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A Gorenstein Ring with Larger Dilworth Number than Sperner Number

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Abstract. A counterexample is given to a conjecture of Ikeda by finding a class of Gorenstein rings of embedding dimension 3 with larger Dilworth number than Sperner number. The Dilworth number of $A[Z/pZ \oplus Z/pZ]$ is computed when A is an unramified principal Artin local ring.

Let A be a commutative ring with identity. Let $\mu(I)$ denote the minimal number of elements required to generate the ideal I. If $\mu(I) \leq n$ then we say that I is n-generated, and if every ideal of A is n-generated then we say that A has the n-generator property. The Dilworth number $d(A)$ of an Artin local ring (A, m) was introduced by Watanabe in [11] and is defined as $\max\{\mu(I) \mid I \text{ an ideal of } A\}$. The Sperner number of A is $\mathrm{sp}(A) = \max\{\mu(m^i) \mid i \geq 0\}$. If G is an abelian group then the group ring associated to A and G, denoted $A[G]$, is the ring of elements of the form $\sum_{g\in G} a_g x^g$, where $\{a_g \mid g \in G\}$ is a family of elements of A which are almost all zero. We refer to [5] for elementary properties of semigroup and group rings.

Several recent papers have studied semigroup and group rings with n-generated ideals. Semigroup rings with the 2-generator property are of special interest because of their relationship to the problem of when finitely generated torsion-free modules are isomorphic to direct sums of ideals (For example see [9], [10] and the references listed there). An important tool in the study of such rings is the ability to relate the number of generators of an arbitrary ideal to the number of generators of powers of the maximal ideal. A theorem of Watanabe [6, Theorem 4.2] says that if (A, m) is an Artin local ring of embedding dimension at most two, *i.e.*, $\mu(m) \leq 2$ then $d(A) = \mathrm{sp}(A)$. This theorem makes it possible to compute the Dilworth and Sperner number of $A[G]$ where G is a finite cyclic p-group and A is a principal Artin local ring [12, Proposition 2.5] and [3, Theorem 1]). When G is not cyclic the Dilworth number of $A[G]$ has only been determined in the case where A is a field. The Dilworth and Sperner numbers of $A[Z/pZ \oplus Z/pZ]$ where (A, m) is a principal Artin local ring and $p \in m \setminus m^2$ were computed for the cases $p = 2$ and $p = 3$ in [1] and [2]. In Theorem 1(a), (b) we compute these numbers for general p.

In [6] examples are given to show that Watanabe's theorem is not true for $\mu(m) \geq 4$ even if A is Gorenstein. Ikeda also makes the following conjecture:

Conjecture Let (A, m) be a Gorenstein Artin local ring of embedding dimension at most three. Then $d(A) = \mathrm{sp}(A)$.

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In Theorem 1(c), we give a class of such rings R for which $d(R) > \mathrm{sp}(R)$. One example of such a ring is $R = (Z/p^iZ)[Z/pZ \oplus Z/pZ]$. Corollary 9 of [4] states that if G is a finite group (not necessarily abelian) and K is a commutative ring then K is Gorenstein if and only if $K[G]$ is Gorenstein. For abelian groups, Lantz [7, Theorem, p. 196] generalizes this result to groups of finite torsion-free rank. Thus R provides a counterexample to Ikeda's conjecture.

In Theorem 1 we compute the Sperner and Dilworth numbers of the group rings $A[Z/pZ \oplus Z/pZ]$ where (A, m) is a principal Artin local ring and $p \in m \setminus m^2$. When $m^2 \neq 0$ and $m^p = 0$ we will show the Sperner number is strictly less than the Dilworth number.

Theorem 1 *Let (A, m) be a principal Artin local ring, $p \in m \setminus m^2$ be a rational prime, $G = Z/pZ \oplus Z/pZ$ and $R = A[G]$. Then*

(a) $\mathrm{sp}(R) = \begin{cases} p+2 & \text{if } m^p \neq 0 \\ p+1 & \text{if } m \neq 0 \text{ and } m^p = 0. \\ p & \text{if } m = 0 \end{cases}$

(b) $d(R) = \begin{cases} p+2 & \text{if } m^2 \neq 0 \\ p+1 & \text{if } m \neq 0 \text{ and } m^2 = 0. \\ p & \text{if } m = 0 \end{cases}$

(c) *If $p > 2$, $m^2 \neq 0$ and $m^p = 0$ then $p + 2 = d(R) > \mathrm{sp}(R) = p + 1$.*

If the maximal ideal m of a local ring A is generated by two elements then the k-th power of the maximal ideal has $k + 1$ natural generators. The next lemma, due to Hassani and Kabbaj, shows that if n is the first power for which one of these natural generators is not required to generate m^n then we get a bound on the number of generators of I in terms of the number of generators of $I + m^{n-1}$. In particular, when I is a power of m, we get the Sperner number of R is less than or equal to n.

Lemma 2 **[2, Lemma 4]** *Let (A, m) be a local ring such that m^n is n-generated, where n is a positive integer. Then for each ideal I of A, $\mu(I) \leq \mu(I + m^{n-1})$.*

When A is a local ring with embedding dimension 2, Watanabe's theorem and Lemma 2 combine to give a bound on the Dilworth number in terms of a specific power of the maximal ideal.

Lemma 3 **[3, Lemma 3]** *Let (A, m) be an Artin local ring with $\mu(m) \leq 2$. Then $d(A) \leq n$ if and only if m^n is n-generated.*

Proof If m^n is n-generated, $\mu(I) \leq \mu(I + m^{n-1})$ for each ideal I of A by Lemma 2. Thus $\mathrm{sp}(A) \leq n$. Since $\mu(m) \leq 2$, $d(A) = \mathrm{sp}(A)$ by Watanabe's Theorem [6, Theorem 4.2]. Thus $d(A) \leq n$ and the converse is obvious. ■

The proof of Theorem 1 uses the isomorphism $A[Z/pZ \oplus Z/pZ] \cong A[x, y]/(1 - x^p, 1 - y^p)$ to represent elements of the group ring. In the proof it is shown that $p(1 - x)$ is a unit times $(1 - x)^p$. Likewise for $p(1 - y)$. Therefore every monomial in the generators of the maximal ideal $p^a(1 - x)^b(1 - y)^c$ of $A[Z/pZ \oplus Z/pZ]$ can be rewritten in one of three ways: (1) $a = 0$, $b > 0$ and $c < p$, (2) $a = b = 0$ or (3) $b = c = 0$. We define the *value* of $f = p^a(1-x)^b(1-y)^c$, denoted $v(f)$, to be $(p-1)a+b+c$ and define an order on monomials by first comparing values, then $b + c$ and finally powers of $(1 - x)$. Though, in general, order is not preserved the next lemma shows that multiplication by monomials preserves order in certain cases.

Lemma 4 *Let $f_1 = p^{a_1}(1-x)^{b_1}(1-y)^{c_1}$, $f_2 = p^{a_2}(1-x)^{b_2}(1-y)^{c_2}$ be monomials in $A[Z/pZ \oplus Z/pZ]$ with $f_1 < f_2$.*

(a) If $a_1 = 0, c_1 < p$ and $g = (1-x)^v$ then $f_1 g < f_2 g$.
(b) If $a_1 = b_1 = 0$ and $g = (1-y)^w$ then $f_1 g < f_2 g$.
(c) If $b_1 = c_1 = 0$ and $g = p^u$ then $f_1 g < f_2 g$.
(d) If $b_1 = c_1 = 0$, $a_2 = 0$, $b_2 \geq 1$ and $g = (1-y)^w$ then $f_1 g < f_2 g$.

Proof If $\nu(f_1) < \nu(f_2)$ then $\nu(f_1 g) < \nu(f_2 g)$ and we are done. So assume that $\nu(f_1) = \nu(f_2)$.

(a) Since $\nu(f_1) = \nu(f_2)$ and $f_1 < f_2$ we have $a_2 = 0, b_1 + c_1 = b_2 + c_2$ and $b_2 > b_1$. Then $f_1 g = (1-x)^{b_1+v}(1-y)^{c_1} < (1-x)^{b_2+v}(1-y)^{c_2} = f_2 g$. For (b) we have $a_2 = 0, c_1 = b_2 + c_2$ and $b_2 > 0$. Thus $f_1 g = (1-y)^{c_1+w} < (1-x)^{b_2}(1-y)^{c_2+w} = f_2 g$. In case (c), $b_1 = c_1 = 0$ and $f_1 < f_2$ implies $a_2 = 0$. Therefore $f_1 g = p^{a_1+u} < p^u(1-x)^{b_2}(1-y)^{c_2} = f_2 g$. For (d) we have $f_1 g = p^{a_1}(1-y)^w = (1-y)^{(p-1)a_1+w} < (1-x)^{b_2}(1-y)^{c_2+w} = f_2 g$. ■

Since $A[Z/pZ \oplus Z/pZ]$ is an Artin local ring with maximal ideal $M = (p, 1-x, 1-y)$, every element $f \in A[Z/pZ \oplus Z/pZ]$ can be written in the form $f = a_0 + a_1 v_1 + \cdots + a_n v_n$ where $v_i = p^a(1-x)^b(1-y)^c$, $v_1 < v_2 < \cdots < v_n$, v_n is the largest non-zero monomial in $A[Z/pZ \oplus Z/pZ]$ and each a_i is zero or a unit in A. If $a_k v_k$ is the first non-zero term of f we say $a_k v_k$ is the *leading monomial* of f, $\nu(f) = \nu(v_k)$ and $\operatorname{depth}(f) = n + 1 - k$. If $B = \{f_1, \ldots, f_r\}$ then $\operatorname{depth}(B) = \sum \operatorname{depth}(f_i)$.

Proof of Theorem 1 Let $M = (p, M_G)$ be the maximal ideal of R where $M_G = (1-x, 1-y)$. Then

$$(1 - x + x)^p = \sum_{k=0}^{p} \binom{p}{k} (1-x)^k x^{p-k}.$$

Since $x^p = 1$,

$$\begin{aligned} (1-x)^p &= -\sum_{k=1}^{p-1} \binom{p}{k} (1-x)^k x^{p-k} \\ &= -p(1-x)(1+v) \end{aligned}$$

where $v \in (1-x) \subseteq M$. Since $1 + v$ is a unit in R, $p(1-x) \in \big((1-x)^p\big)$ and hence $M^k = (p^k) + M_G^k$ for all $k \geq 1$. Furthermore we have

$$M^p = (p^p) + pM_G + \big((1-x)^{p-1}(1-y), \ldots, (1-x)(1-y)^{p-1}\big)$$
$$M^{p+1} = (p^{p+1}) + pM_G^2 + \big((1-x)^{p-1}(1-y)^2, \ldots, (1-x)^2(1-y)^{p-1}\big)$$
$$M^{p+2} = (p^{p+2}) + pM_G^3 + \big((1-x)^{p-1}(1-y)^3, \ldots, (1-x)^3(1-y)^{p-1}\big)$$

which means $\mu(M^{p+2}) \leq p + 2$ and, if $m^p = 0$, then $\mu(M^{p+1}) \leq p + 1$.

(a) By Lemma 2, $\mu(M^k) \le \mu(M^k + M^{p+1})$ for all $k \ge 1$. Therefore

$$\mathrm{sp}(R) = \max\{\mu(M), \dots, \mu(M^{p+1})\} \le p + 2.$$

If $p^p \ne 0$ then we claim that $\{p^p, (1-x)^p, (1-x)^{p-1}(1-y), \dots, (1-y)^p\}$ is a minimal generating set for M^p. Indeed, it is clear that $p^p, (1-x)^p, (1-y)^p$ cannot be omitted as generators. If $(1-x)^k(1-y)^{p-k}$ is not required as a generator of M^p for $1 \le k \le p-1$ we pass to the ring $R/(p) \cong \big(A/(p)\big)[X,Y]/(X^p, Y^p)$ where X, Y are indeterminates to get that $\bar{X}^k\bar{Y}^{p-k}$ is not required as a generator of $(\bar{X}, \bar{Y})^p$, a contradiction. Therefore $\mathrm{sp}(R) = p+2$ if $m^p \ne 0$.

If $m \ne 0$ and $m^p = 0$ then $p^p = 0$. Thus $\mu(M^{p+1}) \le p+1$ and $\mathrm{sp}(R) = \max\{\mu(M), \dots, \mu(M^p)\} \le p+1$. As before $\{(1-x)^p, (1-x)^{p-1}(1-y), \dots, (1-y)^p\}$ is a minimal generating set for M^p. Therefore $\mathrm{sp}(R) = p+1$.

Finally, if A is a field of characteristic p, then $\mathrm{sp}(R) = d(R) = p$ by [8, Theorem 1.2].

(b) Let I be a proper ideal in R and let $\{f_1, \dots, f_r\}$ be a set of generators of I of minimal depth. Let $E = \{(a,b,c) \mid p^a(1-x)^b(1-y)^c$ is a leading monomial of f_i for some $i\}$. Let $E_+ = \{(0,b,c) \mid (0,b,c) \in E$ and $c < p\}$, let $E_y = \{(0,0,c) \in E\}$ and $E_p = \{(a,0,0) \in E\}$. Let $E_+ = \{(0,b_1,c_1), \dots, (0,b_m,c_m)\}$ where $0 \le c_1 \le c_2 \le \cdots \le c_m \le p-1$. If $c_i = c_j$ and say $b_i \ge b_j$ for some $i \ne j$ then, if f_i and f_j are the generators whose leading monomials are $(1-x)^{b_i}(1-y)^{c_i}$ and $(1-x)^{b_j}(1-y)^{c_j}$ respectively, replace f_i with $f_i' = f_i - (1-x)^{b_i-b_j} f_j$. Since $f_i' > f_i$, $\{f_1, \dots, f_i', \dots, f_r\}$ is a generating set with smaller depth, a contradiction. Therefore $0 \le c_1 < c_2 < \cdots < c_m \le p-1$. Since $c_1, \dots, c_m$ are distinct, E_+ has at most p elements. If E_y has more that one element, say $(0,0,d_1)$ and $(0,0,d_2)$ with $d_1 \le d_2$ then we multiply the generator with the smaller leading monomial by $(1-y)^{d_2-d_1}$ and subtract to get a generating set of lower depth. We similarly have that E_p has at most one element. Therefore $d(R) \le p+2$.

Next we assume that $m^2 \ne 0$ and show $d(R) \ge p+2$. For this consider the ideal $I = (p^2) + M_G^p$. We claim the set

$$\{p^2, p(1-x), p(1-y), (1-x)^{p-1}(1-y), \dots, (1-x)(1-y)^{p-1}\}$$

is a minimal set of generators for I. If p^2 can be expressed in terms of the other generators, we apply the augmentation map to get $p^2 = 0$. If, say $p(1-x)$, can be expressed in terms of the other generators, we apply the map sending $y \to 1$ to get $p \in (p^2)$, a contradiction. We use a similar argument for $p(1-y)$. If any of the other generators can be omitted we pass to $\big(A/(p)\big)[Z_p \oplus Z_p]$ to get a contradiction as before. Thus $d(R) \ge p+2$. This shows $d(R) = p+2$.

The case $m = 0$ was considered in part (a). Therefore assume $m \ne 0$, but $m^2 = 0$. From (a) we have $p+1 = \mathrm{sp}(R) \le d(R)$. If E_+ has less than p elements then $d(R) \le p+1$ and we are done. So assume the cardinality of E_+ is p. If both $E_p = \{(1,0,0)\}$ and $E_y = \{(0,0,c)\}$ are non-empty then we consider two cases, $c \ge p$ and $c < p$. If $c \ge p$ then let $f = (1-x)^b(1-y)^{p-1} + \cdots$ be the generator corresponding to $(0,b,p-1) \in E_+$. If the generator with leading term p has the form $f_p = p + a(1-y)^{p-1} + \cdots$ with $a \ne 0$ then multiply f_p by $a^{-1}(1-x)^b$ and subtract from f. Lemma 4(a) and the relation $p(1-x)^b = (1-x)^{p-1+b} > (1-x)^b(1-y)^{p-1}$ show that we get a generating set of lower depth. If $a = 0$ then we multiply f_p by $(1-y)^{c-p+1}$ and subtract from the generator $(1-y)^c + \cdots$

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to get a generating set of lower depth. If $c < p$ then there exists an element of the form $(0, b, c) \in E_+$. Now multiply the generator $(1 - y)^c + \cdots$ by $(1 - x)^b$ and subtract from the generator $(1 - x)^b(1 - y)^c + \cdots$ to get a generating set of lower depth. Therefore in both cases either E_p or E_y is empty and $d(R) \leq p + 1$.

(c) Finally, assume $p > 2, m^2 \neq 0$ and $m^p = 0$. From parts (a) and (b) we have $p + 2 = d(R) > \mathrm{sp}(R) = p + 1$. ■

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