4
Spontaneous symmetry breaking

4.1 General background

As already mentioned in previous chapters, the nuclear structure exhibits many similarities with the electron structure of metals. In both cases, one is dealing with systems of fermions which may be characterized in a first approximation in terms of independent particle motion. However in both systems, important correlations in the particle motion arise from the action of the forces between particles. In particular, it is well established that nucleons moving close to the Fermi energy in time-reversal states have the tendency to form Cooper pairs which eventually condense (Bohr, Mottelson and Pines (1958), Bohr and Mottelson (1975)). This phenomenon, which has its parallel in low-temperature superconductivity, modifies the structure of nuclei in an important way. In particular it influences the occupation numbers of single-particle levels around the Fermi surface (Chapter 3), the moment of inertia of deformed nuclei (Chapter 3), the lifetime of alpha and cluster decay and fission processes (Chapter 7), the depopulation of superdeformed configurations (Chapter 6) and the cross-sections of two-nucleon transfer reactions (Chapter 5).

While one does not expect the transition between the normal and the superfluid phases of the atomic nucleus to be sharp because of finite size effects and the central role played by fluctuations (see Chapter 6), there is a strong analogy between phenomena in nuclei and the corresponding phenomena in bulk superconductors. Spontaneous symmetry breaking is important in both nuclei and superconductors. We focus our attention on one of the fingerprints of the broken symmetry, namely the consequences it has for the energy level spectra of the systems.

The phenomenon of spontaneous symmetry breaking had been known for a long time before the formulation of the BCS theory of superconductivity in 1957. An example is the Jahn–Teller effect in solid state physics; if the symmetry of a crystal is such that ground-state degeneracy of electron states at a crystal site is not the Kramers minimum then it is energetically favourable for the crystal to distort in such a way as to lower the symmetry enough to remove the degeneracy. The same phenomenon is the origin of deformed shapes in nuclei (see Reinhardt and Otten (1984)). The Hartree–Fock single-particle states in a spherical potential for a nucleus with neutron and proton numbers far from closed shells are degenerate or almost degenerate. The energy is reduced by allowing the self-consistent potential to deform to remove the degeneracy. Even though the nuclear Hamiltonian is rotationally invariant the Hartree–Fock wavefunction of a deformed nucleus is not an eigenstate of angular momentum. The theory produces a nucleon density distribution which is deformed and has a definite orientation in space. A rotation applied to a Hartree–Fock state produces an equivalent state with the same energy as the original state. This idea is the basis of Bohr and Mottelson's (1953) and Nilsson's (1955) theory of deformed nuclei.

The situation is similar with Bardeen, Cooper and Schrieffer's (1957a,b) theory of superconductivity. The Hamiltonian of the BCS theory commutes with the electron number operator $\hat{N}$. The ground state of a finite superconductor should be an eigenstate of $\hat{N}$ but the BCS wavefunction does not have this property. A gauge transformation applied to the BCS ground state produces another, different, BCS state with the same ground-state energy. There are an infinite number of equivalent states connected by gauge transformations. The BCS theory predicts that there is an energy gap 2Δ between the ground state and excited two-quasiparticle states. Anderson (1958) investigated corrections to the BCS theory using the random phase approximation (RPA). He found a dispersion relation predicting a phonon-like collective mode related to zero sound with energies within the BCS energy gap. He related this collective excitation to the gauge symmetry breaking. Similar results were obtained by a different method at about the same time by Bogoliubov *et al*. (1958). His approach was based on a development of his quasiparticle theory (Bogoliubov (1958b)).

The connection between the gauge symmetry breaking and Anderson's collective states (Anderson (1958)) was studied in more detail by Nambu (1959). He argued that the phonon-like collective states are essential to the gauge-invariant character of the theory and that they are a necessary consequence of the gauge invariance. He showed that gauge invariance, the energy gap and the collective states are related to each other. In a subsequent paper Nambu (1960) extended his ideas to a γ_5-invariant theory with zero-mass fermions. There γ_5-symmetry breaking (or chiral symmetry breaking) leads to non-zero baryon masses (analogous to the BCS energy gap) and zero-mass pseudoscalar mesons (analogous to Anderson's collective states). Nambu's ideas were incorporated in the Nambu, Jona-Lasinio (1961a,b) model of baryons and mesons which was motivated by the BCS theory of superconductors.

Nambu's (1959) paper is based on a mean field generalization of the BCS theory with RPA corrections (see also Eguchi and Nisijima (1995)). Goldstone (1961) extended Nambu's work in a paper on symmetry breaking with the title 'Field theories with "superconductor" solutions'. He considered several simple covariant models and conjectured that, whenever the original Lagrangian has a continuous symmetry group and the new solutions have a reduced symmetry, then the theory must contain massless bosons. The models he considered were renormalizable, so that Goldstone's result might be very general, and apply not only to approximate mean field solutions, but also to exact solutions.

Goldstone's conjecture was put on a firmer footing by Goldstone, Salam and Weinberg (1962). They proved by three different methods that, if there is a continuous symmetry transformation under which the Lagrangian is invariant, then either the vacuum state is also invariant, or there must exist spinless bosons of zero mass. In particle physics these bosons are called Goldstone bosons. We refer to them as Anderson, Goldstone, Nambu (AGN) bosons because analogous excitations were discovered in theories of superconductivity by Anderson and Nambu and their relation to gauge symmetry breaking was recognized by those authors.

In the following discussion we distinguish between large and small systems, or more properly between three-dimensional (3D-) and zero-dimensional (0D-) systems. We make use of the random phase approximation treatment of pairing developed by Anderson (1958) for the case of a large neutral system and by Högaasen-Feldman (1961) and Bes and Broglia (1966) for the case of the atomic nucleus (see also Scadron (1985) and Broglia *et al.* (2000)).

4.1.1 Infinite systems and finite systems

As discussed in Section 1.7, in normal metals at low temperature the coherence length ξ is of the order of 10^3 Å. This quantity is much larger than the spacing between electrons ($r_s \approx 1$–3 Å) where

$$r_s = \left(\frac{3}{4\pi n}\right)^{1/3}, \tag{4.1}$$

is the Wigner–Seitz radius, while n is the electron density of the system. At the same time, the quantity ξ is also much smaller than the physical dimension L of a typical macroscopic sample. The inequalities $r_s \ll \xi \ll L$ are typical of three-dimensional (3D-) superconductors. In keeping with these results, within the region occupied by any given pair will be found the centre of mass of many (of the order of 10^6) pairs. In a superconductor the pair phase $\phi(\vec{r})$ (gauge angle) is approximately constant over spatial regions characteristic of the correlations in the superconducting phase, and a supercurrent with gauge-invariant velocity $\boldsymbol{v}_s = -\hbar/2m_e(\boldsymbol{\nabla}\phi - 2e/\hbar c\boldsymbol{A})$ where $\boldsymbol{A}$ is the vector potential, can be defined.

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It is obtained by multiplying the wavefunctions of all the effectively interacting particles by approximately the same phase factor. In an anisotropic superfluid such as ^{3}He-A the order parameter not only has a phase but also has an orientation, the preferred direction of $\boldsymbol{l}$, the relative orbital angular momentum of the $l = 1$ Cooper pairs. In this system the superfluid velocity depends not only on the spatial change of the phase ϕ, but also on that of $\boldsymbol{l}$. Parametrizing $\boldsymbol{l}$ by the azimuthal (β) and the polar (α) angles, $\boldsymbol{v}_s$ now takes the form $\boldsymbol{v}_s = \hbar/2m_3(\nabla\phi - \cos\beta\nabla\alpha)$, where m_3 is the mass of the ^{3}He atom (Vollhardt and Wölfle (1990)).

The situation is quite different in nuclei, where $\xi \approx 30$ fm (see equation (1.39)), a quantity which is much larger than the average distance between nucleons ($\approx$2 fm) (see Appendix C). On the other hand, pairs must be located inside the nucleus (radius $R \approx 5$–7 fm for medium heavy nuclei). Thus a nucleus can be viewed as a 0D-system, where the phenomenon of quantized superflow observed in infinite 3D-systems does not seem to have a counterpart. Superflow may, on the other hand, play an important role in the dynamics of nuclear matter occurring in neutron stars (see Section 1.10 and Ruderman (1972), Anderson *et al*. (1982), Pines *et al*. (1980), (1992) and references therein).

The BCS solution of the pairing problem in a finite nucleus has been presented in Chapter 3 and Appendix G. In the next section we will discuss the RPA (collective) modes which are built on it (see Appendices I and J) with special reference to spontaneous symmetry breaking. Then in Section 4.3 we will make a comparison with Anderson's (1958) derivation of collective modes in infinite 3D neutral superconductor and comment on the similarities and differences between the finite and infinite cases.

4.2 Pairing in atomic nuclei (0D systems; $\xi \gg R$)

The present section is concerned with gauge symmetry breaking in a system of neutrons or protons interacting with a pairing force. We begin it by dividing the Hamiltonian into a mean field part and a fluctuating part. The ground state of the mean field part is represented by a BCS wavefunction. The original Hamiltonian is invariant with respect to rotations in gauge space but the mean field Hamiltonian and the BCS wavefunction are not (see Section 3.8). There is a discussion of the transformation properties under rotations in gauge space. The next step is to derive the RPA equations for the fluctuations about the mean field wavefunctions. The RPA equations can be solved exactly for the simple pairing problem. The gauge invariance of the original Hamiltonian requires that the RPA equations must have a zero-frequency mode. This mode comes out automatically from the explicit solution of the RPA equations, but it can also be found by a general argument from gauge symmetry (Section 4.2.3). The zero-frequency mode is related to pair addition and removal processes (Section 4.2.4). The nucleon number dependence of the ground-state energy is contained in the Fermi

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energy, which gives a linear dependence, and a 'moment of inertia' which gives a quadratic dependence. The arguments are illustrated by a simple schematic model in Section 4.2.5, and by Weinberg's (1996) discussion of symmetry breaking in macroscopic systems. Section 4.2.6 presents a comparison with experiment.

4.2.1 Deformation in gauge space: mean-field approximation

Our Hamiltonian describes the motion of independent particles interacting through a pairing force,

$$H = H_{\mathrm{sp}} + H_{\mathrm{p}}. \tag{4.2}$$

Here

$$H_{\mathrm{sp}} = \sum_{\nu>0} (\varepsilon_\nu - \lambda)(a_\nu^\dagger a_\nu + a_{\bar{\nu}}^\dagger a_{\bar{\nu}}), \tag{4.3}$$

is the single-particle Hamiltonian. The operator $a_\nu^\dagger$ creates a particle (fermion) with quantum numbers ν. For spherical nuclei, ν stands for n, l, j and m, i.e. the number of nodes, the orbital angular momentum, the total angular momentum and its projection, respectively. The state $|\bar{\nu}\rangle$ is obtained from the state $|\nu\rangle$ by the operation of time reversal. The condition $\nu > 0$ means $m > 0$, where m is the magnetic quantum number. The single-particle energies ε_ν are measured from the Fermi energy λ. The pairing Hamiltonian

$$H_{\mathrm{p}} = -G P^\dagger P, \tag{4.4}$$

is written in terms of the pair operator

$$P^\dagger = \sum_{\nu>0} a_\nu^\dagger a_{\bar{\nu}}^\dagger, \tag{4.5}$$

which creates a pair of particles in time-reversal states. In a spherical nucleus these are coupled to angular momentum zero. The BCS solution of this Hamiltonian provides a mean-field approximation to H, where the pairing gap parameter,

$$\Delta = G\alpha_0, \tag{4.6}$$

plays a central role in determining the properties of the system. The quantity

$$\alpha_0 = \langle \mathrm{BCS}|P^\dagger|\mathrm{BCS}\rangle \tag{4.7}$$

is the average value of the pair transfer operator in the pairing mean-field ground state $|\mathrm{BCS}\rangle$.

As a function of these parameters, the total Hamiltonian

$$H = H_{\mathrm{MF}} + H_{\mathrm{fluct}}, \tag{4.8}$$

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can be written as a sum of a mean field term

$$H_{\mathrm{MF}} = H_{\mathrm{sp}} - \Delta(P^{\dagger} + P) + \frac{\Delta^2}{G} \tag{4.9}$$

and a fluctuation term

$$H_{\mathrm{fluct}} = -G(P^{\dagger} - \alpha_0)(P - \alpha_0). \tag{4.10}$$

BCS theory assumes $\alpha_0 \gg (P^{\dagger} - \alpha_0)(P - \alpha_0)$ and solves the reduced Hamiltonian H_{MF} making an ansatz for $|\mathrm{BCS}\rangle$.

As in Chapter 3 we make a special choice of gauge and define a standard BCS wavefunction as

$$|\mathrm{BCS}\rangle_{\mathcal{K}} = \prod_{\nu>0}(U_\nu + V_\nu a_\nu^{\dagger} a_{\bar{\nu}}^{\dagger})|0\rangle, \tag{4.11}$$

where U_ν and V_ν are real. This wavefunction does not have a fixed number of particles and selects a privileged orientation in gauge space. The Hamiltonian H (equations (4.2) or (4.8)) is invariant with respect to rotations in gauge space generated by the operator

$$\mathcal{G}(\phi) = \mathrm{e}^{-\frac{\mathrm{i}\hat{N}}{2}\phi}, \tag{4.12}$$

where $\hat{N} = \sum_\nu a_\nu^{\dagger} a_\nu$ is the particle number operator. The state

$$|\mathrm{BCS}\,(\phi)\rangle_{\mathcal{K}} = \mathcal{G}(\phi)\prod_{\nu>0}(U_\nu + V_\nu a_\nu^{\dagger} a_{\bar{\nu}}^{\dagger})|0\rangle \tag{4.13}$$

$$= \prod_{\nu>0}(U_\nu + \mathrm{e}^{-\mathrm{i}\phi} V_\nu a_\nu^{\dagger} a_{\bar{\nu}}^{\dagger})|0\rangle \tag{4.14}$$

is obtained from the standard state $|\mathrm{BCS}\rangle_{\mathcal{K}}$ by rotating it through an angle ϕ in gauge space. The new BCS state has the same energy and a similar structure as $|\mathrm{BCS}\rangle_{\mathcal{K}}$. The rotated state can be written in another way as

$$|\mathrm{BCS}\,(\phi)\rangle_{\mathcal{K}} = |\mathrm{BCS}\rangle_{\mathcal{K}'} = \prod_{\nu>0}(U_\nu + V_\nu a_\nu'^{\dagger} a_{\bar{\nu}}'^{\dagger})|0\rangle, \tag{4.15}$$

in terms of rotated creation operators

$$\begin{aligned} a_\nu'^{\dagger} &= \mathcal{G}(\phi) a_\nu^{\dagger} \mathcal{G}^{-1}(\phi)\,, \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2}\phi} a_\nu^{\dagger}. \end{aligned} \tag{4.16}$$

This allows us to define an intrinsic (body-fixed) coordinate frame $\mathcal{K}'$ (see Fig. 4.1), in terms of the primed operators.

The state $|\mathrm{BCS}(\phi)\rangle_{\mathcal{K}}$ with gauge angle ϕ with respect to the laboratory coordinate system $\mathcal{K}$ has the angle $\phi = 0$ with respect to the intrinsic system $\mathcal{K}'$.

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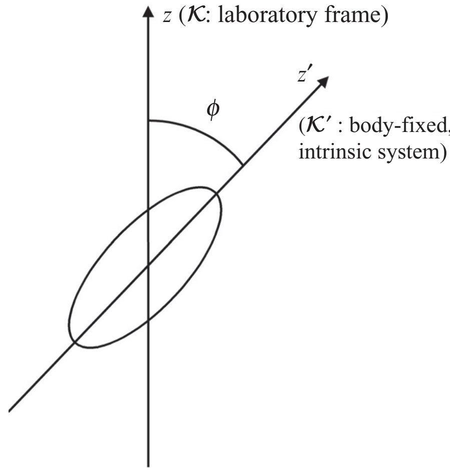

Figure 4.1. Schematic representation of a deformation in gauge space defining a privileged orientation z' in the two-dimensional space and thus an intrinsic, body-fixed, coordinate system of reference $\mathcal{K}'$, making an angle ϕ with the laboratory frame of reference $\mathcal{K}$.

The mean-field pairing Hamiltonian becomes diagonal in the quasiparticle basis, i.e.

$$H_{\mathrm{MF}} = U + H_{11}, \tag{4.17}$$

where

$$U = 2\sum_{\nu>0}(\varepsilon_\nu - \lambda)V_\nu^2 - \frac{\Delta^2}{G} \tag{4.18}$$

and

$$H_{11} = \sum_\nu E_\nu \alpha_\nu^\dagger \alpha_\nu. \tag{4.19}$$

The quasiparticle creation operator

$$\alpha_\nu^\dagger = U_\nu a_\nu^\dagger - V_\nu a_{\bar{\nu}} \tag{4.20}$$

is defined in terms of the BCS theory U_ν and V_ν occupation numbers, the state $|\mathrm{BCS}\rangle$ is the quasiparticle vacuum. The quasiparticle energy is

$$E_\nu = \sqrt{(\varepsilon_\nu - \lambda)^2 + \Delta^2}\,. \tag{4.21}$$

We shall see that restoration of symmetry is obtained by diagonalizing the residual interaction acting among the quasiparticles associated with the terms H_p'' in the expression (Anderson (1958), Bes and Broglia (1966), Broglia (1985))

$$H_{\mathrm{fluct}} = H_p' + H_p'' + C, \tag{4.22}$$

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where

$$H'_{\mathrm{p}} = -\frac{G}{4}\left(\sum_{\nu>0}(U_\nu^2 - V_\nu^2)(\Gamma_\nu^\dagger + \Gamma_\nu)\right)^2 \tag{4.23}$$

and

$$H''_{\mathrm{p}} = \frac{G}{4}\left(\sum_{\nu>0}(\Gamma_\nu^\dagger - \Gamma_\nu)\right)^2, \tag{4.24}$$

with

$$\Gamma_\nu^\dagger = \alpha_\nu^\dagger \alpha_{\bar{\nu}}^\dagger \, . \tag{4.25}$$

The term C in equation (4.22) stands for constant terms, as well as for terms proportional to the number of quasiparticles, and which consequently vanish when acting on the BCS ground state (see Appendices I and J). Neglecting terms proportional to the number of quasiparticles is an important approximation in the RPA and it has to be done consistently. The structure displayed by H'_{p} and H''_{p} is a consequence of the fact that, neglecting terms of type C, one can write

$$P^\dagger + P = \sum_{\nu>0}(U_\nu^2 - V_\nu^2)(\Gamma_\nu^\dagger + \Gamma_\nu) \tag{4.26}$$

and

$$P^\dagger - P = \sum_{\nu>0}(U_\nu^2 + V_\nu^2)(\Gamma_\nu^\dagger - \Gamma_\nu) = \sum_{\nu>0}(\Gamma_\nu^\dagger - \Gamma_\nu). \tag{4.27}$$

In other words, there are two fields which can create (annihilate) two quasiparticles, namely U_ν^2 and V_ν^2. These fields can be combined in a symmetric $(U_\nu^2 + V_\nu^2)$ and in an antisymmetric $(U_\nu^2 - V_\nu^2)$ fashion with respect to the Fermi surface.

Making use of the approximate commutation relation (see Appendix A, Section A.4)

$$[\Gamma_\nu, \Gamma_{\nu'}^\dagger] = \delta(\nu, \nu'), \tag{4.28}$$

which neglects terms proportional to the number of quasiparticles, the solutions of

$$\widetilde{H} = H_{\mathrm{MF}} + H'_{\mathrm{p}} + H''_{\mathrm{p}}, \tag{4.29}$$

in particular the collective modes, can be obtained in the harmonic approximation (RPA), through the equations of motion (see Appendix A)

$$[\widetilde{H}, \Gamma_n^\dagger] = \hbar\omega_n \Gamma_n^\dagger \tag{4.30}$$

and

$$[\Gamma_n, \Gamma_{n'}^\dagger] = \delta(n, n'), \tag{4.31}$$

where

$$\Gamma_n^\dagger = \sum_\nu (a_{n\nu}\Gamma_\nu^\dagger + b_{n\nu}\Gamma_\nu), \tag{4.32}$$

is the creation operator of the nth vibrational mode. Equations (4.30) and (4.31) lead to a dispersion relation and to the normalization condition of the eigenstates, which determine the frequencies $\omega_n = (C_n/D_n)^{1/2}$, the RPA energies $W_n = \hbar\omega_n$, and the zero point fluctuations $(\hbar/2\omega_n D_n)^{1/2}$ associated with the modes, and thus the restoring force (C_n) and inertia (D_n) parameters for the corresponding harmonic motion.

Note that to neglect C in equation (4.22) is equivalent to a quasi-boson approximation. In fact, defining the conjugate variables

$$q_\nu = \frac{1}{\sqrt{2}}(\Gamma_\nu^\dagger + \Gamma_\nu), \quad p_\nu = -\frac{\mathrm{i}}{\sqrt{2}}(\Gamma_\nu^\dagger - \Gamma_\nu),$$

fulfilling the condition (see equation (4.28))

$$[q_\nu, p_{\nu'}] = \mathrm{i}\delta(\nu, \nu'),$$

one can write

$$H'_\mathrm{p} + H''_\mathrm{p} = -\frac{G}{2}\Big[\Big(\sum_{\nu>0}(U_\nu^2 - V_\nu^2)q_\nu\Big)^2 + \Big(\sum_{\nu>0} p_\nu\Big)^2\Big].$$

This is diagonalized by the transformation (equivalent to equations (4.30) and (4.31))

$$Q_n = \sum_{\nu'} \lambda_{n\nu'} q_{\nu'}, \quad P_n = \sum_{\nu'} \mu_{n\nu'} p_{\nu'},$$

so that

$$\widetilde{H} = \sum_n \Big(\frac{P_n^2}{2D_n} + \frac{C_n}{2}Q_n^2\Big)$$

and

$$[Q_n, P_{n'}] = \mathrm{i}\delta(n, n'), \quad [Q_n, \widetilde{H}] = \mathrm{i}\frac{P_n}{D_n}, \quad [P_n, \widetilde{H}] = -\mathrm{i}C_n Q_n,$$

implying that the eigenvalues are $W_n = \hbar(C_n/D_n)^{1/2}$.

4.2.2 Solution of the RPA equations

In what follows we do not diagonalize the full Hamiltonian $H_\mathrm{MF} + H'_\mathrm{p} + H''_\mathrm{p}$ but discuss two special cases (for the simultaneous diagonalization of H_MF, H'_p and H''_p we refer the reader to Appendix J). As a first case we consider the Hamiltonian (see equations (4.17) and (4.23)) $H_\mathrm{MF} + H'_\mathrm{p}$ where the odd term H'_p

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in the interaction is antisymmetric with respect to the Fermi surface. The even term H_p'' is neglected. Equation (4.30) leads, in this case, to the dispersion relation

$$\sum_{\nu>0} \frac{2E_\nu(U_\nu^2 - V_\nu^2)^2}{(2E_\nu)^2 - (W_n')^2} = \frac{1}{G}. \tag{4.33}$$

Making use of the BCS relation

$$U_\nu^2 - V_\nu^2 = \frac{\varepsilon_\nu - \lambda}{E_\nu}, \tag{4.34}$$

which is equivalent to the BCS gap equation, it can be shown that the lowest energy solution of equation (4.33) is $W_1' = \hbar\omega_1 = 2\Delta$. These pairing vibrations (Bes and Broglia (1966)) have been studied extensively through two-nucleon transfer processes (see e.g. Broglia *et al.* (1973) and references therein) and found to be weakly collective, a property also shared with the pairing vibration of a 3D-system (Anderson (1958)). However, they become very collective in the case of normal nuclei, where multiphonon pairing vibration states have been strongly excited through two-particle transfer reaction (see Chapter 5, see also Section 8.4). These modes have not been observed in normal infinite systems.

The second special case includes the even-interaction H_p'' which is symmetric with respect to the Fermi surface and neglects the odd term H_p'. The Hamiltonian is (see equations (4.17) and (4.24)) $H_{\rm MF} + H_p''$. Equation (4.30) leads to

$$\sum_{\nu>0} \frac{2E_\nu}{(2E_\nu)^2 - (W_n'')^2} = \frac{1}{G}\,. \tag{4.35}$$

Using the gap equation

$$\sum_{\nu>0} \frac{1}{E_\nu} = \frac{2}{G}\,, \tag{4.36}$$

this reduces to

$$\sum_{\nu>0} \frac{1}{2E_\nu} \frac{(W_n'')^2}{(2E_\nu)^2 - (W_n'')^2} = 0. \tag{4.37}$$

The lowest energy solution of this equation is $W_1'' = 0$. The general amplitudes associated with the one-phonon amplitude (see equation (4.32)) are

$$a_{n\nu} = \frac{\Lambda_n''}{2E_\nu - W_n''}\,, \qquad b_{n\nu} = \frac{\Lambda_n''}{2E_\nu + W_n''}. \tag{4.38}$$

The normalization factor

$$\Lambda_n'' = \frac{1}{2}\left(\sum_{\nu>0} \frac{2E_\nu \hbar\omega_n''}{\left((2E_\nu)^2 - (\hbar\omega_n'')^2\right)^2}\right)^{-1/2} \tag{4.39}$$

is proportional to the zero-point fluctuation of the corresponding vibrational mode. In the case of the zero-frequency mode Λ_1'' is infinite. The amplitudes of the zero-frequency mode are $a_{1\nu} = b_{1\nu} \propto 1/E_\nu$ but cannot be normalized.

4.2.3 The zero-frequency mode

There is a more general way of looking at the zero-frequency mode. The operator $\hat{N}$ which counts the number of nucleons is

$$\hat{N} = \sum_\nu a_\nu^\dagger a_\nu = \sum_\nu \left(U_\nu \alpha_\nu^\dagger + V_\nu \alpha_{\bar{\nu}}\right)\left(U_\nu \alpha_\nu + V_\nu \alpha_{\bar{\nu}}^\dagger\right). \tag{4.40}$$

The RPA approximation $\tilde{N}$ for $\hat{N}$ can be written in terms of the quasiboson operators introduced in (4.25) as

$$\begin{aligned}\tilde{N} &= 2\sum_{\nu>0} U_\nu V_\nu (\Gamma_\nu^\dagger + \Gamma_\nu) + N_0 \\ &= \Delta \sum_{\nu>0} \frac{1}{E_\nu}(\Gamma_\nu^\dagger + \Gamma_\nu) + N_0,\end{aligned} \tag{4.41}$$

where $N_0 = 2\sum_{\nu>0} V_\nu^2$ is the average number of particles in the quasiparticle vacuum state. Terms proportional to the number of quasiparticles have been neglected. The operator $\hat{N}$ commutes with the exact Hamiltonian and it is easy to check that $\tilde{N}$ commutes with the RPA Hamiltonian defined in equation (4.29). In fact, because $[(\Gamma_\nu + \Gamma_\nu^\dagger), (\Gamma_{\nu'} + \Gamma_{\nu'}^\dagger)] = 0$, one can show that $[\mathrm{H}_\mathrm{p}', \tilde{N}] = 0$. Furthermore, because $2U_\nu\Delta_\nu = \Delta/E_\nu$ and the quasiparticle energies satisfy the gap equation (4.36) one can demonstrate that $[\mathrm{H}_\mathrm{MF} + \mathrm{H}_\mathrm{p}'', \tilde{N}] = 0$ (see Appendix I). There are two conclusions to be drawn from these results. One is that particle number conservation is restored, by taking into account the fluctuations of the pairing mean field around the static deformation α_0, in the RPA (in particular those associated with H_p''). The other is that the operator $(\hat{N} - N_0)$ is the creation operator of the zero-frequency mode of the RPA equation of motion (4.30).

In fact, the one-phonon state associated with the zero-frequency mode is

$$|1''\rangle = \Gamma_1^\dagger|0''\rangle \sim \Lambda_1'' \sum_{\nu>0} \frac{1}{2E_\nu}(\Gamma_\nu^\dagger + \Gamma_\nu)|0''\rangle \tag{4.42}$$

$$= \frac{\Lambda_1''}{2\Delta}(\hat{N} - N_0)|0''\rangle, \tag{4.43}$$

where $|0''\rangle$ is the ground state of the RPA Hamiltonian. The first line in the above equation is obtained from (4.38) by putting $W_1'' = 0$, and the second line is from equation (4.41). In equation (4.42) $\Gamma_1^\dagger$ is a boson creation operator and should be finite. On the other hand the normalization constant $\Lambda_1'' \to \infty$ for the zero energy mode. This is possible only if in (4.43) $(\tilde{N} - N_0) \to 0$ which is another demonstration that particle number conservation is restored in the RPA approximation. This, together with equation (4.45), are the basic equations which testify to the fact that gauge symmetry is being restored.

Because a finite (rigid) rotation in gauge space can be generated by a series of infinitesimal operations of the type defined in equation (4.12), i.e.

$$\mathcal{G}(\delta\phi) \approx 1 - \mathrm{i}\frac{\hat{N}}{2}\delta\phi\,, \tag{4.44}$$

the state $|1''\rangle$ in equation (4.43) is obtained by a gauge rotation of the ground state.

The zero-point amplitude associated with this state, proportional to the quantity Λ_1'', diverges (see equation (4.39)) but nonetheless defines a finite inertia for pairing rotations (see (I.34)). By a proper inclusion of these fluctuations (of the orientation angle in gauge space, see also discussion at the end of Appendix I) one can restore gauge invariance to the $|\mathrm{BCS}\rangle_{\mathcal{K}'}$ state. In fact the states,

$$\begin{aligned}
|N\rangle &\sim \int \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}\frac{N}{2}\phi} |\mathrm{BCS}\rangle_{\mathcal{K}'} \\
&= \left(\prod_{\nu>0} U_\nu\right) \int \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}\frac{N}{2}\phi} \\
&\qquad \times \left(1 + \mathrm{e}^{-\mathrm{i}\phi} \sum_{\nu>0} c(\nu) a_\nu^\dagger a_{\bar{\nu}}^\dagger + \mathrm{e}^{-2\mathrm{i}\phi} \left(\sum_{\nu>0} c(\nu) a_\nu^\dagger a_{\bar{\nu}}^\dagger\right)^2 + \cdots \right) |0\rangle \\
&\sim \left(\sum_{\nu>0} c(\nu) a_\nu^\dagger a_{\bar{\nu}}^\dagger\right)^{\frac{N}{2}} |0\rangle\,,
\end{aligned} \tag{4.45}$$

where

$$c(\nu) = \frac{V_\nu}{U_\nu}, \tag{4.46}$$

are states with fixed number N of particles.* They are the members of a pairing rotational band (rotations in gauge space) (Bes and Broglia (1966), see also Belyaev (1972)). Examples of such a rotational band are provided by the ground state of even–even nuclei with many particles outside the closed shell (see Section 4.2.5 and Fig. 4.2). The operation carried out in equation (4.45) is number projection. It can be viewed as a change of representation between the conjugate variables N and ϕ, from the ϕ-representation to the N-representation (see Anderson (1964)).

* See Section 6.6 for a discussion of alternative techniques of projection devised to restore particle number conservation.

The BCS wavefunction $|\mathrm{BCS}\rangle_{\mathcal{K}'}$ has a definite orientation in gauge space. The RPA ground state $|0''\rangle$ in equation (4.42) and the zero-frequency mode built on it has a uniform distribution in ϕ-space, corresponding to the RPA representation of the number projected states $|N\rangle$ given in equation (4.45).

The inertia associated with pairing rotational bands is obtained by recognizing that the normalization quantity Λ''_n is also the particle-vibration coupling strength of the nth mode. Thus

$$\frac{D''_1}{\hbar^2} = 4 \sum_{\nu>0} \frac{U_\nu^2 V_\nu^2}{E_\nu}. \tag{4.47}$$

This result is derived and discussed in Appendix I. It coincides with the cranking model moment of inertia (Ring and Schuck (1980), equation (3.91)).

$$\frac{\mathcal{J}}{\hbar^2} = 2 \sum_{\nu>0} \frac{\left|\langle \nu\bar{\nu}|\hat{N}|\mathrm{BCS}\rangle\right|^2}{2E_\nu}. \tag{4.48}$$

We shall see that, although this moment of inertia is finite, the associated rotational energies are much smaller than typical quasiparticle energies, as expected for a collective mode.

4.2.4 Two-particle transfer reaction

The basic feature characterizing a family of states as belonging to a rotational (or vibrational) band is the fact that there exists an operator $\hat{O}$ whose matrix elements between members of the band, aside from displaying very simple relations, are conspicuosly enhanced with respect to the value of the same operator between pure particle states. Consequently, the (external) field associated with the operator $\hat{O}$ constitutes the specific probe to excite the band. In particular, in the case of rotations in normal space of quadrupole-deformed nuclei, it is the E2-operator which displays large matrix elements, while Coulomb excitation is the specific probe of dynamic and static nuclear deformations, and of the associated collective bands.

The specific probes of the pairing modes are two-nucleon transfer reactions (see e.g. Lane (1964), Mottelson (1977), Broglia (1985c), Broglia *et al.* (1985c)) and references therein). In fact, the existence of a large static pair deformation (pairing gap) manifests itself very directly in the pattern of two-particle transfer intensities, in keeping with the fact that

$$\langle \mathrm{BCS}|P^\dagger|\mathrm{BCS}\rangle = \frac{\Delta}{G}. \tag{4.49}$$

Consequently, the two-particle cross-section between members of the pairing rotational band is

$$\sigma_{\mathrm{rot}} \sim \left(\frac{\Delta}{G}\right)^2 \sim \frac{A}{4}, \tag{4.50}$$

where use was made of the empirical values $\Delta \approx 12\,\mathrm{MeV}/\sqrt{A}$ and $G \approx 25\,\mathrm{MeV}/A$ (Bohr and Mottelson (1975)). The two-particle transfer cross-section associated with a typical two-quasiparticle state is

$$\sigma_{2\mathrm{qp}} \sim \langle \nu\bar{\nu}|P^{\dagger}|\mathrm{BCS}\rangle^2 = U_\nu^4 \approx 1\ . \tag{4.51}$$

From the ratio

$$\bar{R} = \frac{\sigma_{\mathrm{rot}}}{\sigma_{2\mathrm{qp}}} \sim \frac{A}{4} \tag{4.52}$$

one expects that, in superfluid nuclei, a large fraction of the cross-section associated with the transfer of two nucleons in time-reversal states connects members of the same pairing rotational band.

Note that the total two-particle transfer cross-section, once Q-value effects are eliminated, is the same for a system of nucleons which move independently of each other in the mean field as it is for the same system of nucleons interacting via a pairing force; i.e. the same before and after the pairing interaction is switched on. The basic difference introduced by the presence of U and V factors in the corresponding cross-sections is that of concentrating a large fraction of the original strength on the ground-state transition (see Broglia *et al.* (1972a)).

4.2.5 A schematic model

Let us consider particles moving in a single j-shell with pair degeneracy $\Omega = (2j+1)/2$. The BCS occupation numbers can be written directly as (Appendix H, see also Section 3.7)

$$V = \left(\frac{N}{2\Omega}\right)^{1/2}, \quad U = \left(1 - \frac{N}{2\Omega}\right)^{1/2}, \tag{4.53}$$

and lead to a gap and a Fermi energy

$$\Delta = \frac{G}{2}\sqrt{N(2\Omega - N)} \tag{4.54}$$

and

$$\lambda = -\frac{G}{2}(\Omega - N), \tag{4.55}$$

respectively. The quasiparticle energy is given by

$$E = \frac{G\Omega}{2}. \tag{4.56}$$

Typical superfluid nuclei, i.e. nuclei with many particles outside the closed shell and thus with a large number of 0^+ pairs (Cooper pairs) in the ground state, display a large pairing gap. Within the present simplified model, this situation corresponds to $N \approx \Omega$. In fact, the pairing gap given in equation (4.54) acquires its largest value $\Delta = G\Omega/2$ for $N = \Omega$. In what follows we shall have this situation in mind in discussing the properties of the excitation spectrum. The ground-state energy (see Appendices H, I) is

$$\begin{aligned} E_0 = U + \lambda N &= \lambda N + \frac{G}{4}N^2 \\ &= \lambda N + \frac{\hbar^2}{2\mathcal{J}}N^2, \end{aligned} \tag{4.57}$$

where $n = N/2$ is the number of pairs, and where the moment of inertia is determined by the relation

$$\frac{2\mathcal{J}}{\hbar^2} = \frac{4}{G}. \tag{4.58}$$

This result coincides with that obtained from equation (4.48) making use of the occupation numbers and quasiparticle energy provided by equations (4.53) and (4.56) respectively.

From the ratio,

$$\frac{\hbar^2/2\mathcal{J}}{2E} \approx \frac{1}{\Omega} \ll 1. \tag{4.59}$$

This indicates that the rotational excitations have an energy which is much smaller than that associated with the quasiparticle energies, the ratio approaching zero, as $N = \Omega \to \infty$. In this connection, it is illuminating to quote part of the discussion in Weinberg (1996) on spontaneously broken global symmetries, where he uses a chair as an example of a macroscopic system:

> spontaneous symmetry breaking actually occurs only for idealized systems that are infinitely large. The appearance of broken symmetry for a chair arises because it has a macroscopic moment of inertia $\mathcal{J}$, so that its ground state is part of a tower of rotationally excited states whose energies are separated by only tiny amounts, of the order of $\hbar^2/\mathcal{J}$. This gives the state vector of a chair an exquisite sensitivity to external perturbations; even very weak external fields will shift the energy much more than the energy difference of these rotational levels. In consequence, any rotationally asymmetric external field will cause the ground state or any other state of the chair with definite angular momentum rapidly to develop components with other angular momentum quantum numbers. The states of the

> chair that are relatively stable with respect to small external perturbations are not those with definite angular momentum quantum numbers, but rather those with a definite orientation, in which the rotational symmetry of the underlying theory is broken.

Weinberg's arguments are true for an atomic nucleus only in the limit $N \to \infty$ when deformation and rotation are rigorously defined. Nevertheless when one observes a (pairing) rotational spectrum one can talk about a privileged direction in gauge space, which can be clamped down in a collision between two superfluid nuclei (see Broglia and Winther (1991)) resulting in the transfer of a Cooper pair (this is a Josephson-like phenomenon, see Anderson (1972), Anderson (1964) p. 134; see also Appendix L).

Broken symmetries in relativistic theories and in many-body systems imply an Anderson–Goldstone–Nambu (AGN) boson (zero-mass particle or phonon branch respectively). The analogous property in the case of the RPA description of pairing in atomic nuclei is the $\hbar\omega_1'' = 0$ solution (see Section 4.2 as well as equation (4.57)) and the associated pairing rotational band built out of the ground state of systems with N, $N \pm 2$, $N \pm 4, \ldots,$ particles. As shall be seen in the next subsection, there exists strong experimental evidence which testifies to the validity of this picture.

4.2.6 Comparison with experiment

In Fig. 4.2 we summarize the experimental information concerning the ground-state energies of one of the longest sequences of isotopes of nuclei with many nucleons outside a closed shell, that associated with the Sn-isotopes ($^{A}_{50}$Sn) (Broglia *et al*. (1973), Broglia (1985c), Bes and Broglia (1977)). The data can be rather accurately fitted, after a linear term has been removed, with the parabola corresponding to an energy parameter $\hbar^2/2\mathcal{J} = 0.1$ MeV, in overall agreement with the simple estimate provided by the prefactor of N^2 in equation (4.57) ($G/4 \approx 28/4A$ MeV ≈ 0.07 MeV, $A \approx 100$, see equation (2.27) and Appendix H). Also displayed in Fig. 4.2 is systematic information on the transfer of two neutrons in time-reversal states (single Cooper pair transfer). The average value of $\bar{R} = 24.4$ is in overall agreement with the simple estimate provided by equation (4.52) ($\bar{R} = 25$ for $A \approx 100$, see also Appendix H).

The diagonalization of the total Hamiltonian $H = H_{\mathrm{MF}} + H_{\mathrm{p}}' + H_{\mathrm{p}}''$ in the RPA has still a root at $\omega_1 = 0$ (corresponding to the $\omega_1'' = 0$ root of the Hamiltonian $H_{\mathrm{MF}} + H_{\mathrm{p}}''$), orthogonal to all the other two-quasiparticle like states (pairing vibrations), and which are somewhat modified by the Coriolis coupling associated with the rotation of the system in gauge space as a whole (see Appendix J). Pairing vibrations of superfluid nuclei correspond to the odd solution discussed by Anderson in his RPA treatment of superconductivity

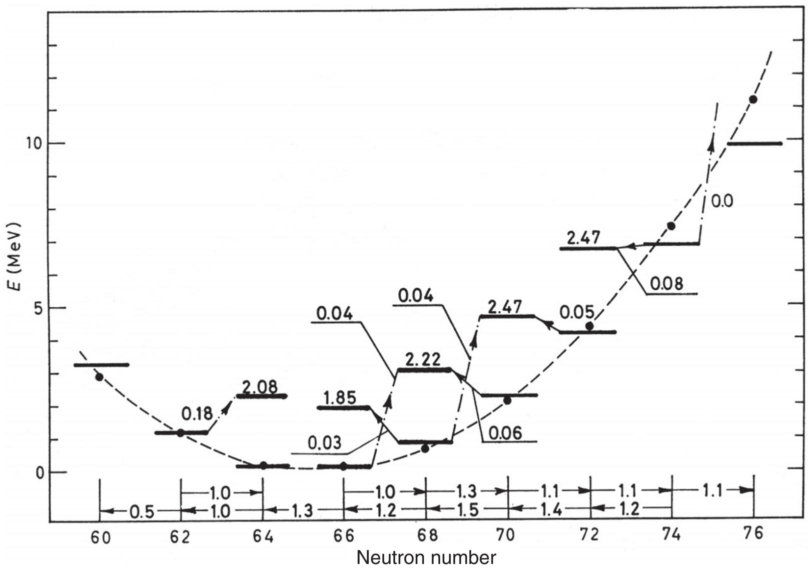

Figure 4.2. Experimental energies of the $J^\pi = 0^+$ states of the even Sn isotopes excited in two-particle transfer reactions ((t, p) and (p, t)). The heavy drawn lines represent the values of the expression $E = -B(\mathrm{Sn}) + E_{\mathrm{exc}} + 8.58N + 45.3$ (MeV), where the binding energies $B(A)$ (in MeV) are taken from Wapstra and Gove (1971). The dashed line represents the parabola $0.10(N - 65.4)^2$ Also displayed is the excited pairing rotational band associated with the pairing vibrational mode. In all cases where more than one $J^\pi = 0^+$ state has been excited below 3 MeV in two-neutron transfer processes, the energy $\sum_i \sigma(0_i)E(0_i^+)/\sum_i \sigma(0_i^+)$ of the centroid is quoted, as well as the corresponding cross-section $\sum_i \sigma(0_i^+)$. The quantity $\sigma(0_i^+)$ is the relative cross-sections with respect to the ground-state cross-sections. The numbers along the abscissa are the ground-state (p, t) and (t, p) cross-sections normalized to the $^{116}\mathrm{Sn} \leftrightarrow {}^{118}\mathrm{Sn(gs)}$ cross-section. The (t, p) and (p, t) data utilizing in constructing this figure were taken from Bjerregaard *et al.* (1968), Bjerregaard *et al.* (1969), Flynn *et al.* (1970), Flemming *et al.* (1970).

(Anderson (1958)), lying at the top of the pairing gap. These solutions are, as a rule, both in the 0D and in the 3D systems, almost pure two-quasiparticle states (see Broglia *et al.* (1977)). This is the reason why we should not refer to them further. As will be discussed in Chapter 5 pairing vibrations play an important role in closed shell (normal) nuclei.

4.3 Infinite 3D neutral superconductors ($\xi \ll L$)

In this section we follow Anderson (1958) and study the correlated two-quasiparticle excitations associated with the Hamiltonian

$$\mathcal{H} = \sum_{k,\sigma} \varepsilon_k a^\dagger_{k\sigma} a_{k\sigma} + \frac{1}{2} \sum_{k \neq k', q} \sum_{\sigma,\sigma'} V(\vec{k}, \vec{k}') a^\dagger_{k',\sigma'} a^\dagger_{-k'+q,\sigma} a_{-k+q,\sigma} a_{k,\sigma}. \quad (4.60)$$

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Here the (Galilean invariant) Coulomb and induced interactions have been lumped together in $V(k, k')$, while σ denotes the projection of the electron spin. Making use of the RPA equations of motion for the operators

$$\hat{b}_k^Q = a_{-k-Q\downarrow} a_{k\uparrow}, \tag{4.61}$$

$$\hat{\rho}_k^Q = a^\dagger_{k+Q\uparrow} a_{k\uparrow}, \tag{4.62}$$

and for the corresponding Hermitian, and time-reversal conjugate operators, Anderson (1958) obtains for the even solution of the Hamiltonian $\mathcal{H}$,

$$\begin{vmatrix} 1 - 2V_{\rm D} f & l \\ 2V_{\rm D} h & 1 - g \end{vmatrix} = 0\,, \tag{4.63}$$

where

$$f = \sum_k \frac{\omega_{kQ} n_{kQ}}{(\nu_k^Q)^2 - \nu^2}, \tag{4.64}$$

$$g = \sum_k \frac{(-V)\nu_k^Q \cos^2\left[\frac{1}{2}(\theta_k - \theta_{k+Q})\right]}{(\nu_k^Q)^2 - \nu^2}, \tag{4.65}$$

$$h = \sum_k \frac{(-V)(b_k + b_{k+Q})}{(\nu_k^Q)^2 - \nu^2}, \tag{4.66}$$

$$l = \sum_k \frac{\omega_{kQ}^2 (b_k + b_{k+Q})}{\nu^2 - (\nu_k^Q)^2}, \tag{4.67}$$

and where $V_{\rm D}$ indicates the ‘direct’, unscreened interaction. In the above equations one has used the definitions

$$b_k = \langle a_{-k\downarrow} a_{k\uparrow} \rangle = b_k^* = U_k V_k, \tag{4.68}$$

$$\omega_{kQ} = \varepsilon_{k+Q} - \varepsilon_k, \tag{4.69}$$

$$n_{kQ} = n_{k+Q} - n_k, \tag{4.70}$$

$$\nu_k^Q = E_k + E_{k+Q}, \quad \left(E_k = (\varepsilon_k^2 + \Delta_k^2)^{1/2}\right), \tag{4.71}$$

$$\cos\theta_k = U_k^2 - V_k^2. \tag{4.72}$$

The collective modes associated with the secular equation (4.63) have entirely different behaviour, depending on whether we consider the charged or neutral case. In the charged case, $V_{\rm D}$ is singular and large, and f determines the frequencies. In the neutral case the frequencies are mostly determined by g.

4.3.1 Neutral superconductor

Following Anderson (1958) we assume $V_{\rm D} = V$ as $Q \to 0$. Since $f \sim Q^2$, and thus $1 - 2V_{\rm D} f \approx 1$, equation (4.63) can be approximated as

$$1 - g = 2Vhl. \tag{4.73}$$

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Because also $l \sim Q^2$, in the limit $Q \to 0$ the dispersion relation reads (see (4.35))

$$\sum_k \frac{(-V)2E_k}{(2E_k)^2 - \nu^2} = 1\,, \tag{4.74}$$

where $2E_k \approx \nu_k^Q$. In keeping with the BCS gap equation, equation (4.74) admits a solution with $\nu = 0$.

In the following we use the approximation $\omega_{kQ}^2 \approx \frac{1}{3}\left(\frac{k_F Q}{m}\right)^2$. The factor $1/3$ comes from the average $\langle\cos^2\gamma\rangle$ where γ is the angle between $\vec{k}$ and $\vec{Q}$. Expanding h and l to the lowest non-vanishing order in Q^2 and ν^2, and g to first order,

$$h \approx \sum_k \frac{(-V)2U_kV_k}{(2E_k)^2} = 2\Delta\sum_k \frac{(-V)}{(2E_k)^3}, \tag{4.75}$$

$$l \approx -\sum_k \frac{\omega_{kQ}^2 2U_kV_k}{(2E_k)^2} = -\frac{1}{3}k_F^2\frac{Q^2}{m^2}\sum_k \frac{2U_kV_k}{(2E_k)^2}, \tag{4.76}$$

$$g \approx 1 + \nu^2\sum_k \frac{(-V)}{(2E_k)^3} - \sum_k \frac{(-V)\omega_{kQ}^2}{(2E_k)^3}, \tag{4.77}$$

the dispersion relation equation (4.73) becomes

$$\begin{aligned} 1 - &\left(1 + \nu^2\sum_k \frac{(-V)}{(2E_k)^3} - \sum_k \frac{(-V)\omega_{kQ}^2}{(2E_k)^3}\right) \\ &= 2V\left(2\Delta\sum_k \frac{(-V)}{(2E_k)^3}\right)\left(-\sum_k \frac{\omega_{kQ}^2 2U_kV_k}{(2E_k)^2}\right), \end{aligned} \tag{4.78}$$

which can also be written as

$$\nu = \frac{1}{\sqrt{3}}v_F Q\left(1 + 4V\Delta\sum_k \frac{2U_kV_k}{(2E_k)^2}\right)^{1/2}, \tag{4.79}$$

where the assumption has been made that $\Delta_k = \Delta$. The AGN-phonon velocity $v_F/\sqrt{3}$ seems to be a kinematical 'ideal gas' effect, which has also been derived in a different way by Bogoliubov *et al.* (1958). It is curious that the term which modifies the 'ideal gas' velocity in equation (4.79) is related to the pairing moment of inertia (see Appendix H, equation (H.17) and Appendix I, equation (I.24)) and (4.47),

$$4V\Delta\sum_k \frac{2U_kV_k}{(2E_k)^2} = V\frac{\mathcal{I}}{\hbar^2} = V\frac{\partial N}{\partial\lambda}. \tag{4.80}$$

An example of an AGN boson in a neutral system is provided by the fourth sound in superfluid ^{3}He, which corresponds to the oscillatory motion of the

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superfluid phase in a confined geometry (superleak) where the normal fluid is clamped. The corresponding sound velocity $C_4^2 = C_1^2 \bar{\rho}_s/\rho$, where $\bar{\rho}_s$ is the superfluid density and ρ the total density of the system, is proportional to the first sound velocity (Vollhardt and Wölfle (1990)), $C_1^2 = \frac{1}{3}v_F^2(1 + F_0^s)(1 + \frac{1}{3}F_1^s)$, where F_l^s are the spin symmetric $l = 0$ and $l = 1$ Landau parameters (see Section 10.5.1).

Let us now return to the main subject discussed above, namely the relation between the solutions of the dispersion relations given in equations (4.35) and (4.74) (see also equations (J.27) and (4.63))), solutions which look suggestively similar (see also equations (4.57) and (4.79)). It has been argued that in relativistic theory, as well as in 3D many-body systems, the $Q \to 0$ is a proper solution of the problem (zero-mass particle and phonon branch respectively), while in a 0D system like the nucleus, it is a spurious solution to be eliminated in terms of a pairing rotational band whose inertia is that of the $\omega_1 = 0$ root or spurious state (see equation (4.47)). To this line of reasoning one could argue that, had we used a more powerful technique than RPA to diagonalize the Hamiltonian $H = H_0 + H_p' + H_p''$, we would have obtained the modified two-quasiparticle-like states (pairing vibrations), and the pairing rotational band, without further ado.

For a discussion of these subjects which goes beyond the RPA, we refer the reader to Bes and Kurchan (1990).