GRAPHS WITH GIVEN GROUP AND GIVEN GRAPH-THEORETICAL PROPERTIES

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1. Introduction. In 1938 Frucht **(2)** proved the following theorem:

(1.1). THEOREM. *Given any finite group G there exist infinitely many non-isomorphic connected graphs X whose automorphism group is isomorphic to G.*

Later, the same author showed **(3)** that this theorem still holds, if the words "connected graphs X" are replaced by "connected regular graphs X of degree 3." There is, of course, no reason to assume that such graphs play any distinguished rôle, and that similar theorems do not hold for degrees > 3. Indeed it can be shown that (1.1) holds with "connected graphs X" replaced by "connected regular graphs X of degree n, where n is any integer $\geqslant 3$."

It is only natural, then, to investigate whether the property that a graph X be regular of degree n is the only graph-theoretical property of X which can be prescribed together with the automorphism group. Consider the following properties $P_j (j = 1,2,3,4)$ of X:

P_1: The connectivity **(6)** of X is n, where n is an integer $\geqslant 1$.
P_2: The chromatic number **(1)** of X is n, where n is an integer $\geqslant 2$.
P_3: X is regular of degree n, where n is an integer $\geqslant 3$.
P_4: X is spanned by a graph $\tilde{Y}$ homeomorphic to a given connected graph Y.

Call a graph X *fixed-point-free* if there is no vertex x of X which is invariant under all automorphisms of X.

The following theorem contains the main results of this paper:

(1.2) THEOREM. *Given a finite group G of order > 1 and an integer j, $1 \leqslant j \leqslant 4$, there exist infinitely many non-homeomorphic connected fixed-point-free graphs X such that* (i) *the automorphism group of X is isomorphic to G, and* (ii) *X has property P_j.*

The principal tool in deriving these results is the graph multiplication "$\times$" defined in **(5)**. A typical proof of the statements of (1.2) runs as follows:

(a) Construct a connected fixed-point-free prime graph X' (for a definition of "prime" **(5**, (1.3)**)**) whose automorphism group is isomorphic to G.

(b) Construct a connected prime graph $X'' \,\underline{\subset}\!\!|\!\!\underline{\simeq}\, X'$ with trivial automorphism

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group and certain graph theoretical properties P_j' which are such that the product $X' \times X''$ has property P_j.

(c) Apply **(5**, Theorem (3.2)**)**, with the result that

The automorphism group of $X' \times X''$ is isomorphic to the automorphism group of X', that is, isomorphic to G.

By a *graph* X we mean an ordered triple $X = (V, E, f)$, where V and E are two disjoint sets (the sets of vertices and edges of X), and f is a function of E into the set V^* of unordered pairs of distinct elements of V such that if $e^* \in V^*$ there is *at most one* $e \in E$ with $fe = e^*$. To indicate that V and E are the sets of vertices and edges of a graph $X = (V, E, f)$ we shall write $V = V(X)$, $E = E(X)$. Edges will be written as unordered pairs of vertices (indicated by brackets). To describe a graph X it clearly suffices to give the set $V(X)$ and a certain set $E(X)$ of unordered pairs of elements of $V(X)$. All graphs considered in this paper are *finite*.

Let X be a graph. By $G(X)$ we denote the *automorphism group* of X. We can consider $G(X)$ as a group of one-one mappings of $V(X)$ onto itself.

2. Definition and properties of the graph product.

(2.1) Definition: Let X, Y be graphs. By the *product* $X \times Y$ of X and Y is meant the following graph Z:

$$V(Z) = V(X) \times V(Y);$$

$[(x,y), (x',y')]$, where $x, x' \in V(X)$, $y, y' \in V(Y)$, is an edge of Z if $x = x'$ and $[y,y'] \in E(Y)$, or $y = y'$ and $[x,x'] \in E(X)$.

If we identify isomorphic graphs the multiplication thus defined is clearly *associative* and *commutative*. It has a *unit*, viz. the graph consisting of a single vertex and no edge.

(2.2) Lemma. *The product of connected graphs is connected. The product of any graph by a disconnected graph is disconnected.*

(2.3) Lemma. *If X is m-ply connected, and Y is n-ply connected, then $X \times Y$ is $(m + n)$-ply connected.*

Proof. We shall use a theorem of Whitney **(6**, Theorem 7**)**. X is m-ply connected implies: Given any pair of distinct vertices x, x' of X there exist m paths X_j of X such that

$$V(X_j) \cap V(X_k) = \{x, x'\}, \qquad j \neq k.$$

Y is n-ply connected means: Given any pair of distinct vertices y, y' of Y there exist n paths Y_j of Y such that

$$V(Y_j) \cap V(Y_k) = \{y, y'\}, \qquad j \neq k.$$

To show: Given any pair of distinct vertices (x,y), (x',y') of $Z = X \times Y$ there exist $m + n$ paths Z_j of Z such that

$$V(Z_j) \cap V(Z_k) = \{(x,y), (x',y')\}, \qquad j \neq k.$$

We have to consider two cases.

Case (1). Given (x,y), $(x',y') \in V(Z)$, where $x \neq x'$, $y \neq y'$. At most one of the paths X_j (Y_j) consists of a single edge. In that case let the notation be so chosen that X_m (Y_n) is that path. Let

$$\begin{aligned} V(X_j) &= \{x, x_2^{(j)}, x_3^{(j)}, \ldots, x'\}, && j \leqslant m; \\ V(Y_j) &= \{y, y_2^{(j)}, y_3^{(j)}, \ldots, y'\}, && j \leqslant n. \end{aligned}$$

Define paths Z_j, Z_{m+k} of Z as follows:

$$\begin{aligned} V(Z_j) &= \{(x,y), (x_2^{(j)},y), (x_2^{(j)},y_2^{(n)}), (x_2^{(j)},y_3^{(n)}), \ldots, \\ &\qquad (x_2^{(j)},y'), (x_3^{(j)},y'), \ldots, (x',y')\}, && j \leqslant m-1; \\ V(Z_m) &= \{(x,y), (x_2^{(m)},y), \ldots, (x',y), (x',y_2^{(n)}), \ldots, (x',y')\}; \\ V(Z_{m+k}) &= \{(x,y), (x,y_2^{(k)}), (x_2^{(m)},y_2^{(k)}), (x_3^{(m)},y_2^{(k)}), \ldots, \\ &\qquad (x',y_2^{(k)}), (x'y_3^{(k)}), \ldots, (x',y')\}, && k \leqslant n-1; \\ V(Z_{m+n}) &= \{(x,y), (x,y_2^{(n)}), \ldots, (x,y'), (x_2^{(m)},y'), \ldots, (x',y')\}. \end{aligned}$$

Case (2). Given (x,y), $(x',y) \in V(Z)$, where $x \neq x'$. Let y' be any vertex of Y distinct from y. Using the same notation as in case (1), define Z_j, Z_{m+k} as follows:

$$\begin{aligned} V(Z_j) &= \{(x,y), (x_2^{(j)},y), \ldots, (x',y)\}, && j \leqslant m; \\ V(Z_{m+k}) &= \{(x,y), (x,y_2^{(k)}), (x_2^{(m)},y_2^{(k)}), (x_3^{(m)},y_2^{(k)}), \ldots, \\ &\qquad (x',y_2^{(k)}), (x',y)\}, && k \leqslant n. \end{aligned}$$

In both cases the $m + n$ paths Z_j, Z_{m+k} of Z have the required properties.

(2.4) LEMMA. *Let X, Y be graphs of connectivity m and n respectively. If there is an $x \in V(X)$ of degree m, and a $y \in V(Y)$ of degree n, then the connectivity of $X \times Y$ is $m + n$.*

Proof. Let V_x, V_y, $V_{(x,y)}$ be the sets of those vertices of X, Y and $X \times Y$ which are joined with $x \in V(X)$, $y \in V(Y)$, and $(x,y) \in V(X \times Y)$ respectively. Then the definition of the graph product implies that

$$V_{(x,y)} = (V_x \times \{y\}) \cup (\{x\} \times V_y).$$

Hence the degree of (x,y) in $X \times Y$ is $m + n$. It follows that every subgraph I of $X \times Y$ with $V(I) = V_{(x,y)}$ is an isthmoid[1] of order $m + n$ of $X \times Y$ (with one component of $(X \times Y) - I$ consisting of the vertex (x,y) alone). Hence the connectivity of $X \times Y$ is $\leqslant m + n$. By (2.3) the connectivity of $X \times Y$ is $\geqslant m + n$, and this proves Lemma (2.4).

[1]An *isthmoid* of a graph X is a subgraph I of X such that $X - I$ is disconnected. $X - I$ is the maximal subgraph X' of X with $V(X') = V(X) - V(I)$.

(2.5) LEMMA. *If X and Y are regular of degree m and n respectively, then $X \times Y$ is regular of degree $m + n$.*

The proof of this Lemma is contained in the proof of (2.4).

(2.6) LEMMA. *Let $\chi(X)$, $\chi(Y)$, $\chi(X \times Y)$ be the chromatic numbers of X, Y and $X \times Y$ respectively. Then $\chi(X \times Y) = \max(\chi(X), \chi(Y))$.*

Proof. The maximal subgraphs X_y, Y_x of $X \times Y$ with

$$V(X_y) = V(X) \times \{y\},\ y \in V(Y),$$
$$V(Y_x) = \{x\} \times V(Y),\ x \in V(X),$$

are isomorphic to X and Y respectively. Hence $\chi(X \times Y) \geqslant m$, where $m = \max(\chi(X), \chi(Y))$. Let c_X, c_Y be m-colorings of X and Y respectively (an m-coloring of X is a function c_X of $V(X)$ into J_m, the group of integers (mod m), such that $[x,x'] \in E(X)$ implies $c_X(x) \neq c_X(x')$; likewise for Y). Define a function c of $V(X \times Y)$ into J_m by

$$c(x,y) = c_X(x) + c_Y(y),\ x \in V(X),\ y \in V(Y).$$

c is an m-coloring of $X \times Y$. To show:

$$[(x,y),\ (x',y')] \in E(X \times Y) \rightarrow c(x,y) \neq c(x',y');$$

$[(x,y),(x',y')] \in E(X \times Y) \rightarrow x = x'$, $[y,y'] \in E(Y)$, or $y = y'$, $[x,x'] \in E(X)$. It suffices to consider the first case: $x = x' \rightarrow c_X(x) = c_X(x')$; $[y,y'] \in E(Y) \rightarrow c_Y(y) \neq c_Y(y')$. Hence

$$c(x,y) = c_X(x) + c_Y(y) \neq c_X(x') + c_Y(y') = c(x',y').$$

Since c is an m-coloring of $X \times Y$, it follows that $\chi(X \times Y) \leqslant m$.

(2.7) LEMMA. *Let X, Y, Z be connected graphs such that* (i) $\alpha_0(Z) \geqslant 2\,\alpha_0(X) - 2$ *(α_0 = number of vertices)*; (ii) *Z contains a Hamiltonian circuit H*; (iii) *Z is spanned by a graph $\bar{Y}$ homeomorphic to Y with $E(\bar{Y}) \cap E(H) \neq \square$ (= the empty set). Then $X \times Z$ is spanned by a graph $\tilde{Y}$ homeomorphic to Y.*

Proof. Let X' be a (connected) spanning tree of X, and let $E(X') = \{e_1, \ldots, e_{m-1}\}$, $m = \alpha_0(X)$. For each $x_i \in V(X')$ define $E_i = \{k | x_i$ is incident with e_k, $e_k \in E(X')\}$. Since X' is a tree, we can assume that $x_1 \in V(X')$ is of degree 1 in X'. Let

$$V(Z) = V(\bar{Y}) = \{z_1, \ldots, z_n\},\ n \geqslant 2m - 2,$$

and let the notation be so chosen that

(1) $E(H) = \{[z_1,z_2],\ [z_2,z_3],\ \ldots,\ [z_{n-1},z_n],\ [z_n,z_1]\}$, and
(2) $[z_1,z_2] \in E(\bar{Y}) \cap E(H)$. Let H_i, $\bar{Y}_i$ be given by

$$V(H_i) = V(\bar{Y}_i) = \{x_i\} \times V(\bar{Y}),$$
$$E(H_i) = \{[(x_i,z_j),\ (x_i,z_k)] | [z_j,z_k] \in E(H)\},$$
$$E(\bar{Y}_i) = \{[(x_i,z_j),\ (x_i,z_k)] | [z_j,z_k] \in E(\bar{Y})\}.$$

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Notice that

$$\bigcup_{i=1}^{m} V(H_i) = V(X \times Z).$$

Consider the following subgraph P of $X \times Z$:

$$V(P) = \bigcup_{i=2}^{m} V(H_i) \cup \{(x_1,z_1), (x_1,z_2)\},$$

$$E(P) = \bigcup_{i=2}^{m} (E(H_i) - \{[(x_i,z_{2k-1}), (x_i,z_{2k})] | k \in E_i\}) \cup$$
$$\bigcup_{k=1}^{m-1} \{[(x^{(k)},z_{2k-1}), (y^{(k)},z_{2k-1})], [(x^{(k)},z_{2k}), (y^{(k)},z_{2k})]\},$$

where $[x^{(k)},y^{(k)}] = e_k (k = 1, \ldots, m-1)$. It can be easily checked that (1) P is connected, (2) the degree of (x_1,z_1) and (x_1,z_2) in P is 1, (3) the degree in P of any other vertex of P is 2. Hence P is a path joining (x_1,z_1) and (x_1,z_2), and containing all vertices of $(X \times Z) - \bar{Y}_1$. Now let $\tilde{Y}$ be given by

$$V(\tilde{Y}) = V(X \times Z),$$
$$E(\tilde{Y}) = (E(\bar{Y}_1) - \{[(x_1,z_1), (x_1,z_2)]\}) \cup E(P).$$

Then clearly $\tilde{Y}$ spans $X \times Z$, and is homeomorphic to Y.

(2.8) LEMMA. *Every connected graph X containing a vertex or an edge which is not contained in a 4-circuit of X is prime.*

Proof. Suppose $X = Y \times Z$, where $\alpha_0(Y), \alpha_0(Z) \geqslant 2$. Let

$$(y,z) \in V(X), y \in V(Y), z \in V(Z).$$

Since X is connected, both Y and Z are connected; hence by (2.5) the degree of y in Y and the degree of z in Z must be $\geqslant 1$. Let y', z' be vertices joined with y and z in Y and Z respectively. Then the subgraph C of X given by

$$V(C) = \{(y,z), (y,z'), (y',z'), (y',z)\},$$
$$E(C) = \{[(y,z), (y,z')], [(y,z'), (y',z')], [(y',z'), (y',z)], [(y',z), (y,z)]\}$$

is a 4-circuit of X containing (y,z). The same proof applies to edges.

(2.9) LEMMA. *The product of a fixed-point-free graph X by any graph Y is fixed-point-free.*

Proof. Let $x \in V(X)$. Since X is fixed-point-free, there is a $\phi \in G(X)$ such that $\phi x \neq x$. Then the function ϕ^* given by $\phi^*(x,y) = (\phi x,y)$ is an automorphism of $X \times Y$, and $\phi^*(x,y) \neq (x,y)$ for all $y \in V(Y)$. Hence $X \times Y$ is fixed-point-free.

(2.10) LEMMA. **(5**, (3.2)**)**. *If X and Y are relatively prime, then $G(X \times Y) \cong G(X) \times G(Y)$.*

For a definition of "relatively prime" cf. **(5**, (1.3)**)**.

3. Existence of graphs with given group and given graph theoretical properties. We shall now prove the four theorems stated as Theorem (1.2). It should be emphasized that the constructions given in this paragraph are by no means the only possible ones. They have been chosen mainly to demonstrate the usefulness of graph multiplication.

(3.1) *Definition:* Let X be a graph without isolated vertices. By $\tilde{X}$ we mean the graph defined by

(1) $V(\tilde{X}) = \{(x,e) \in V(X) \times E(X) | x \text{ is incident with } e\}$;

(2) given (x,e), $(x',e') \in V(\tilde{X})$, then $[(x,e), (x',e')] \in E(\tilde{X})$ if and only if $x = x'$, $e \neq e'$, or $x \neq x'$, $e = e'$.

The following properties of $\tilde{X}$ are obvious from the definition.

(3.2) LEMMA. *Let X be as in* (3.1). (i) *If X is connected or cyclically connected, then so also is $\tilde{X}$.* (ii) *If X is regular of degree $n \geqslant 1$, then $\tilde{X}$ is likewise of degree n.* (iii) *If no component of X is a circuit, then X and $\tilde{X}$ are not homeomorphic. If X is an n-circuit, then $\tilde{X}$ is a $2n$-circuit.* (iv) *If X is connected, then $\tilde{X}$ is prime.*

(3.3) LEMMA. *Let X be as in* (3.1). *If X is fixed-point-free and without fixed edge*[2], *then so also is $\tilde{X}$. If no component of X is a circuit, then $G(X) \cong G(\tilde{X})$.*

Proof. Given $\phi \in G(X)$ define $\tilde{\phi}$: $V(\tilde{X}) \to V(\tilde{X})$ by $\tilde{\phi}(x,e) = (\phi x, \phi e)$. Then clearly $\tilde{\phi} \in G(\tilde{X})$, and $\phi \to \tilde{\phi}$ is an isomorphism of $G(X)$ into $G(\tilde{X})$.

Define an equivalence relation $\sim$ on $V(\tilde{X})$ by $(x,e) \sim (x',e')$ if and only if $x = x'$. Let $\mathbf{X}$ be the graph given by

(i) $V(\mathbf{X}) = V(\tilde{X})/\sim$;

(ii) $[\mathbf{x},\mathbf{x}'] \in E(\mathbf{X})$, where $\mathbf{x}$, $\mathbf{x}' \in V(\mathbf{X})$, if and only if there exist $(x, e) \in \mathbf{x}$ and $(x',e') \in \mathbf{x}'$ such that $[(x,e), (x',e')] \in E(\tilde{X})$.

Then clearly $\mathbf{X} \cong X$. By p denote the natural projection of $V(\tilde{X})$ onto $V(\mathbf{X})$.

$G(\tilde{X})$ preserves the relation $\sim$. Let $(x_1,e_1) \sim (x_2,e_2)$, so that $x_1 = x_2$, and let $\tilde{\psi} \in G(\tilde{X})$. Put $\tilde{\psi}(x_i,e_i) = (x_i',e_i')$ $(i = 1,2)$. To show that $x_1' = x_2'$. $(x_1,e_1) \sim (x_2,e_2) \to (x_1,e_1) = (x_2,e_2)$ or $\epsilon = [(x_1,e_1), (x_2,e_2)] \in E(\tilde{X})$. Hence $(x_1',e_1') = (x_2',e_2')$, and hence $x_1' = x_2'$, or $\epsilon' = [(x_1',e_1'), (x_2',e_2')] \in E(\tilde{X})$. In the latter case either

(1) $x_1' = x_2'$, $e_1' \neq e_2'$, or

(2) $x_1' \neq x_2'$, $e_1' = e_2'$.

We have to show that (2) leads to a contradiction. Assume (2). It is easily seen that then there is no 3-circuit of $\tilde{X}$ containing ϵ'. Hence there is no 3-circuit of $\tilde{X}$ containing ϵ. Therefore $x_1 = x_2$ is of degree 2 in X, which in turn implies that (x_i,e_i) $(i = 1,2)$ are of degree 2 in $\tilde{X}$. Since no component of X is a circuit, no component of $\tilde{X}$ is a circuit (cf. (3.2) (ii), (iii)). Hence $\tilde{X}$ contains a vertex

$$(y_1,e_{i_1})$$

[2]An edge e of X is *fixed*, if $\phi e = e$ for all $\phi \in G(X)$.

and a path $\tilde{P}$ with

$$V(\tilde{P}) = \{(y_1,e_{i1}), \ldots, (y_n,e_{in})\}, \quad E(\tilde{P}) = \{\epsilon_1, \ldots, \epsilon_{n-1}\},$$

such that

(α) $$(y_{n-1},e_{i_{n-1}}) = (x_1,e_1), \quad (y_n,e_{i_n}) = (x_2,e_2);$$

(β) $$(y_1,e_{i1})$$

is of degree $\neq 2$ in $\tilde{X}$;

(γ) $$(y_k,e_{ik})$$

is of degree 2 in $\tilde{X}$ for all $k \neq 1$.

We show that

$$n = 2m + 1,\ e_{i_{2k-1}} = e_{i_{2k}},\ y_{2k} = y_{2k+1}, \qquad k \leqslant m.$$

For the proof notice that $(y,e) \in V(\tilde{X})$ and $y \in V(X)$ are always of the same degree. $\epsilon_1 \in E(\tilde{X})$ implies

(a) $$y_1 = y_2,\ e_{i_1} \neq e_{i_2},$$

or

(b) $$y_1 \neq y_2,\ e_{i_1} = e_{i_2}.$$

(a) is impossible because

$$(y_1,e_{i_1}) \text{ and } (y_2,e_{i_2}),$$

and hence y_1 and y_2, have different degrees. Hence (b) must hold. $\epsilon_2 \in E(\tilde{X})$ implies

(c) $$y_2 = y_3,\ e_{i_2} \neq e_{i_3},$$

or

(d) $$y_2 \neq y_3,\ e_{i_2} = e_{i_3}.$$

Suppose (d) holds. Then (b) and (d) imply

$$e_{i_1} = e_{i_2} = e_{i_3},$$

which is incident with y_1, y_2, y_3. Two of these vertices must be equal: (b) and (d) imply $y_1 = y_3$. But then

$$(y_1,e_{i_1}) = (y_3,e_{i_3}),$$

so that

$$(y_2,e_{i_2})$$

is of degree 1, a contradiction. Hence (c) must hold. The rest of the assertion follows in a similar way by induction. We shall express the fact that $n = 2m + 1$ by saying that the "distance" of ϵ from

$$(y_1,e_{i_1})$$

is odd. Since ϵ and ϵ' are similar under $\tilde{\psi}$, there is a vertex

$$(z_1,e_{j_1})$$

and a path $\tilde{Q}$, similar under $\tilde{\psi}$ to

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$$(y_1, e_{i1})$$

and $\tilde{P}$ respectively, and such that (α), (β), (γ) are satisfied with respect to ϵ'. By the same argument as above it then follows that the distance of ϵ' from

$$(z_1, e_{j1})$$

is even. But this contradicts the similarity of ϵ and ϵ'.

Given $\tilde{\psi} \in G(\tilde{X})$ define

$$\psi\colon V(\mathbf{X}) \to V(\mathbf{X}) \text{ by } \psi x = p\tilde{\psi}(x,e),$$

where $(x,e) \in p^{-1}\mathbf{x}$, $\mathbf{x} \in V(\mathbf{X})$. Since $\tilde{\psi}$ preserves equivalence, ψ is in $G(\mathbf{X})$, and $h\colon G(\tilde{X}) \to G(\mathbf{X})$ given by $h\tilde{\psi} = \psi$ is a homomorphism. Consider

$$\operatorname{Ker} h = \{\tilde{\psi} | \tilde{\psi}(x,e) \sim (x,e)\}.$$

Let $e = [x,y] \in E(X)$. Then $[(x,e), (y,e)] \in E(\tilde{X})$. For $\tilde{\psi} \in \operatorname{Ker} h$ put

$$\tilde{\psi}(x,e) = (x_1,e_1), \quad \tilde{\psi}(y,e) = (y_1,e_2).$$

Then $x = x_1$, $y = y_1$, and $[(x_1,e_1), (y_1,e_2)] \in E(\tilde{X})$. Hence

(1) $x_1 = y_1$, $e_1 \neq e_2$, or

(2) $x_1 \neq y_1$, $e_1 = e_2 = [x_1,y_1]$.

(1) is impossible since it implies $x = y$; (2) implies $e_1 = e_2 = e$, so that $\tilde{\psi}(x,e) = (x,e)$. Hence $\operatorname{Ker} h = 1$, and h is an isomorphism.

The assertion about fixed vertices and edges follows from the fact that $\tilde{\phi}$ given by $\tilde{\phi}(x,e) = (\phi x, \phi e)$ is in $G(\tilde{X})$.

All constructions in this paragraph are based on the following theorem:

(3.4) THEOREM. *Given a finite group G of order > 1, there exist infinitely many non-homeomorphic cyclically connected fixed-point-free prime graphs X_i containing no fixed edge, and such that $G(X_i) \cong G$.*

Proof. By (**3**, Theorem 4.1) there exists at least one such graph, X_1. By induction, let $X_{i+1} = \tilde{X}_i$, $i \geqslant 1$. Then by (3.2) and (3.3) all X_i have the required properties. Since X_1 is regular of degree 3, no X_i is a circuit.

(3.5) THEOREM. *Given a finite group G of order > 1 and a positive integer n, there exist infinitely many non-homeomorphic fixed-point-free graphs X of connectivity n whose automorphism group is isomorphic to G.*

Proof. Given any graph X denote the connectivity of X by $c(X)$. For $n = 1$, (3.5) has been proved in (**2**, §2). We can therefore assume that $n \geqslant 2$.

Case (1). $n = 2$. Let X' be a graph with the properties stated in (3.4). In particular, $c(X') \geqslant 2$. By subdividing each edge e of X' by a vertex x_e we obtain a graph X with $c(X) = 2$. X is prime, since no circuit of X is of order < 6 (cf. (2.8)). Since X' is not a circuit, $G(X) \cong G(X') \cong G$. X is fixed-point-free because X' is fixed-point-free and contains no fixed edge.

Case (2): $n \geqslant 3$. Let Y_k, $k \geqslant 1$, be the graph given by

$$V(Y_k) = \{0,1,\ldots,k+5\},$$
$$E(Y_k) = \{[0,1],\ [0,2],\ [2,3],\ [0,4],\ [4,5],\ldots,[k+4,\ k+5]\}.$$

Then (i) 1 is a vertex of degree 1 of Y_k; (ii) $c(Y_k) = 1$; (iii) $G(Y_k) = 1$; (iv) Y_k and $Y_{k'}$ are relatively prime if $k \neq k'$. It follows from (2.4) that

$$Y^{(m)} = Y_1 \times \ldots \times Y_m$$

is a graph of connectivity m, and from (2.10) that $G(Y^{(m)}) = 1$. By (2.5), $(1,\ldots,1)$ is a vertex of degree m of $Y^{(m)}$.

Let X be as in case (1). Then X and $Y^{(n-2)}$, $n \geqslant 3$, are relatively prime, and satisfy the hypotheses of (2,4). Hence $c(X \times Y^{(n-2)}) = n$, and by (2.10),

$$G(X \times Y^{(n-2)}) \cong G(X) \times G(Y^{(n-2)}) \cong G.$$

By (2.9), $X \times Y^{(n-2)}$ is fixed-point-free.

(3.6) THEOREM. *Given a finite group G of order > 1 and an integer $n \geqslant 2$, there exist infinitely many non-homeomorphic connected fixed-point-free graphs X of chromatic number n whose automorphism group is isomorphic to G.*

Proof. Case (1): $n = 2$. Let X be as in (3.5), case (1). Every circuit of X is of even order; hence by a well-known theorem (**4**, p. 170), $\chi(X) = 2$.

Case (2). $n \geqslant 3$. Let P_i, $i = 1,\ldots,n$, be the graph with

$$V(P_i) = \{p_1,\ldots,p_i\}, \quad E(P_i) = \{[p_j,p_{j+1}],\ j = 1,\ldots,i-1\}.$$

Consider the complete n-graph $C^{(n)}$. Denote its vertices by $x_1,\ldots,x_n$. Identify the vertex x_i of $C^{(n)}$ with the vertex p_i of P_i, $i = 1,\ldots,n$. The graph C_n so obtained is prime (since it is connected, and contains vertices which do not belong to any 4-circuit of C_n), has chromatic number $\chi(C_n) = n$, and $G(C_n) = 1$.

Let X be as in case (1). Then by (2.10), $G(X \times C_n) \cong G$, by (2.9), $X \times C_n$ is fixed-point-free; and by (2.6), $\chi(X \times C_n) = n$.

(3.7) THEOREM. *Given a finite group G of order > 1 and an integer $n \geqslant 3$, there exist infinitely many non-homeomorphic connected fixed-point-free graphs X which are regular of degree n, and whose automorphism group is isomorphic to G.*

Proof. For $n = 3$ part of (3.7) has been proved in (**3**). The proof given here for $n \geqslant 4$ is patterned after that of (**3**, Theorem 4.1).

We first show that there exists an infinite sequence of cyclically connected non-isomorphic prime graphs $Y_1, Y_2, \ldots$, which are regular of degree 3, and for which $G(Y_i) = 1 (i = 1,2,\ldots)$. By (**3**, Theorem 2.3) there exists at least one such graph Y_1. By induction, let $Y_{i+1} = \tilde{Y}_i$, $i \geqslant 1$. Then by (3.2), (3.3) the Y_i's have the required properties.

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Let X be a fixed-point-free graph of degree n which is relatively prime to $Y_1, \ldots, Y_k$, and such that $G(X) \cong G$, where G is a given finite group of order > 1. By (2.9),

$$W_k = X \times Y_1 \times \ldots \times Y_k$$

is fixed-point-free, and by (2.10), $G(W_k) \cong G$. By (2.5), W_k is regular of degree $n + 3k$. Hence (3.7) is proved if we show the following: There exist infinitely many non-isomorphic connected fixed-point-free graphs $X_j^{(n)}$ $(j = 1,2, \ldots)$, which are regular of degree $n = 3,4,5$, relatively prime to all Y_i, and such that $G(X_j^{(n)}) \cong G$ for all j.

Let $G = \{\tau\}$, and let $X_1^{(3)}$ be the graph given in **(3**, Theorem 4.1**)**. $V(X_1^{(3)}) = \{x_j^\tau, j \leqslant m, \tau \in G\}$, where $m = 2h + 4$, $E(X_1^{(3)})$ as given in **(3**, p. 374**)**, by quadratic forms. Define $X_1^{(4)}$, $X_1^{(5)}$ as follows:

$$\begin{aligned} V(X_1^{(4)}) &= V(X_1^{(3)}) \cup \{y_j^\tau, j \leqslant m, \tau \in G\}, \\ E(X_1^{(4)}) &= E(X_1^{(3)}) \cup \{[x_j^\tau, y_j^\tau]\ (j \leqslant m),\ [y_j^\tau, y_{j+1}^\tau],\ (j \leqslant m-1), \\ &\qquad [y_j^\tau, y_{m-j+1}^\tau]\ (j \leqslant h+1)\ [y_1^\tau\ y_{h+2}^\tau],\ [y_{h+3}^\tau, y_m^\tau], \\ &\qquad \tau \in G\}; \\ V(X_1^{(5)}) &= V(X_1^{(4)}) \cup \{z_j^\tau\ \ (j \leqslant m), \tau \in G\}, \\ E(X_1^{(5)}) &= E(X_1^{(4)}) \cup \{[x_j^\tau, z_j^\tau], [y_j^\tau, z_j^\tau]\ (j \leqslant m),\ [z_j^\tau, z_{j+1}^\tau] \\ &\qquad (j \leqslant m-1),\ [z_j^\tau, z_{m-j+1}^\tau]\ (j \leqslant h+1),\ [z_1^\tau, z_{h+3}^\tau],\ [z_{h+2}^\tau, z_m^\tau], \tau \in G\}. \end{aligned}$$

It is easily checked that $X_1^{(n)}$ $(n = 4,5)$ is of degree n, and that if $\phi \in G(X_1^{(n)})$, and

$$\phi x_1^{\tau_0} = x_1^{\tau_0}$$

for some $\tau_0 \in G$, then $\phi x_j^\tau = x_j^\tau$, $\phi y_j^\tau = y_j^\tau$, $\phi z_j^\tau = z_j^\tau$, for $j \leqslant m$ and $\tau \in G$. An argument similar to that in **(3)** then shows that $G(X_1^{(n)}) \cong G (n = 3,4,5)$.

By induction, let $X_{j+1}^{(n)} = \tilde{X}_j^{(n)}$ $(j \geqslant 1, n = 3,4,5)$. Then by (3.2), (3.3), $X_{j+1}^{(n)}$ is prime, regular of degree n, fixed-point-free, and

$$G(X_{j+1}^{(n)}) \cong G(X_j^{(n)}) \cong G, \qquad j \geqslant 1.$$

Clearly $X_j^{(n)}$ and Y_i are non-isomorphic for $i,j = 1,2, \ldots$, and $n = 3,4,5$; hence the $X_j^{(n)}$ are relatively prime to the Y_i, and this is what we set out to prove.

(3.8) THEOREM. *Let Y be a connected graph, and let G be a finite group of order > 1. Then there exist infinitely many non-homeomorphic fixed-point-free graphs X such that* (i) *$G(X) \cong G$, and* (ii) *X is spanned by a graph $\tilde{Y}$ homeomorphic to Y.*

Proof. Let $V(Y) = \{y_1, \ldots, y_r\}$. Take a spanning tree T of Y. Let e_1 be an edge of T incident with y_1. Subdivide e_1 by a new vertex z_1. Let T_1, Y_1 be the graphs obtained by this subdivision from T and Y respectively. Let

e_2 be an edge of T_1 incident with y_2. Subdivide e_2 by a new vertex z_2, obtaining graphs T_2, Y_2. Continuing this process we finally obtain a graph Y_r with

$$V(Y_r) = \{y_1,z_1,y_2,z_2, \ldots , y_r,z_r\}, [y_i,z_i] \in E(Y_r), \qquad i \leqslant r.$$

Define H, $\bar{Y}$, Z by

$$V(H) = V(\bar{Y}) = V(Z) = V(Y_r) \cup \bigcup_{i=1}^{r} \{y_{i1}, \ldots , y_{is_i}\},$$

$$E(H) = E \cup \{[y_1,z_1], \ [z_1,y_2], \ [y_2,z_2], \ldots , [z_{r-1},y_r], [y_r,z_r], [z_r,y_1]\},$$

$$E(\bar{Y}) = E \cup E(Y_r), E(Z) = E(H) \cup E(Y_r),$$

where

$$E = \bigcup_{i=1}^{r} \{[y_i,y_{i1}], [y_{i1},y_{i2}], \ldots , [y_{is_i-1},y_{is_i}], [y_{is_i},z_i]\},$$

$s_1, \ldots , s_r$ being positive integers to be chosen as specified below. Clearly $\bar{Y}$ spans Z and is homeomorphic to Y. H is a Hamiltonian circuit of Z, and $E(H) \cap E(\bar{Y}) \neq \square$. In Z each z_i is of degree 3. Let P_i be that path of H which joins z_i and z_{i+1}, and contains y_{i+1} (subscripts to be taken modulo r). All vertices of P_i except z_i, z_{i+1}, and possibly y_{i+1} are of degree 2 in Z. Let the s_i be so chosen that (1) $s_i > a$, where a is a given positive integer, and (2) $\alpha_0(P_{i+1}) > \alpha_0(P_i) > \alpha_0(P)$, for $i = 1, \ldots , r - 1$, and all paths P of Z not containing a vertex y_{ij}. It follows from (2) that Z is prime (since no y_{ij} belongs to a 4-circuit of Z), and that $G(Z) = 1$.

Given a finite group G of order > 1, let X be a graph with the properties stated in (3.4), and let Z be the graph constructed above with $a = 2\alpha_0(X) - 2$. Then X, Y, Z satisfy the hypotheses of (2.7), and it follows that $X \times Z$ is spanned by a graph $\tilde{Y}$ homeomorphic to Y. By (2.9), $X \times Z$ is fixed-point-free. X and Z are non-isomorphic, and since both graphs are prime, $G(X \times Z) \cong G(X) \cong G$.

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