SPARSE FUSION SYSTEMS

ADAM GLESSER

*Department of Mathematics, California State University Fullerton,
Fullerton, CA 92831, USA* (algesser@fullerton.edu)

Abstract   We define sparse saturated fusion systems and show that, for odd primes, sparse systems are constrained. This simplifies the proof of the Glauberman–Thompson p-Nilpotency Theorem for fusion systems and a related theorem of Stellmacher. We then define a more restrictive class of saturated fusion systems, called extremely sparse systems, that are constrained for all primes.

1. Introduction

For those concerned with fusion in finite groups, a p-nilpotent group is as trivial as it gets. As such, p-nilpotency criteria are inherently interesting. Following up on the work in his dissertation, John Thompson proved, for odd primes p, a p-nilpotency criterion for finite groups that reduces the problem to a p-local question [**21**], namely, checking the fusion in two subgroups $\mathrm{N}_G(\mathrm{J}(P))$ and $\mathrm{C}_G(\mathrm{Z}(P))$. Recently, Díaz *et al.* proved a generalization of this theorem for fusion systems [**8**].

In [**14**], Kessar and Linckelmann state and prove a generalization to fusion systems of Glauberman's improved version of Thompson's p-nilpotency result: one that reduces the question of p-nilpotency in G to that of the p-nilpotency of $\mathrm{N}_G(\mathrm{Z}(\mathrm{J}(P)))$. Motivated, presumably, by Gorenstein's treatment in [**10**], this result is referred to as the Glauberman—Thompson p-Nilpotency Theorem.

In this paper we aim to shorten the proof of this last result, showing that it is a consequence of the fusion system version of Thompson's p-nilpotency criterion and Glauberman's p-nilpotency criterion for groups. This simplification follows from the following observation: minimal counter-examples to statements whose conclusion is that a fusion system is trivial tend to only have the trivial subsystem as a proper subsystem on the same p-group. We call such a fusion system *sparse*. In § 3, we prove that, for p odd, any sparse fusion system is constrained, i.e. it contains a normal centric subgroup; this implies that the system comes from a finite group and, thus, is subject to fusion results for finite groups. For $p = 2$, this result does not hold in general and we will give an example of a fusion system exhibiting this deficiency.

To demonstrate the ubiquity of sparse fusion systems, we present several further examples. In §4.2, we generalize a result of Navarro to fusion systems. This result holds for all primes and strengthens Navarro's original result. Furthermore, when applied to the situation of p-blocks, we obtain a new nilpotency criterion for blocks generalizing the classical result (see [**15**] and [**6**]) that a block with inertial index 1 and abelian defect group is nilpotent. In this section we also generalize a recent result of Weigel [**22**] on *slim p-groups*. Finally, in §5, we consider saturated fusion systems for which a proper subsystem on *any* p-subgroup is trivial. These *extremely sparse* fusion systems are always constrained (even for $p = 2$) and we give a classification of these systems along with an example of their use.

The genesis of this paper is a question posed by Radha Kessar asking for metatheorems that decide when a result from group theory will hold for fusion systems. Consider this a very tiny first step in that direction.

2. Preliminaries

We begin with a brief summary of saturated fusion systems. The concept of a (saturated) fusion system is originally due to Puig (in unpublished notes) and the approach used in this paper is the one adopted by Broto *et al.* [**2**]. For more detail, proper motivation or the proofs of any theorems given without justification, we refer the reader to [**16**].

2.1. Saturated fusion systems

Let p be a prime, let P be a finite p-group, let $\mathcal{F}$ be a category whose objects are the subgroups of P and, for $Q, R \leqslant P$, let $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ be a subset of the injective group homomorphisms from Q to R. Composition of morphisms is given as the usual composition of group homomorphisms. Denote the $\mathcal{F}$-isomorphism class of Q by $Q^{\mathcal{F}}$.

(1) A subgroup $Q \leqslant P$ is *fully $\mathcal{F}$-normalized* (respectively, *fully $\mathcal{F}$-centralized*) if $|\mathrm{N}_P(Q)| \geqslant |\mathrm{N}_P(R)|$ (respectively, $|\mathrm{C}_P(Q)| \geqslant |\mathrm{C}_P(R)|$) for all $R \in Q^{\mathcal{F}}$.

(2) The category $\mathcal{F}$ is a *saturated fusion system* on P if the following hold for all subgroups $Q, R \leqslant P$.

   (a) If $Q \leqslant R$, then the inclusion map from Q to R is a morphism in $\mathcal{F}$.

   (b) If $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, then the induced isomorphism from Q to $\phi(Q)$ and its inverse are also in $\mathcal{F}$.

   (c) $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R)$, where the former set denotes the group homomorphisms from Q to R induced by conjugation with an element of P.

   (d) (The Sylow axiom.) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

   (e) (The extension axiom.) If $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ such that $\phi(Q)$ is fully $\mathcal{F}$-normalized, then ϕ extends to a morphism in $\mathrm{Hom}_{\mathcal{F}}(\mathrm{N}_{\phi}, P)$. Here, N_{ϕ} denotes the inverse image in $\mathrm{N}_P(Q)$ of $\mathrm{Aut}_P(Q) \cap (\phi^{-1} \mathrm{Aut}_P(\phi(Q))\phi)$.

Example 2.1. Let G be a finite group with Sylow p-subgroup P. For $g \in G$, let c_g denote the automorphism of G given by conjugation by g and, for $Q, R \leqslant P$, set

$$\mathrm{Hom}_{\mathcal{F}}(Q, R) = \{c_g|_Q \mid g \in G, \ Q^g \leqslant R\}.$$

It is a straightforward application of the Sylow theorems to show that this gives a saturated fusion system on P and we denote it by $\mathcal{F}_P(G)$. Recall that a group G is *p-nilpotent* if it has a normal p-complement, i.e, if $G = PO_{p'}(G)$. In the language of fusion systems, G is p-nilpotent if and only if $\mathcal{F}_P(G) = \mathcal{F}_P(P)$. In general, we call a saturated fusion system $\mathcal{F}$ on P *trivial* if $\mathcal{F} = \mathcal{F}_P(P)$.

Part of the motivation for the above definition is that it allows us to mimic quite a bit of local group theory, including normalizers, centralizers and Alperin's Fusion Theorem.

2.2. Subsystems and quotient systems

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. A subcategory $\mathcal{G}$ of $\mathcal{F}$ is a *saturated subsystem* of $\mathcal{F}$ if there exists a subgroup Q of P such that $\mathcal{G}$ is a saturated fusion system on Q. Puig defined subcategories, corresponding to local subgroups in finite group theory, $\mathrm{N}_{\mathcal{F}}(Q)$, $\mathrm{N}_P(Q)\mathrm{C}_{\mathcal{F}}(Q)$ and $\mathrm{C}_{\mathcal{F}}(Q)$ whose objects are the subgroups of $\mathrm{N}_P(Q)$, $\mathrm{N}_P(Q)$ and $\mathrm{C}_P(Q)$, respectively, and where

$$\mathrm{Hom}_{\mathrm{N}_{\mathcal{F}}(Q)}(R, S) = \{\phi \in \mathrm{Hom}_{\mathcal{F}}(R, S) \mid \exists \varphi \in \mathrm{Hom}_{\mathcal{F}}(QR, QS) \colon \varphi|_R = \phi\},$$

$$\mathrm{Hom}_{\mathrm{N}_P(Q)\mathrm{C}_{\mathcal{F}}(Q)}(R, S) = \{\phi \in \mathrm{Hom}_{\mathcal{F}}(R, S) \mid \exists \varphi \in \mathrm{Hom}_{\mathcal{F}}(QR, QS) \colon$$
$$\varphi|_R = \phi, \ \varphi|_Q \in \mathrm{Aut}_P(Q)\},$$

$$\mathrm{Hom}_{\mathrm{C}_{\mathcal{F}}(Q)}(R, S) = \{\phi \in \mathrm{Hom}_{\mathcal{F}}(R, S) \mid \exists \varphi \in \mathrm{Hom}_{\mathcal{F}}(QR, QS) \colon$$
$$\varphi|_R = \phi, \ \varphi|_Q = \mathrm{id}_Q\}$$

for subgroups R and S of these respective groups. If Q is fully $\mathcal{F}$-centralized, then $\mathrm{N}_P(Q)\mathrm{C}_{\mathcal{F}}(Q)$ and $\mathrm{C}_{\mathcal{F}}(Q)$ are saturated subsystems of $\mathcal{F}$. If Q is fully $\mathcal{F}$-normalized, then $\mathrm{N}_{\mathcal{F}}(Q)$ is saturated. In the special case where $\mathrm{N}_{\mathcal{F}}(Q) = \mathcal{F}$, we say that Q is *normal* in $\mathcal{F}$ and write $Q \lhd \mathcal{F}$. If $\mathrm{C}_{\mathcal{F}}(Q) = \mathcal{F}$, we say that Q is *central* in $\mathcal{F}$. When Q is fully $\mathcal{F}$-normalized, all of the above systems are saturated and we get the following chain of saturated subsystems of $\mathcal{F}$:

$$\mathrm{C}_{\mathcal{F}}(Q) \subseteq \mathrm{N}_P(Q)\mathrm{C}_{\mathcal{F}}(Q) \subseteq \mathrm{N}_{\mathcal{F}}(Q) \subseteq \mathcal{F}.$$

The largest normal and central subgroups of $\mathcal{F}$ are denoted by $\mathrm{O}_p(\mathcal{F})$ and $\mathrm{Z}(\mathcal{F})$, respectively. For more details and proofs, see [**16**, §3].

It may happen that a subgroup Q is normal in P but not normal in $\mathcal{F}$. In this context, there are a couple of gradations worth mentioning. If Q is stabilized by every $\mathcal{F}$-morphism defined on Q, then Q is called *weakly $\mathcal{F}$-closed*. Furthermore, if the image, under any $\mathcal{F}$-morphism, of every subgroup of Q remains in Q, then Q is *strongly $\mathcal{F}$-closed*. Therefore, if $Q \leqslant P$, then

$$Q \lhd \mathcal{F} \implies Q \text{ strongly } \mathcal{F}\text{-closed} \implies Q \text{ weakly } \mathcal{F}\text{-closed} \implies Q \trianglelefteq P.$$

When Q is strongly $\mathcal{F}$-closed, Puig defined a category $\mathcal{F}/Q$ whose objects are the subgroups of P/Q and whose morphisms are induced from $\mathcal{F}$ and stabilize Q and proved that it is a saturated fusion system. We omit a precise definition here as we will only need quotient systems in the context of the following proposition of Kessar and Linckelmann. For more details on $\mathcal{F}/Q$, we recommend the recent article by Craven [**7**], where several technical flaws in earlier treatments are overcome and $\mathcal{F}/Q$ is proven to be a saturated fusion system.

Our goal in many cases is to reduce to the case where $\mathcal{F} = P\mathrm{C}_{\mathcal{F}}(Q)$ for some $Q \trianglelefteq P$. To do so, we use the following result of Kessar and Linckelmann.

Proposition 2.2 (Kessar and Linckelmann [14, Proposition 3.4]). *Let $\mathcal{G} \subseteq \mathcal{F}$ be saturated fusion systems on a finite p-group P. If Q and R are normal subgroups of P such that $Q \leqslant R$ and $\mathcal{F} = P\mathrm{C}_{\mathcal{F}}(Q)$, then*

$$\mathcal{G} = \mathrm{N}_{\mathcal{F}}(R) \text{ if and only if } \mathcal{G}/Q = \mathrm{N}_{\mathcal{F}/Q}(R/Q).$$

2.3. Alperin's Fusion Theorem

The theorem we refer to here as Alperin's Fusion Theorem is a generalization to fusion systems of a theorem first proved by Alperin and improved upon by Goldschmidt and Puig. This version uses $\mathcal{F}$-essential subgroups: a class of subgroups that is, in some sense, minimal when it comes to generating fusion. Recall that a proper subgroup H of a finite group G is *strongly p-embedded* if it contains a non-trivial Sylow p-subgroup P of G and $H \cap P^x = 1$ for any $x \in G \setminus H$. In particular, if $\mathrm{O}_p(G) \neq 1$, then G has no strongly p-embedded subgroup.

Definition 2.3. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P and let Q be a subgroup of P.

(1) Q is $\mathcal{F}$-*centric* if $\mathrm{C}_P(R) \leqslant R$ for all $R \in Q^{\mathcal{F}}$.

(2) Q is $\mathcal{F}$-*essential* if Q is $\mathcal{F}$-centric and $\mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$ has a strongly p-embedded subgroup.

We start with a few useful trivialities and then a lesser-known property of $\mathcal{F}$-essential subgroups and the extension axiom.

Lemma 2.4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P.*

(1) *P is not $\mathcal{F}$-essential.*

If $Q < P$ is $\mathcal{F}$-essential, then

(2) *$\mathrm{Aut}_{\mathcal{F}}(Q)$ is not a p-group,*

(3) *$\mathrm{Aut}_{\mathcal{F}}(Q)$ does not have a normal Sylow p-subgroup (in particular, a cyclic p-group is never essential), and*

(4) *if $R \in Q^{\mathcal{F}}$ is a fully $\mathcal{F}$-normalized subgroup of P, then there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ such that $\mathrm{N}_{\varphi} = Q$.*

Proof. The Sylow axiom in the definition of a saturated fusion system implies that $\mathrm{Out}_{\mathcal{F}}(P)$ is a p'-group and so, by definition, it cannot have a strongly p-embedded subgroup. Thus, P is not $\mathcal{F}$-essential, proving (1). By definition, a p-group cannot have a strongly p-embedded subgroup and so $\mathrm{Out}_{\mathcal{F}}(Q)$ is not a p-group. This implies the existence of a p'-automorphism for Q in $\mathcal{F}$ proving (2). If $\mathrm{Aut}_{\mathcal{F}}(Q)$ has a normal Sylow p-subgroup, then (as it must contain $\mathrm{Aut}_P(Q)$) $\mathrm{O}_p(\mathrm{Out}_{\mathcal{F}}(Q)) \neq 1$; this is a contradiction since it implies $\mathrm{Out}_{\mathcal{F}}(Q)$ has no strongly p-embedded subgroup. To prove (4), note that as Q is $\mathcal{F}$-essential, we may choose $A, B \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ such that $A \cap B = \mathrm{Aut}_Q(Q)$. Furthermore, without loss of generality, we assume that $\mathrm{Aut}_P(Q) \leqslant A$. Since R is fully $\mathcal{F}$-normalized, $\mathrm{Aut}_P(R)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R) \cong \mathrm{Aut}_{\mathcal{F}}(Q)$ and so if $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, then $\phi^{-1} \mathrm{Aut}_P(R)\phi \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$. Thus, there exists $\psi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\psi^{-1}\phi^{-1} \mathrm{Aut}_P(R)\phi\psi = B$. Set $\varphi = \phi\psi$. If $c \colon \mathrm{N}_P(Q) \to \mathrm{Aut}_P(Q)$ denotes the homomorphism sending an element $u \in \mathrm{N}_P(Q)$ to the automorphism c_u given by conjugation by u, then

$$\mathrm{N}_{\varphi} = c^{-1}(\mathrm{Aut}_P(Q) \cap \varphi^{-1} \mathrm{Aut}_P(R)\varphi) \leqslant c^{-1}(A \cap B) = c^{-1}(\mathrm{Aut}_Q(Q)) = Q\mathrm{C}_P(Q) = Q,$$

where the last equality follows since Q is $\mathcal{F}$-centric. $\square$

Another important property of $\mathcal{F}$-essential subgroups is that they always contain $\mathrm{O}_p(\mathcal{F})$.

Proposition 2.5 (Broto *et al.* [3, Proposition 1.6]). *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. If Q is an $\mathcal{F}$-essential subgroup of P, then $\mathrm{O}_p(\mathcal{F}) \leqslant Q$.*

It is worth pointing out that, for non-trivial P, every $\mathcal{F}$-centric subgroup properly contains $\mathrm{Z}(P)$ so that, in fact, every $\mathcal{F}$-essential subgroup contains $\mathrm{Z}(P)\mathrm{O}_p(\mathcal{F})$.

We now state Alperin's Fusion Theorem. Morally, it tells us that a saturated fusion system is determined by the $\mathcal{F}$-automorphisms of P and the $\mathcal{F}$-essential subgroups of P.

Theorem 2.6 (Alperin's Fusion Theorem). *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Every $\mathcal{F}$-isomorphism is the composition of finitely many morphisms of the form $\phi \colon Q \to R$ where there exist $Q, R \leqslant S \leqslant P$ such that $S = P$ or S is $\mathcal{F}$-essential and there exists $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\alpha|_Q = \phi$.*

By [8, Proposition 2.10], a saturated fusion system $\mathcal{F}$ is generated by the $\mathcal{F}$-automorphisms of a set of representatives of the $\mathcal{F}$-isomorphism classes of $\mathcal{F}$-essential subgroups of P. This motivates the following definition.

Definition 2.7. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. The *essential rank* of $\mathcal{F}$ is the number of $\mathcal{F}$-isomorphism classes of $\mathcal{F}$-essential subgroups of P. The essential rank of $\mathcal{F}$ is denoted by $\mathrm{rank}_{\mathrm{e}}(\mathcal{F})$.

The structure of a saturated fusion system $\mathcal{F}$ on a finite p-group P with essential rank 0 is particularly straightforward.

Lemma 2.8. *If $\mathcal{F}$ is a saturated fusion system on a finite p-group P, then the following are equivalent.*

(1) $P \lhd \mathcal{F}$.

(2) $\mathrm{rank}_{\mathrm{e}}(\mathcal{F}) = 0$.

(3) $\mathcal{F} = \mathcal{F}_P(P \rtimes \mathrm{Out}_{\mathcal{F}}(P))$.

Proof. If $P \lhd \mathcal{F}$, then $\mathrm{O}_p(\mathcal{F}) = P$ and so, by Lemma 2.4 and Proposition 2.5, there are no $\mathcal{F}$-essential subgroups of P. Conversely, if there are no $\mathcal{F}$-essentials, then Alperin's Fusion Theorem implies that each $\mathcal{F}$-automorphism extends to P and P is normal in $\mathcal{F}$. This shows that (1) is equivalent to (2). As P is clearly normal in $\mathcal{F}_P(P \rtimes \mathrm{Out}_{\mathcal{F}}(P))$, it remains to show that (1) implies (3). In this case, by (2) and Alperin's Fusion Theorem, $\mathcal{F}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(P)$. The result now follows since $\mathrm{Aut}_P(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. $\square$

2.4. Constrained fusion systems

In [**3**, Proposition 4.3], Broto *et al.* prove that if $\mathcal{F}$ is a saturated fusion system, then for every fully $\mathcal{F}$-normalized $\mathcal{F}$-centric subgroup Q of P, there exists a unique finite group G (up to isomorphism) such that $\mathrm{O}_{p'}(G) = 1$, $\mathrm{C}_G(\mathrm{O}_p(G)) \leqslant \mathrm{O}_p(G)$ and such that $\mathrm{N}_{\mathcal{F}}(Q) = \mathcal{F}_{\mathrm{N}_P(Q)}(G)$. In this case there is an exact sequence

$$1 \to \mathrm{Z}(Q) \to G \to \mathrm{Aut}_{\mathcal{F}}(Q) \to 1.$$

In particular, if Q is normal in $\mathcal{F}$, then $\mathcal{F} = \mathcal{F}_P(G)$. In this case, where $\mathcal{F}$ has a normal $\mathcal{F}$-centric subgroup, $\mathcal{F}$ is called *constrained*. By Lemma 2.8, any saturated fusion system with essential rank 0 is constrained. The importance of being constrained is that it reduces some questions about fusion systems to questions about groups (see [**9**], [**8**] or [**14**] for some recent examples).

3. Sparse fusion systems

We begin by proving a useful lemma due to Onofrei and Stancu. This will help us reduce to the case where the fusion system is of the form $\mathcal{F} = PC_{\mathcal{F}}(Q)$ for some $Q \lhd \mathcal{F}$.

Lemma 3.1 (Onofrei and Stancu [17, Lemma 3.7]). *Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let $Q \leqslant P$. If $Q \lhd \mathcal{F}$, then*

$$\mathcal{F} = \langle PC_{\mathcal{F}}(Q), \mathrm{N}_{\mathcal{F}}(QC_P(Q))\rangle.$$

Proof. Let T be an $\mathcal{F}$-essential subgroup of P and take $\varphi \in \mathrm{Aut}_{\mathcal{F}}(T)$. As Q is weakly $\mathcal{F}$-closed, we have $\theta = \varphi|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$, and since $TQC_P(Q) \leqslant \mathrm{N}_\theta$, there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(TQC_P(Q), P)$ such that $\psi|_Q = \varphi|_Q$. Then

$$\varphi = (\varphi \circ (\psi|_T)^{-1}) \circ \psi|_T;$$

$\varphi \circ (\psi|_T)^{-1}$ is a morphism in $PC_{\mathcal{F}}(Q)$; $\psi|_T$ is a morphism in $\mathrm{N}_{\mathcal{F}}(QC_P(Q))$ because $\psi(QC_P(Q)) = QC_P(Q)$. Thus, φ is a morphism in $\langle PC_{\mathcal{F}}(Q), \mathrm{N}_{\mathcal{F}}(QC_P(Q))\rangle$. By Alperin's Fusion Theorem, it follows that $\mathcal{F} = \langle PC_{\mathcal{F}}(Q), \mathrm{N}_{\mathcal{F}}(QC_P(Q))\rangle$. $\square$

Proposition 3.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. If Q and R are normal subgroups of P such that $Q \leqslant R$ and $\mathcal{F} = \mathrm{PC}_{\mathcal{F}}(Q)$, then*

(1) $\mathrm{N}_{\mathcal{F}}(R)$ *is trivial if and only if* $\mathrm{N}_{\mathcal{F}/Q}(R/Q)$ *is trivial; in particular, $\mathcal{F}$ is trivial if and only if $\mathcal{F}/Q$ is trivial; and*

(2) $R \lhd \mathcal{F}$ *if and only if* $R/Q \lhd \mathcal{F}/Q$.

Proof. We obtain (1) by applying Proposition 2.2 with $\mathcal{G} = \mathcal{F}_P(P)$. Statement (2) is obtained when applying Proposition 2.2 with $\mathcal{G} = \mathcal{F}$. □

Definition 3.3. A non-trivial saturated fusion system $\mathcal{F}$ on a finite p-group P is called *sparse* if the only proper subfusion system of $\mathcal{F}$ on P is $\mathcal{F}_P(P)$.

The motivation for defining this class of fusion systems is that a (putative) minimal counter-example to a theorem whose conclusion is that a saturated fusion system is trivial will often be a sparse fusion system. A tangible example of a sparse fusion system is the fusion system of S_4 on D_8. In the picture below, the fusion of D_8 on D_8 is described by the circled dots, while the additional fusion of S_4 on D_8 is described by the dashed line:
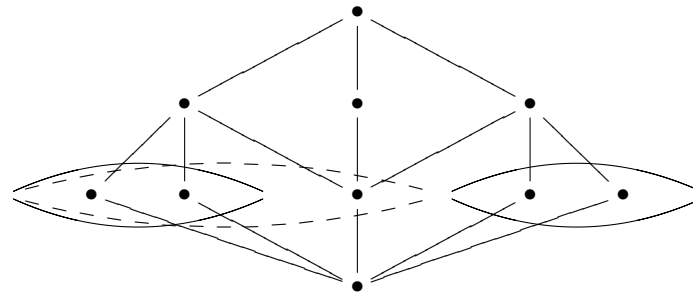
Sparse fusion systems are necessarily ubiquitous objects. Take any non-trivial saturated fusion system $\mathcal{F}$ on a finite p-group P and consider the lattice of subsystems of $\mathcal{F}$ on P. Any minimal non-trivial subsystem in this lattice is a sparse fusion system. In particular, for any finite p-group on which there is at least one non-trivial saturated fusion system, there is a sparse fusion system on that p-group.

For a finite p-group P, the Thompson subgroup, $\mathrm{J}(P)$, is the subgroup of P generated by the abelian subgroups of P of maximal order. A classical result of Thompson is that a group G with Sylow p-subgroup P is p-nilpotent if and only if $\mathrm{N}_G(\mathrm{J}(P))$ and $\mathrm{C}_G(\mathrm{Z}(P))$ are p-nilpotent. This was recently extended to saturated fusion systems in [**8**]. Recall from [**14**] that a saturated fusion system is S_4-free if all of the groups arising from the normalizers of $\mathcal{F}$-centric, fully $\mathcal{F}$-normalized subgroups of P, as in [**3**, Proposition 4.3], are S_4-free.

Theorem 3.4 (Díaz *et al*. [8, Theorem 4.5]). *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P where p is odd or $\mathcal{F}$ is S_4-free. If $\mathrm{N}_{\mathcal{F}}(\mathrm{J}(P)) = \mathcal{F}_P(P) = \mathrm{C}_{\mathcal{F}}(\mathrm{Z}(P))$, then $\mathcal{F} = \mathcal{F}_P(P)$.*

We use this result to detect constraint in sparse fusion systems.

Theorem 3.5. *Let $\mathcal{F}$ be a sparse fusion system on a finite p-group P.*

(1) *Let $Q \leqslant P$ such that $Q \lhd \mathcal{F}$. If $Q\mathrm{C}_P(Q)$ is not normal in $\mathcal{F}$, then $\mathcal{F} = P\mathrm{C}_{\mathcal{F}}(Q)$. In this case, $Q \cap \mathrm{Z}(P) \leqslant \mathrm{Z}(\mathcal{F})$.*

(2) *If p is odd or $\mathcal{F}$ is S_4-free, then $\mathcal{F}$ is constrained.*

Proof. If $Q\mathrm{C}_P(Q)$ is not normal in $\mathcal{F}$ and $\mathcal{F}$ is sparse, we have $\mathrm{N}_{\mathcal{F}}(Q\mathrm{C}_P(Q)) = \mathcal{F}_P(P)$. By Lemma 3.1, $\mathcal{F} = P\mathrm{C}_{\mathcal{F}}(Q)$, proving (1). If $\mathrm{Z}(P)$ and $\mathrm{J}(P)$ are not normal in $\mathcal{F}$, then, as $\mathcal{F}$ is sparse, their $\mathcal{F}$-normalizers are all trivial. By [**8**, Theorem 4.5], we conclude that $\mathcal{F} = \mathcal{F}_P(P)$, a contradiction proving that $\mathrm{O}_p(\mathcal{F}) \neq 1$. Set $Q = \mathrm{O}_p(\mathcal{F})$. If Q is $\mathcal{F}$-centric, then $\mathcal{F}$ is constrained. Therefore, we assume that Q is not $\mathcal{F}$-centric. This implies that Q is a proper subgroup of $Q\mathrm{C}_P(Q)$ and, as $Q = \mathrm{O}_p(F)$, we have that $Q\mathrm{C}_P(Q)$ is not normal in $\mathcal{F}$. Applying (1), we get $\mathcal{F} = P\mathrm{C}_{\mathcal{F}}(Q)$. If $\mathcal{F}/Q$ is trivial, then so is $\mathcal{F}$ by Proposition 3.2, giving a contradiction. If $\mathcal{F}$ is S_4-free, then [**14**, Proposition 6.3] implies that $\mathcal{F}/Q$ is also S_4-free. Thus, by [**8**, Theorem 4.5], at least one of $\mathrm{C}_{\mathcal{F}}(\mathrm{Z}(P/Q))$ and $\mathrm{N}_{\mathcal{F}}(\mathrm{J}(P/Q))$ is not trivial, so we may assume that there is a normal subgroup R of P properly containing Q such that $\mathrm{N}_{\mathcal{F}/Q}(R/Q)$ is not trivial. Using Proposition 2.2 again, $\mathrm{N}_{\mathcal{F}}(R)$ is not trivial and so, since $\mathcal{F}$ is sparse, $R \lhd \mathcal{F}$, contradicting the maximality of Q. This proves (2). □

In [**7**], a saturated fusion system $\mathcal{F}$ on a finite p-group P is called *p-solvable* if there exists a chain of strongly $\mathcal{F}$-closed subgroups $1 = P_0 \leqslant P_1 \leqslant \cdots \leqslant P_n = P$ such that $P_i/P_{i-1} \leqslant \mathrm{O}_p(\mathcal{F}/P_{i-1})$ for all $1 \leqslant i \leqslant n$. When such a chain exists, the length of a minimal possible chain satisfying the above is called the *p-length* of $\mathcal{F}$. It is easy to see from the above proof that a sparse fusion system is p-solvable with p-length 2 when p is odd or the fusion system is S_4-free.

A careful reading of the proofs of Lemma 3.1 and Theorem 3.5 shows that a slightly weaker condition on Q will suffice: namely, we only require that Q be a weakly $\mathcal{F}$-closed subgroup contained in every subgroup T of some conjugation family for $\mathcal{F}$. For example, if $Q \lhd \mathcal{F}$, then, by Proposition 2.5, Q is contained in every $\mathcal{F}$-essential subgroup.

Note that the statement of Theorem 3.5 (2) only considers the case where $\mathcal{F}$ is S_4-free when $p = 2$ but that the result holds for $\mathcal{F}_{D_8}(S_4)$. This led us, in an earlier version of this paper, to conjecture that all sparse fusion systems are constrained. However, David Craven pointed out a family of counter-examples to this conjecture. Take, for example, the fusion system on D_{16} afforded by $\mathrm{PGL}(2, 7)$. This system is easily seen to be sparse (see [**16**, Example 8.8] for details) and not constrained. It is still an open question as to whether there exist any sparse exotic fusion systems.

4. p-nilpotency criteria

4.1. ZJ and the Stellmacher functor

We use Theorem 3.5 to give a streamlined version of the proofs of [**14**, Theorem A] and [**17**, Theorem 1.3].

Theorem 4.1. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. The following are equivalent:*

(1) $\mathcal{F} = \mathcal{F}_P(P)$,

(2) $\mathrm{N}_\mathcal{F}(\mathrm{Z}(\mathrm{J}(P))) = \mathcal{F}_P(P)$,

(3) $\mathrm{N}_\mathcal{F}(\mathrm{W}(P)) = \mathcal{F}_P(P)$.

Here $\mathrm{J}(P)$ denotes the Thompson subgroup of P, and W denotes the Stellmacher functor (as in [19]).

Proof. It is clear that (1) implies both (2) and (3). We will prove that (2) implies (1) and the proof that (3) implies (1) will be the same almost verbatim (referring, of course, to Stellmacher's result instead). Let $\mathcal{F}$ be a minimal counter-example with respect to the number $|\mathcal{F}|$ of morphisms in $\mathcal{F}$. If $\mathcal{G}$ is a proper subfusion system of $\mathcal{F}$ on P, then $\mathrm{N}_\mathcal{G}(\mathrm{Z}(\mathrm{J}(P))) \subseteq \mathrm{N}_\mathcal{F}(\mathrm{Z}(\mathrm{J}(P))) = \mathcal{F}_P(P)$ and so by the minimality of $\mathcal{F}$ we have $\mathcal{G} = \mathcal{F}_P(P)$. In particular, $\mathcal{F}$ is sparse. By Theorem 3.5, $\mathcal{F}$ is constrained and so by [3, Proposition 4.3] there exists a finite group G with Sylow p-subgroup P such that $\mathcal{F} = \mathcal{F}_P(G)$. This implies that

$$\mathcal{F}_P(\mathrm{N}_G(\mathrm{Z}(\mathrm{J}(P)))) = \mathrm{N}_\mathcal{F}(\mathrm{Z}(\mathrm{J}(P))) = \mathcal{F}_P(P).$$

By Glauberman's and Thompson's p-Nilpotency Theorem for groups, $\mathcal{F} = \mathcal{F}_P(G) = \mathcal{F}_P(P)$: a contradiction. □

4.2. A theorem of Navarro

We now generalize a result of Navarro and then translate it to fusion systems. In the following, P' denotes the derived subgroup of P, i.e. the smallest normal subgroup of P with abelian quotient, and $\Phi(P)$ denotes the Frattini subgroup of P, i.e. the smallest normal subgroup of P with elementary abelian quotient.

Theorem 4.2 (Navarro). *Let G be a finite group with Sylow p-subgroup P. If $\mathrm{N}_G(P)$ is p-nilpotent, then $\mathrm{N}_G(Q)$ is p-nilpotent for every subgroup $P' \leqslant Q \leqslant \Phi(P)$.*

The original statement of Navarro's theorem is slightly weaker: namely, it makes the stronger assumption that $\mathrm{N}_G(P) = P$ and only considers the case where $Q = P'$. The following proof is based on one given by I. M. Isaacs (personal communication) for Navarro's original statement. The inspiration for considering this stronger version is in Remark 4.5.

Proof. As $P' \leqslant Q$, P is a Sylow p-subgroup of $\mathrm{N}_G(Q)$. Also, $\mathrm{N}_{\mathrm{N}_G(Q)}(Q) = \mathrm{N}_G(Q)$ and so, without loss of generality, we assume that $Q \trianglelefteq G$. As

$$[P, \mathrm{N}_G(P)] = [P, P\mathrm{C}_G(P)] = P' \leqslant Q,$$

we have $P/Q \leqslant \mathrm{Z}(\mathrm{N}_G(P)/Q) = \mathrm{Z}(\mathrm{N}_{G/Q}(P/Q))$, and so by Burnside's Normal p-Complement Theorem [10, Theorem 7.4.3], P/Q has a normal complement in G/Q. Let

$Q \trianglelefteq K \trianglelefteq G$ such that $G = PK$ and $P \cap K = Q$. The Schur–Zassenhaus Theorem [**10**, Theorem 6.2.1] implies that there exists $L \leqslant K$ such that $K = QL$ and $Q \cap L = 1$:
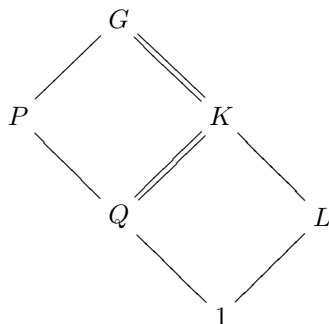
As Q is solvable, any two such complements are K-conjugate. By the Frattini argument, $G = K\mathrm{N}_G(L) = QL\mathrm{N}_G(L) = Q\mathrm{N}_G(L)$. Dedekind's Lemma now implies that $P = Q\mathrm{N}_P(L)$ and, since $Q \leqslant \Phi(P)$, this gives $P = \mathrm{N}_P(L)$. As $G = PK = PQL = PL$, we have $L \trianglelefteq G$ and so L is a normal p-complement for G, i.e. G is p-nilpotent. $\qquad\square$

Proposition 4.3. *Let $\mathcal{F}$ be a fusion system on a finite p-group P such that $\mathrm{N}_{\mathcal{F}}(P)$ is trivial. If $P' \leqslant Q \leqslant P$ such that $\mathcal{F} = P\mathrm{C}_{\mathcal{F}}(Q)$, then $\mathcal{F}$ is trivial.*

Proof. By Burnside's Fusion Theorem (see [**16**, Theorem 3.8]), $\mathcal{F}/Q = \mathrm{N}_{\mathcal{F}/Q}(P/Q)$, and by Proposition 2.2 $\mathrm{N}_{\mathcal{F}/Q}(P/Q)$ is trivial. Therefore, $\mathcal{F}/Q$ is trivial and, applying Proposition 2.2 again, $\mathcal{F}$ is trivial. $\qquad\square$

Theorem 4.4. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. If $\mathrm{N}_{\mathcal{F}}(P)$ is trivial, then $\mathrm{N}_{\mathcal{F}}(Q)$ is trivial for every subgroup $P' \leqslant Q \leqslant \Phi(P)$.*

Proof. Without loss of generality, we may assume that $Q \vartriangleleft \mathcal{F}$. Let $\mathcal{F}$ be a minimal counter-example with respect to $|\mathcal{F}|$ (so that $Q > 1$). If Q is $\mathcal{F}$-centric, then $\mathcal{F}$ is constrained. By [**3**, Proposition 4.3], $\mathcal{F}$ is the fusion system of a finite group G with Sylow p-subgroup P satisfying $\mathrm{N}_G(P) = P\mathrm{C}_G(P)$. The result now follows from Theorem 4.2. So we assume that $Q < Q\mathrm{C}_P(Q)$ and that $Q\mathrm{C}_P(Q) \not\vartriangleleft \mathcal{F}$. As $\mathcal{F}$ is sparse and $\mathrm{N}_{\mathcal{F}}(P)$ is trivial, Theorem 3.5 and Proposition 4.3 imply that $\mathcal{F} = P\mathrm{C}_{\mathcal{F}}(Q) = \mathcal{F}_P(P)$: a contradiction. $\qquad\square$

Remark 4.5. For any choice of Q, the proof of Theorem 4.4 only requires Navarro's original theorem (where $\mathrm{N}_G(P) = P$) and not the full strength of Theorem 4.2. In fact, if Q is a normal $\mathcal{F}$-centric subgroup of P, then $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G with $Q \trianglelefteq G$, $\mathrm{C}_G(Q) \leqslant Q$ and, since $\mathrm{Aut}_{\mathcal{F}}(P)$ is a p-group, $\mathrm{N}_G(P) = P\mathrm{C}_G(P)$. Consequently, $\mathrm{C}_G(P) \leqslant \mathrm{C}_G(Q) = \mathrm{Z}(Q) \leqslant P$ and so $\mathrm{N}_G(P) = P$. This gives an excellent example of a statement about groups being used to prove a result in the context of fusion systems and thereby obtaining a stronger result about groups.

4.3. Navarro's theorem for blocks

Recall that if b is a p-block of a finite group G over an algebraically closed field and if (P, e_P) is a maximal b-Brauer pair, then for every subgroup Q of P, there exists a unique block e_Q of $C_G(Q)$ such that $(Q, e_Q) \leqslant (P, e_P)$ (for details about this inclusion, see [1] or [5]). The group G acts on the set of b-Brauer pairs by conjugation and this gives rise to a saturated fusion system on P where, for $Q, R \leqslant P$, conjugation by an element $g \in G$ is in the fusion system if it respects the b-Brauer pair structure, i.e. if $(Q, e_Q)^g \leqslant (R, e_R)$. The $\mathcal{F}$-automorphism groups of subgroups of P are easily seen to be $\mathrm{Aut}_{\mathcal{F}}(Q) \cong N_G(Q, e_Q)/C_G(Q)$, and $|\mathrm{Out}_{\mathcal{F}}(P)|$ is called the *inertial index* of b. The block b is called *nilpotent* if the corresponding saturated fusion system is trivial. For more details and a presentation of the structure of nilpotent blocks, we refer the reader to [20]. For an explicit proof that a block gives rise to a saturated fusion system, see [13]. The following corollary generalizes the classical result (see [15] or [5]) that a block with inertial index 1 and abelian defect group is nilpotent.

Corollary 4.6. *Let b be a p-block of a finite group G with inertial index 1 and maximal b-Brauer pair (P, e_P). Let $P' \leqslant Q \leqslant \Phi(P)$ and let e_Q be the unique block of $C_G(Q)$ such that $(Q, e_Q) \leqslant (P, e_P)$. If f_Q is a block of $N_G(Q)$ covering e_Q and if it is in Brauer correspondence with b, then f_Q is nilpotent. In particular, if $Q \trianglelefteq G$, then b is nilpotent.*

$$
\begin{array}{ll}
G & b \\
| & \\
N_G(Q) & f_Q \\
| & \\
N_G(Q, e_Q) & e_Q \\
| & \\
C_G(Q) & e_Q
\end{array}
$$
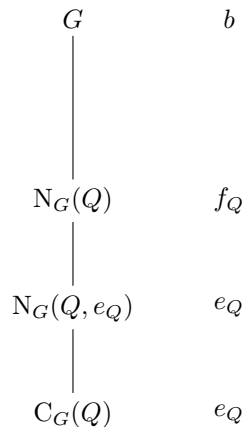
Proof. Let $\mathcal{F} = \mathcal{F}_{(P,e_P)}(G, b)$ be the saturated fusion system on P corresponding to the block b and the maximal b-Brauer pair (P, e_P). The condition that b has inertial index 1 is equivalent to the condition that $N_{\mathcal{F}}(P)$ is trivial. Therefore, by Theorem 4.4, $\mathcal{F}_P(P) = N_{\mathcal{F}}(Q) = \mathcal{F}_{(P,e_P)}(N_G(Q, e_Q), e_Q)$ and hence e_Q is nilpotent as a block of $N_G(Q, e_Q)$. As Q is a normal p-subgroup of $N_G(Q)$, $f_Q = \mathrm{tr}_{N_G(Q,e_Q)}^{N_G(Q)}(e_Q)$ is a block of $N_G(Q)$ covering e_Q and is in Brauer correspondence with b. Now, [12, Proposition 2.13] implies that $\mathcal{F}_{(P,e_P)}(N_G(Q, e_Q), e_Q) = \mathcal{F}_{(P,e_P)}(N_G(Q), f_Q)$, completing the proof. $\qquad\square$

4.4. Slim p-groups

In [**22**], Weigel introduces the following definition. Set $Y_1 = C_p \wr C_p$ and, for $m > 1$, define Y_m to be the pull-back in the diagram:

$$
\begin{array}{ccc}
C_p \wr C_p & \longrightarrow & C_p \\
\uparrow & & \uparrow \\
\vdots & & \Big| \\
Y_m & \cdots\!\!\!\rightarrow & C_{p^m}
\end{array}
$$

Definition 4.7. With the notation as above, a finite p-group P is *slim* if Y_m is not a subgroup of P for all $m \geqslant 1$.

Theorem 4.8 (Weigel [22]). *Let G be a finite group and let P be a Sylow p-subgroup of G. If*

(1) *p is odd and P is slim or*

(2) *$p = 2$ and P is D_8-free,*

then G is p-nilpotent if and only if $\mathrm{N}_G(P)$ is p-nilpotent.

The following elementary lemma connects Theorem 3.5 with the present context.

Lemma 4.9. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P and let H be a finite group with p-subgroup Q. If P is Q-free, then $\mathcal{F}$ is H-free.*

Proof. Let S be a fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric subgroup of P and let G be the unique finite group with Sylow p-subgroup $\mathrm{N}_P(S)$ guaranteed by [**3**, Proposition 4.3]. If G is Q-free, then it is H-free, so assume that $K/L \cong Q$ is a section of G. A Sylow p-subgroup R of K is conjugate to a subgroup of $\mathrm{N}_P(S) \leqslant P$ and hence is Q-free. On the other hand, since K/L is a p-group, $Q \cong K/L \cong RL/L \cong R/(R \cap L)$: a contradiction. $\square$

The restatement of Theorem 4.8 for fusion systems is clear and a minimal counter-example to such a result is easily seen to be sparse. Moreover, if P is D_8-free, then the previous lemma shows that any saturated fusion system on P is S_4-free. So, regardless of p, Theorem 3.5 implies that the minimal counter-example is constrained.

Theorem 4.10. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. If*

(1) *p is odd and P is slim or*

(2) *$p = 2$ and P is D_8-free,*

then $\mathcal{F} = \mathcal{F}_P(P)$ if and only if $\mathrm{N}_{\mathcal{F}}(P) = \mathcal{F}_P(P)$.

Restricting to blocks, we get the following corollary, again generalizing the result that a block with inertial index 1 and abelian defect group is nilpotent.

Corollary 4.11. *Let b be a p-block of a finite group G with defect group P and inertial index 1. If p is odd and P is slim or if $p = 2$ and P is D_8-free, then b is nilpotent.*

5. Extremely sparse fusion systems

In the spirit of §3, we define an even more restrictive situation: namely, that of an *extremely sparse fusion system*.

Definition 5.1. A non-trivial saturated fusion system $\mathcal{F}$ on a finite p-group P is called *extremely sparse* if the only proper subsystem of $\mathcal{F}$ on any subgroup Q of P is $\mathcal{F}_Q(Q)$.

Clearly, if a fusion system is extremely sparse, then it is sparse. We will see that every extremely sparse fusion system is constrained, regardless of p. In fact, we offer two proofs of this result: the first because it seems that Lemma 2.4 (4) may be of independent interest and the second because it gives a clearer picture of this situation.

Theorem 5.2. *Every extremely sparse fusion system is constrained.*

Proof. Let $\mathcal{F}$ be an extremely sparse fusion system on a finite p-group P. By Lemma 2.8, we may assume that P has a fully $\mathcal{F}$-normalized, $\mathcal{F}$-essential subgroup Q. By Lemma 2.4 (4), there exists $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\mathrm{N}_{\varphi} = Q$. As any morphism in $\mathcal{F}_{\mathrm{N}_P(Q)}(\mathrm{N}_P(Q))$ extends to P, we conclude that $\varphi \notin \mathrm{Aut}_{\mathrm{N}_P(Q)}(Q)$ and hence $\mathrm{N}_{\mathcal{F}}(Q)$ is non-trivial. Since $\mathcal{F}$ is extremely sparse, it follows that $\mathrm{N}_{\mathcal{F}}(Q) = \mathcal{F}$. As Q is $\mathcal{F}$-essential, it is $\mathcal{F}$-centric and so $\mathcal{F}$ is constrained. $\square$

Now we offer a different proof: one that provides a simple classification of all extremely sparse fusion systems.

Theorem 5.3. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. If $\mathcal{F}$ is extremely sparse, then $\mathcal{F} = \mathcal{F}_P(P \rtimes A)$, where A is a cyclic group of order q for some prime $q \neq p$.*

Proof. If Q is an $\mathcal{F}$-essential subgroup of P, then, by Lemma 2.4 (2), there exists a non-trivial p'-element α in $\mathrm{Out}_{\mathcal{F}}(Q)$. It follows that $\mathcal{F}_Q(Q)$ is a proper subsystem of $\mathcal{F}_Q(Q \rtimes \langle \alpha \rangle)$. However, Lemma 2.4 (1) implies $Q \neq P$, and $\mathcal{F}_Q(Q \rtimes \langle \alpha \rangle)$ is therefore a proper subsystem of $\mathcal{F}$, contradicting the extreme sparseness of $\mathcal{F}$. Therefore, $\mathrm{rank}_{\mathrm{e}}(\mathcal{F}) = 0$ and, by Lemma 2.8, $\mathcal{F} = \mathcal{F}_P(P \rtimes \mathrm{Out}_{\mathcal{F}}(P))$. As $\mathcal{F}$ is non-trivial, there exists a non-trivial q-automorphism β of P for some prime $q \neq p$. The sparseness of $\mathcal{F}$ implies that $\mathcal{F} = \mathcal{F}_P(P \rtimes \langle \beta \rangle)$. $\square$

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. The *focal subgroup* of $\mathcal{F}$ is defined as

$$[P, \mathcal{F}] = \langle x^{-1}\varphi(x) \mid x \in P, \ \varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, P) \rangle.$$

As with the analogue for group theory, the focal subgroup of a fusion system controls the existence of subsystems of p-power index with abelian quotient. For more detail and properties we refer the reader to [**4**] and [**9**]. In the former reference, the reader can find a definition of $O^p(\mathcal{F})$. Here it suffices to say that $O^p(\mathcal{F})$ is the unique saturated subfusion system of $\mathcal{F}$ on $[P, O^p(\mathcal{F})]$ with p-power index and that $\mathcal{F}$ is the unique saturated subfusion system of $\mathcal{F}$ on P with p-power index.

Corollary 5.4. *Let $\mathcal{F}$ be an extremely sparse fusion system on a finite p-group P and let A be a p'-group of $\mathcal{F}$-automorphisms of P such that $\mathcal{F} = \mathcal{F}_P(P \rtimes A)$.*

(1) *If $Q < P$, then $\mathrm{N}_A(Q) = \mathrm{C}_A(Q)$.*

(2) *$[P, \mathcal{F}] = P$.*

(3) *$O^p(\mathcal{F}) = \mathcal{F}$.*

Proof. Let $1 \neq \alpha \in A$ such that $\alpha(Q) = Q$. This implies that $\alpha|_Q$ is a p'-automorphism in $\mathrm{Aut}_{\mathcal{F}}(Q)$. Since $Q < P$, the subsystem $\mathcal{F}_Q(Q \rtimes \langle \alpha \rangle)$ is a proper subsystem of $\mathcal{F}$ and, hence, is trivial: an impossibility unless α is the identity on Q. This proves (1). For $u \in P$,

$$\alpha(u^{-1}\alpha(u)) = \alpha(u)^{-1}\alpha(\alpha(u)) \in [P, \alpha],$$

and so α normalizes $[P, \alpha]$. As α induces the identity on $P/[P, \alpha]$, α cannot centralize $[P, \alpha]$ (otherwise it would be the identity on P). By (1) we conclude that $P = [P, \alpha] \leqslant [P, \mathcal{F}]$, proving (2). Finally, as $P/[P, O^p(\mathcal{F})]$ controls the existence of a saturated subsystem of $\mathcal{F}$ with p-power index and $P/[P, \mathcal{F}]$ controls the existence of a saturated subsystem of $\mathcal{F}$ with p-power index on an abelian quotient, $[P, \mathcal{F}] = P$ implies $[P, O^p(\mathcal{F})] = P$ and so there is no proper saturated subsystem with p-power index. This gives (3). $\qquad\square$

For a saturated fusion system $\mathcal{F}$ on a finite p-group P and $Q \leqslant P$, set $[Q, \mathcal{F}; 0] = Q$ and, for positive integers i, define

$$[Q, \mathcal{F}; i] = [[Q, \mathcal{F}; i-1], \mathcal{F}].$$

As $[Q, \mathcal{F}; i] \leqslant [Q, \mathcal{F}; i-1]$ for any positive integer i, we may define

$$[Q, \mathcal{F}; \infty] = \bigcap_{i=0}^{\infty} [Q, \mathcal{F}; i].$$

Note that if $\mathcal{G}$ is a subfusion system of $\mathcal{F}$ on P, then $[P, \mathcal{G}; i] \leqslant [P, \mathcal{F}; i]$ for all i. Similarly, if $Q \triangleleft \mathcal{F}$, then $[P/Q, \mathcal{F}; i] \leqslant [P, \mathcal{F}; i]Q/Q$ for all i. Our final corollary generalizes a well-known result of groups (see [**11**, Theorem 4.3] or [**18**, Proposition 12.4]) to fusion systems.

Corollary 5.5. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. If $[P, \mathcal{F}; \infty] = 1$, then $\mathcal{F} = \mathcal{F}_P(P)$.*

Proof. Let $\mathcal{F}$ be a minimal counter-example with respect to $|\mathcal{F}|$, the number of morphisms in $\mathcal{F}$. If $\mathcal{G}$ is a proper subfusion system of $\mathcal{F}$ on $Q \leqslant P$, then $[Q, \mathcal{G}; \infty] \leqslant [P, \mathcal{F}; \infty] = 1$ and so, by the minimality of $\mathcal{F}$, we have $\mathcal{G} = \mathcal{F}_Q(Q)$. Therefore, $\mathcal{F}$ is extremely sparse. However, the previous corollary implies that $[P, \mathcal{F}; \infty] = P$. We conclude that $P = 1$ and $\mathcal{F}$ is trivial: a contradiction. $\qquad\square$

Note that, unlike many of the previous proofs, this one does not use the original theorem from group theory and therefore gives an alternate proof of that result.

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