that a particular space is or is not k-(pre)transitive. The only known classes of k-(pre)transitive spaces are those of the generalized ordered spaces, $k = 3$ [**15**], the (pre)orthocompact semi-stratifiable spaces, $k = 3$, [**12**] and the spaces with ortho-(pair-)bases, $k = 2$ as will be seen below. More on this subject can be found in [**5**], [**16**] and [**9**].

3. The following construction was used in [**15**] to prove that each generalized ordered space is 3-transitive.

Given a neighbournet U in X, we define a new neighbournet U^+ in X such that for each $x \in X$,

$$U^+\{x\} = \cap \{U(G) | \, G \text{ is a neighbourhood of } x\}.$$

LEMMA 1. *For each neighbournet U in X*

(i) $U \subset V^+ \subset U^2$.

(ii) $(U^+)^+ = U^+$.

(iii) *If each $U\{x\}$ is open then* $((U^+)^2)^+ = (U^+)^2$.

Proof. (i) is obvious. Since for each open set G, $U^+(G) = U(G)$, it follows that $(U^+)^+ = U^+$ and if each $U\{x\}$ is open, then

$$(U^+)^2(G) = U^+(U^+(G)) = U^+(U(G)) = U(U(G)) = U^2(G),$$

and it follows that $((U^+)^2 = (U^2)^+$, i.e., (ii) and (iii) are proved.

It follows from Lemma 1 (i) that in order to prove that each space X with an ortho-(pair-)base is 2-(pre)transitive, it is sufficient to show that for each neighbournet U in X there is a (normal) transitive neighbournet $V \subset U^+$.

Notice that a neighbournet U contains a normal neighbournet if and only if U is normal.

PROPOSITION 3. *The following are equivalent for a space X.*

https://doi.org/10.4153/CJM-1981-110-3 Published online by Cambridge University Press

(i) *For each neighbournet U in X, U^+ is normal.*

(ii) *For each neighbournet U in X, there is a neighbournet V in X such that $V^2 \subset U^+$.*

(iii) *For each neighbournet U in X, there is an interior preserving collection Q of pairs $\langle G', G'' \rangle$ of open sets, $G' \subset G''$, such that for each $x \in X$ there exists $\langle G', G'' \rangle \in Q$ with $x \in G'$, $G'' \subset U^+\{x\}$.*

Proof. (i) $\Rightarrow$ (ii) is obvious. In order to prove (ii) $\Rightarrow$ (i), let $U = U_1$ be a neighbournet in X. There is a neighbournet $V = U_2$ in X, such that $V^2 \subset U^+$, and such that all $V\{x\}$ are open. By Lemma 1 (ii) $(V^2)^+ \subset U^+$. By Lemma 1 (iii) $((V^+)^2)^+ \subset U^+$. We have proved that for each neighbournet U_1 in X there exists a neighbournet U_2 in X such that $(U_2{}^+)^2 \subset U_1{}^+$. Repeating similar arguments for each $n = 2, 3, \ldots$, we obtain a sequence $\langle U_n \rangle$ of neighbournets, $n = 1, 2, \ldots$ such that for each n, $(U_{n+1}{}^+)^2 \subset U_n{}^+$; hence, $U_1{}^+$ is normal.

For (ii) $\Rightarrow$ (iii) let $V^2 \subset U^+$ and each $V\{x\}$ be open. We set

$$Q = \{\langle V\{x\}, V^2\{x\}\rangle | x \in X\}.$$

For (iii) $\Rightarrow$ (ii) let $x \in X$ and $\langle G'(x), G''(x)\rangle \in Q$ such that $x \in G'(x)$, $G''(x) \subset U^+\{x\}$. We define a neighbournet V in X such that each $V\{x\}$ is given by

$$V\{x\} = G'(x) \cap (\cap \{G'' | \langle G', G'' \rangle \in Q, x \in G'\}).$$

It follows that $V^2 \subset U^+$.

PROPOSITION 3′. *The following are equivalent for a space X.*

(i) *For each neighbournet U in X, there is a transitive neighbournet $V \subset U^+$.*

(ii) *For each neighbournet U in X, there is an interior-preserving collection C of open sets such that for each $x \in X$ there exists $G \in C$ with $x \in G \subset U^+\{x\}$.*

Proof. For (i) $\Rightarrow$ (ii) let all $V\{x\}$ be open. We set $C = \{V\{x\} | x \in X\}$. It follows that C is interior-preserving. For (ii) $\Rightarrow$ (i) we define a neighbournet V such that each $V\{x\} = \cap \{G | G \in C, x \in G\}$.

We will also use the following property of neighbournets U^+, the proof of which is straightforward.

LEMMA 2. *Let U be a neighbournet in X, and $G \subset X$. Then $F = \{x | G \subset U^+\{x\}\}$ is relatively closed in G.*

4. THEOREM 1. *In each space with an ortho-pair-base for each neighbournet U, U^+ is normal.*

THEOREM 1′. *In each space with an ortho-base for each neighbournet U there is a transitive neighbournet $V \subset U^+$.*

Proof. Let $\leqq$ be a well order on a space X. We will simultaneously prove Theorems 1 and 1′ assuming for both Theorems that there is an ortho-pair-base $\mathscr{P}$ in X, and assuming for Theorem 1′, in addition, that $G' = G''$ for each $\langle G', G''\rangle \in \mathscr{P}$; this means that there is an ortho-pair-base in X. It is sufficient to prove that there exists an interior-preserving collection $\mathscr{Q} \subset \mathscr{P}$ such that for each $x \in X$ there is $\langle G', G''\rangle \in \mathscr{Q}$ with $x \in G'$ and $G'' \subset U^+\{x\}$. For Theorem 1 this means that U^+ is normal by Proposition 3. For Theorem 1′, however, this means the existence of a transitive neighbornet $V \subset U^+$ by Proposition 3′, since the collection $\{G | \langle G, G\rangle \in \mathscr{Q}\}$ is interior-preserving.

In fact we shall obtain an interior preserving collection

$$\mathscr{Q} \subseteq \mathscr{P},\ \mathscr{Q} = \{p(x) | x \in X\},\ p(x) = \langle G'(x), G''(x)\rangle,$$

such that $x \in G'(x)$ and $G''(x) \subset U^+\{x\}$. Simultaneously we shall define a set $Y \subset X$, and for each $x \in Y$ sets $Y(x)$ and $F(x)$ which will be used in our argument. The set Y will be defined by stating for each $x \in X$ whether $x \in Y$. The sets

$$Y(x) \subset \{y \in Y | y < x\}$$

will be defined for each $x \in Y$ using induction on $y < x$ by stating whether $y \in Y(x)$. For each $x \in Y$ the set

$$F(x) \subset \{y \in G'(x) | G''(x) \subset U^+\{y\}\}$$

will be a relatively closed subset of $G'(x)$ and $x \in F(x)$. All the definitions will be carried out by induction on $\langle X, \leqq\rangle$ as follows.

Let $x \in X$. If $x \in F(y)$ for some $y \in Y$, $y < x$, we set $p(x) = p(y)$ for the first such y, and state that $x \notin Y$.

Otherwise $x \in Y$. Then put

$$(\mathrm{i})_x\ Y(x) = \{y \in Y | y < x \in G'(y) \text{ and } Y(y) = \{z \in Y(x) | z < y\}\};$$

$$(\mathrm{ii})_x\ p(x) = \langle G'(x), G''(x)\rangle \in \mathscr{P} \text{ such that } x \in G'(x) \text{ and } G''(x) \subset U^+\{x\} \cap G(x), \text{ where } G(x) = \bigcap \{G'(y) - F(x) | y \in Y(x)\};$$

$$(\mathrm{iii})_x\ F(x) = \{y \in G'(x) | G''(x) \subset U^+\{y\}\} - \bigcup \{G'(y) | y < x \notin G'(y)\}.$$

Note that part $(\mathrm{ii})_x$ of the definition can be carried out because $G(x)$ is a neighborhood of x.

If $Y(x)$ has the last element y, $G(x)$ is a neighborhood of x since by $(\mathrm{ii})_y$ and $(\mathrm{i})_x$

$$G'(y) - F(y) \subset \bigcap \{G'(z) - F(z) | z \in Y(y)\} \cap (G'(y) - F(y)) = \bigcup \{G'(z) - F(z) | z \in Y(x), z < y\} \cap (G'(y) - F(y)) = G(x).$$

https://doi.org/10.4153/CJM-1981-110-3 Published online by Cambridge University Press

The set $G'(y) - F(y)$ is a neighborhood of x because since $y \in Y(x)$, $x \in G'(y)$ and since $y < x \in Y$, $x \notin F(y)$ and $F(y)$ is a relatively closed subset of the open set $G'(y)$ by $(\text{iii})_y$ and Lemma 2.

If $Y(x)$ has no last element then $G(x)$ is a neighborhood of x since by $(\text{iii})_y$ and $(\text{i})_x$

$$\begin{aligned} &\bigcap \{G''(y) \mid y \in Y(x)\} \\ &\subset \bigcap\{\bigcap\{G'(z) - F(z) \mid z \in Y(y)\} \mid y \in Y(x)\} \\ &= \bigcap \{G'(y) - F(y) \mid y \in Y(x)\} = G(x). \end{aligned}$$

The set $\bigcap\{G''(y) \mid y \in Y(x)\}$ is a neighborhood of x since

$$\{\langle G'(y), G''(y)\rangle \mid y \in Y(x)\}$$

is a subcollection of the ortho-pair-base $\mathscr{P}$ and it is not a local pair-base, for otherwise for some $y \in Y(x)$, $G''(y) \subseteq U^+x$, and hence for the first such y by $(\text{iii})_y$ $x \in F(y)$, while $y < x$. This is impossible for $x \in Y$.

It is clear now that for each $x \in X$, $x \in G'(x)$ and $G''(x) \subseteq U^+\{x\}$. We complete the proof by showing that

$$Q = \{\langle G'(x), G''(x)\rangle \mid x \in X\} = \{\langle G'(x), G''(x)\rangle \mid x \in Y\}$$

is interior preserving. Since $\mathscr{Q} \subset \mathscr{P}$ and $\mathscr{P}$ is an ortho-pair-base, it is enough to prove that $\mathscr{Q}$ does not contain a local pair-base for a point $x \in X$.

In fact, if y is the first point such that $p(x) = p(y)$ then $y \in Y$ and for $z \in Y$, $z \neq y$, either

$$F(y) \cap G'(z) = \emptyset \text{ or } F(z) \cap G'(z) = \emptyset$$

hence either

$$x \in G'(z) \text{ or } G'(z) \not\subset G'(x).$$

Indeed, let $I = Y(y) \cap Y(z)$. It follows from $(\text{i})_t$, $t \in I$, that I is an initial subset of both $Y(y)$ and $Y(z)$. Let $\tilde{y}$ and $\tilde{z}$ be the first elements of $(Y(y) - I) \cup \{y\}$ and $(Y(z) - I) \cup \{z\}$ respectively. Obviously $I = Y(\tilde{z}) = Y(\tilde{z})$. If $\tilde{z} < \tilde{y} \leqq y$ then $\tilde{z} \notin Y(y)$ and by $(\text{i})_y$ $y \notin G'(\tilde{z})$. Hence

$$F(y) \subset X - G'(z) \subset X - G'(\tilde{z})$$

by $(\text{iii})_y$ and $(\text{ii})_z$. If $\tilde{y} < \tilde{z}$ then similarly

$$F(z) \subseteq X - G'(y).$$

Let now $\tilde{y} = \tilde{z}$. Then either $\tilde{y} = y \in Y(z)$ or $\tilde{z} = z \in Y(y)$, hence either

$$G'(z) \subseteq G''(z) \subseteq X - F(y)$$

https://doi.org/10.4153/CJM-1981-110-3 Published online by Cambridge University Press

by $(\text{ii})_z$ or similarly

$$G'(y) \subseteq X - F(z).$$

From Lemma 1(i) and Proposition 2 we have:

THEOREM 2. *Each space with an ortho-pair-base is 2-pretransitive; hence, each γ-space with an ortho-pair-base is quasi-metrizable.*

THEOREM 2'. *Each space with an ortho-base is 2-transitive, hence each γ-space with an ortho-base is non-archimedean quasi-metrizable.*

Remark. If U is a neighbournet in a space without an ortho-base then U^+ may be non-normal even if the space is 2-transitive [**10**].

5. In light of the results of this paper, the following problems are of interest.

Problem 1. Is each space with an ortho-pair-base k-transitive for some k? Does it have a pair-base?

Problem 2. Is each quasi-metrizable space with an ortho-pair-base non-archimedean quasi-metrizable?

Notice that it is not known whether each preorthocompact developable space is orthocompact or, in other words, whether each quasi-metrizable developable space is non-archimedean quasi-metrizable [**12**].

Added in proof. After this paper was submitted the γ-space problem was solved negatively by Ralph Fox [**8**].

REFERENCES

1. H. R. Bennett, *Quasi-metrizability and the γ-space property in certain generalized ordered spaces*, Topology Proceedings *4* (1979), 1–12.
2. P. Fletcher and W. F. Lindgren, *Transitive quasi-uniformities*, J. Math. Anal. Appl. *39* (1972), 397–405.
3. ——— *Quasi-uniformities with a transitive base*, Pacific J. Math. *43* (1972), 619–631.
4. ——— *Orthocompactness and strong Cech compactness in Moore spaces*, Duke Math. J. *39* (1972), 753–766.
5. ——— *Quasi-uniform spaces*, Marcel-Dekker, to appear.
6. R. Fox, *On metrizability and quasi-metrizability*, to appear.
7. ——— *A short proof of the Junnila quasi-metrization theorem*, Proc. Amer. Math. Soc., to appear.
8. ——— *Solution of the γ-space problem*, Proc. Amer. Math. Soc., to appear.
9. ——— *Pretransitivity and the γ-space conjecture*, to appear.
10. R. Fox and J. Kofner, to appear.
11. G. Gruenhage, *A note on quasi-metrizability*, Can. J. Math. *29* (1977), 360–366.
12. H. Junnila, *Covering properties and quasi-uniformities of topological spaces*, Ph.D. thesis, Virginia Polytech. Inst. and State Univ. (1978).
13. ——— *Neighbournets*, Pacific J. Math. *76* (1978), 83–108.
14. J. Kofner, *On Δ-metrizable spaces*, Math. Notes *13* (1973), 168–174.

15. ——— *Transitivity and the γ-space conjecture in ordered spaces*, Proc. Amer. Math. Soc., to appear.

16. ——— *On quasi-metrizability*, Topology Proceedings (1980).

17. W. F. Lindgren and P. Nyikos, *Spaces with bases satisfying certain order and intersection properties*, Pacific J. Math. *66* (1976), 455–476.

18. ——— *Open problems*, Topology Proceedings *2* (1977), 687–688.

George Mason University,
Fairfax, Virginia