MULTIPLY HARMONIC FUNCTIONS

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1. Introduction

Let Ω and Ω' be two locally compact, connected Hausdorff spaces having countable bases. On each of the spaces is defined a system of harmonic functions satisfying the axioms of M. Brelot [2]. The following is the description of such a system. To each open set of Ω is assigned a vector space of finite continuous functions, called the harmonic functions, on this set. An open set V is called regular if it is non-empty, relatively compact, and if, for any finite continuous function f on the boundary ∂V of V, there exists a unique continuous function on $\overline{V}$, equal to f on ∂V and a harmonic function on V, non-negative if f is non-negative. The restriction to V of this function will be denoted by H_f^V. For any $x \in V$, the functional $f \to H_f^V(x)$ is a non-negative Radon measure ρ_x^V on ∂V. The systems of harmonic functions are assumed to satisfy the following three fundamental axioms.

I. Axiom (1). *The harmonic functions have the sheaf property.*

II. Axiom (2). *Ω and Ω' have bases consisting of their regular domains.*

III. Axiom (3). *On every domain of the spaces, any harmonic function $u \geqq 0$ has the property that either $u \equiv 0$ or u is nowhere zero on the domain of definition. Further, for every point of the domain, the positive harmonic functions taking the value 1 at this point are equi-continuous at this point.*

Definition. A lower semi-continuous, extended real valued function v on an open set V is called *hyperharmonic*, if v never takes the value $-\infty$ and, for any regular domain $W \subset \overline{W} \subset V$, and $x \in W$,

$$v(x) \geqq \int v(y)\, \rho_x^W(dy).$$

A hyperharmonic function on an open set is called a superharmonic function

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if it is not identically $+\infty$ on any connected component. A superharmonic function $p \geqq 0$, is called a potential if any harmonic function $u \leqq p$ also satisfies $u \leqq 0$.

Let there be potentials >0 on each one of the spaces Ω and Ω'. (This assumption is made to avoid trivialities.)

The object of this paper is to consider functions on $\Omega \times \Omega'$ which are superharmonic in each variable for every fixed value of the other. Let δ be an open subset of $\Omega \times \Omega'$, let $MH(\delta)$ [resp. $MS(\delta)$] be the class of all finite continuous (resp. lower semi-continuous and $> -\infty$) functions on δ, that are harmonic (resp. superharmonic) in each variable for every fixed value of the other. The paper can be divided into two parts. The first part deals with the general properties of elements in MH and MS. It is shown that the product of regular domains (which form a base for $\Omega \times \Omega'$) have a special role to play in the discussions. A convergence property for any increasing directed family of multiply harmonic functions (viz. elements of MH) is demonstrated. This leads to the important result that any real valued function v, on any open set $\delta \subset \Omega \times \Omega'$, and hyperharmonic in each variable for every fixed value of the other, is lower semi-continuous, if it is lower bounded on every compact set.

The second part deals with the integral representation of positive multiply harmonic functions on $\Omega \times \Omega'$. It is proved that $(MH)^{+}(\Omega \times \Omega')$ is a lattice for the natural order and that it has a compact base (for the compact convergence topology). Choquet's theorem on integral representation then assures the existence of a unique measure ν_u, corresponding to each positive multiply harmonic function u, on the compact base, charging only the extreme elements such that $u = \int H \nu_u(dH)$. The set of extreme elements of the compact base is shown to be homeomorphic to $\Delta_1 \times \Delta_1'$, where Δ_1 (resp. Δ_1') is the fine boundary [or equivalently the set of minimal harmonic functions belonging to a compact base of the positive harmonic functions] on Ω (respectively Ω').

These results are true for multiply harmonic functions in any finite number of variables; [viz. the functions that are harmonic in each variable for every fixed value of all the other variables]. The same proofs carry over without any substantial change. We have considered here the case of two variables, for the sake of simplicity.

2. For any open set $\omega \subset \Omega \times \Omega'$, let $MH(\omega)$ be the class of all multiply harmonic functions on ω. That is, $MH(\omega)$ is the class of all finite continuous function on ω, that are harmonic in each variable for every fixed value of the other. Corresponding to $\Omega\times\Omega'$, the class will be denoted simply by MH. It is clear that $MH(\omega)$ is a real vector space, for every ω. We shall first prove the following three fundamental properties of these classes. The first of these three properties is an immediate consequence of the local nature of the definition of harmonic functions on Ω and Ω'.

P_1. (*Sheaf property*): *If $u\in MH(\omega)$, then u belongs to $MH(\delta)$ for every open subset δ of ω. Conversely, if u is a finite continuous function on an open set ω and if $u\in MH(\delta)$ for some neighbourhood δ of each point of ω, then u belongs to $MH(\omega)$.*

P_2. (*A base for open sets of $\Omega\times\Omega'$*)

Let $\omega\subset\Omega$ and $\omega'\subset\Omega'$ be regular domains of the respective spaces and $\Gamma=\partial\omega\times\partial\omega'$. For any finite continuous function f on Γ, there exists a function Γ_f on $\bar{\omega}\times\bar{\omega}'$, having the following properties.

1) *$\Gamma_f\geqq 0$ if $f\geqq 0$.*

2) *$\Gamma_f=f$ on Γ and Γ_f is continuous on $\bar{\omega}\times\bar{\omega}'$.*

3) *Γ_f belongs to $MH(\omega\times\omega')$.*

and

4) *$\Gamma_f(x, y)$ is a harmonic function of $x\in\omega$ for every fixed $y\in\partial\omega'$ and a harmonic function of $y\in\omega'$ for every $x\in\partial\omega$.*

Moreover, Γ_f is uniquely determined by f, subject to the above four conditions.

Proof: Uniqueness. Suppose that Γ_f^1 and Γ_f^2 are two functions on $\bar{\omega}\times\bar{\omega}'$, corresponding to a finite continuous function f on Γ, verifying the above four conditions.

For any fixed $y'\in\partial\omega'$, $\Gamma_f^1(x, y')$ and $\Gamma_f^2(x, y')$ are two finite continuous functions on $\bar{\omega}$, harmonic in ω and further $\Gamma_f^1(x, y')=f(x, y')=\Gamma_f^2(x, y')$, for every $x\in\partial\omega$. Since ω is a regular domain, we have, $\Gamma_f^1(x, y')=\Gamma_f^2(x, y')$, for every $x\in\bar{\omega}$. This is true for all $y\in\partial\omega'$. Now, consider $\Gamma_f^1(x, y)$ and $\Gamma_f^2(x, y)$, for any fixed $x\in\bar{\omega}$. By a similar argument, it can be easily seen that $\Gamma_f^1(x, y)=\Gamma_f^2(x, y)$, for every $y\in\bar{\omega}'$ (and for every $x\in\bar{\omega}$). This proves the uniqueness of Γ_f.

Existence. Set, for every finite continuous function f on Γ,

$$
\begin{aligned}
\Phi_f(x, y) &= \int f(z, z')\rho_x^{\omega}(dz)\rho_y^{\omega'}(dz') && \text{if } (x, y)\in\omega\times\omega'. \\
&= \int f(x, z')\rho_y^{\omega'}(dz') && \text{if } (x, y)\in\partial\omega\times\omega'. \\
&= \int f(z, y)\rho_x^{\omega}(dz) && \text{if } (x, y)\in\omega\times\partial\omega'. \\
&= f(x, y) && \text{if } (x, y)\in\Gamma.
\end{aligned}
$$

We shall show that Φ_f meets our requirements. The conditions (1) and (4) are obviously true, and by definition $\Phi_f = f$ on Γ. Let now g and g' be finite continuous functions on $\partial\omega$ and $\partial\omega'$ respectively. Then $\Phi_{gg'}$ is equal to GG', where $G(x) = H_g^{\omega}(x)$ for $x\in\omega$ and $G(x) = g(x)$ for $x\in\partial\omega$ and G' defined similarly. Hence $\Phi_{gg'}$ is continuous on $\bar\omega\times\bar\omega'$. It follows that Φ_f is continuous on $\bar\omega\times\bar\omega'$, for any f which is a finite linear combination of elements of the form gg', since Φ_f is then equal to a finite linear combination of $\Phi_{gg'}$. Since every finite continuous function on Γ can be approximated uniformly by functions of the form f, to complete the proof of the condition (2), it is enough to show that if $\{f_n\}$ is a sequence of finite continuous functions on Γ, converging uniformly to f and such that Φ_{f_n} is continuous on $\bar\omega\times\bar\omega'$ then Φ_f is continuous. Let $M\geqq 1$ be a real number such that $H_1^{\omega}\leqq M$ on $\bar\omega$ and $H_1^{\omega'}\leqq M$ on $\bar\omega'$. Given $\varepsilon>0$, there exists an integer N such that $|f_n - f| < \frac{\varepsilon}{M^2}$ for $n\geqq N$, uniformly on Γ. It is easily seen that if $n\geqq N$, then for every $(x, y)\in\omega\times\omega'$, $|\Phi_{f_n}(x, y) - \Phi_f(x, y)| < \varepsilon$, for every $(x, y)\in\omega\times\partial\omega'\cup\partial\omega\times\omega'$, $|\Phi_{f_n}(x, y) - \Phi_f(x, y)| < \frac{\varepsilon}{M}\leqq \varepsilon$ and $|f_n - f| < \frac{\varepsilon}{M^2}\leqq\varepsilon$ on Γ. This shows that Φ_f is the uniform limit on $\bar\omega\times\bar\omega'$, of Φ_{f_n}; and hence Φ_f is also continuous on $\bar\omega\times\bar\omega'$.

It remains to verify the condition (3). Let $y_0\in\omega'$ and δ be any regular domain $\subset\bar\delta\subset\omega$. Then

$$
\begin{aligned}
\int \Phi_f(z, y_0)\rho_x^{\delta}(dz) &= \int\rho_x^{\delta}(dz)\iint f(\xi, \eta)\,\rho_z^{\omega}(d\xi)\,\rho_{y_0}^{\omega'}(d\eta) \\
&= \int\rho_x^{\delta}(dz)\int\rho_z^{\omega}(d\xi)\int f(\xi, \eta)\,\rho_{y_0}^{\omega'}(d\eta) \\
&= \int\rho_x^{\delta}(dz)\int\Phi_f(\xi, y_0)\,\rho_z^{\omega}(d\xi) && \text{(by definition)} \\
&= \int\rho_x^{\delta}(dz)\, H_{\Phi f(\cdot, y_0)}^{\omega}(z) \\
&= H_{\Phi f(\cdot, y_0)}^{\omega}(x) \\
&= \int\Phi_f(\xi, y_0)\,\rho_x^{\omega}(d\xi)
\end{aligned}
$$

$$= \Phi_f(x, y_0) \qquad \text{(by definition)}$$

This shows that $\Phi_f(x, y)$ is harmonic in $x \in \omega$ for every fixed $y \in \omega'$ and it can be similarly proved that $\Phi_f(x, y)$ is harmonic in $y \in \omega'$ for every fixed $x \in \omega$. This completes the proof.

LEMMA 1. *Let u be a finite valued function defined on an open subset δ of $\Omega \times \Omega'$. If $u \geqq 0$ and harmonic in each variable for each fixed value of the other variable, then u is continuous on δ.*

Proof: Let (x_n, y_n) in δ converge to $(x', y') \in \delta$. For every fixed y, $u(x_n, y) \to u(x', y)$. Let $V \subset \Omega$ and $V' \subset \Omega'$ be regular domains of the respective spaces such that $(x', y') \in V \times V' \subset \overline{V} \times \overline{V}' \subset \delta$. We can assume that $(x_n, y_n) \in V \times V'$ for all $n \geqq 1$. Now $\{u(x_n, y)\}$ is a sequence of positive harmonic functions on V' and further $u(x_n, y) \to u(x', y)$ for every $y \in V'$, where $u(x', y)$ is another positive harmonic function on V'. Hence $u(x_n, y)$ converges locally uniformly to $u(x', y)$ for $y \in V'$. It follows that, given $\varepsilon > 0$, and a neighbourhood W' of y' (with $W' \subset \overline{W}' \subset V'$), there exists an integer N (depending on ε and W') such that

$$|u(x_n, y) - u(x', y)| < \frac{\varepsilon}{2} \text{ for } n \geqq N \text{ and all } y \in W'.$$

Suppose N' is an integer $\geqq N$ such that $y_n \in W'$ for all $n \geqq N'$. Then, for all $n \geqq N'$,

$$|u(x_n, y_n) - u(x', y_n)| < \frac{\varepsilon}{2}. \tag{1}$$

Again by the continuity of $u(x', y)$ on V', there exists an integer $N'' \geqq N'$ such that

$$|u(x', y_n) - u(x', y')| < \frac{\varepsilon}{2} \text{ for } n \geqq N'' \tag{2}$$

From (1) and (2), we have the inequality

$$|u(x', y') - u(x_n, y_n)| < \varepsilon \text{ for } n \geqq N''.$$

This proves the continuity of u at $(x', y') \in \delta$. But this point being arbitrary in δ, it follows that u is continuous on δ. This proves the lemma.

P_3. (*Convergence Property*). *Let $\delta \subset \Omega \times \Omega'$ be a domain and let $\{u_i\}_{i \in I}$ be an increasing directed family of functions in $MH(\delta)$. Then, the upper envelope u of* *this family is either identically* $+\infty$ *on* δ *or* u *belongs to* $MH(\delta)$.

Proof. Let $E_1=\{x\in\delta : u(x)\equiv+\infty\}$ and $E_2=\delta-E_1$. We shall show that both E_1 and E_2 are open.

Let $(x_0, y_0)\in E_1$. Let $V\subset\Omega$ and $V'\subset\Omega'$ be regular domains of the respective spaces such that $(x_0, y_0)\in V\times V'\subset\bar{V}\times\bar{V}'\subset\delta$. Now $\{u_i(x_0, y)\}_{i\in I}$ is an increasing directed family of harmonic functions on V' and $u(x_0, y_0)=\sup_{i\in I} u_i(x_0, y_0)=+\infty$. V' being a domain, it follows that $u(x_0, y)=+\infty$ for all $y\in V'$. Now, fixing $y\in V'$, we can similarly prove that $u(x, y)\equiv+\infty$ on $V\times V'$. Hence $(x_0, y_0)\in V\times V'\subset E_1$. This is true for every point of E_1. This shows that E_1 is an open subset of δ. An exactly similar argument shows that E_2 is also an open subset of δ. Now, δ being a domain, one of E_1 or E_2 has to be void. This shows that either $u\equiv+\infty$ on δ or $u<+\infty$ everywhere on δ.

Suppose now that $u<+\infty$ everywhere on δ. We can assume without loss of generality that $u\geqq 0$. Since u is finite at every point, it follows from the Harnack property (axiom 3′) that u is harmonic in each variable for every fixed value of the other. Hence by the lemma 1, u is also continuous on δ. This completes the proof.

Consequences.

PROPOSITION 1. *Let* $u\in(MH)^+(\delta)$ *where* δ *is a domain. Then either* $u>0$ *everywhere on* δ *or* $u\equiv 0$ *on* δ.

This follows immediately by considering the increasing sequence $\{nu\}$ of $MH(\delta)$-functions.

PROPOSITION 2. *Let* u *be a finite continuous function on an open set* $\delta\subset\Omega\times\Omega'$. *Then* u *is multiply harmonic on* δ, *if and only if, for every pair of regular domains* $\omega\subset\Omega$ *and* $\omega'\subset\Omega'$ *such that* $\bar{\omega}\times\bar{\omega}'\subset\delta$, u *satisfies the condition* $u=\Gamma_u$ *in* $\bar{\omega}\times\bar{\omega}'$.

This is an easy consequence of (P_2).

THEOREM 1. *Let* $\omega\subset\Omega$ *and* $\omega'\subset\Omega'$ *be a pair of regular domains. Then, for any extended real valued function* f *on* $\partial\omega\times\partial\omega'$, *the* $\rho_x^\omega\times\rho_y^{\omega'}$ *summability is independent of* $(x, y)\in\omega\times\omega'$. *And in the case of a summable function* f, $\int f(z, z')(\rho_x^\omega\times\rho_y^{\omega'})(dz\,dz')$ *is an element of* $MH(\omega\times\omega')$.

Proof. Let $\psi>-\infty$ be any lower semi-continuous function on $\partial\omega\times\partial\omega'$.

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There is an increasing sequence $\{f_n\}$ of finite continuous functions on $\partial\omega\times\partial\omega'$ such that ψ is the pointwise limit of f_n. Then

$$\int\psi(z, z')(\rho_x^{\omega}\times\rho_y^{\omega'})(dz\,dz') = \lim_{n\to\infty}\int f_n(z, z')(\rho_x^{\omega}\times\rho_y^{\omega'})(dz\,dz') = \lim_{n\to\infty}\Gamma_{f_n}(x, y)$$

for every $(x, y)\in\omega\times\omega'$. But, since $\{\Gamma_{f_n}\}$ is an increasing sequence, we have, by the property P_3, that the integral of ψ is either $\equiv +\infty$ or an element of $MH(\omega\times\omega')$. Now, it is clear that, for any extended real valued function f on $\partial\omega\times\partial\omega'$, $\overline{\int} f(z, z')(\rho_x^{\omega}\times\rho_y^{\omega'})(dz\,dz')$ is identically $+\infty$ or $-\infty$ or else an element of $MH(\omega\times\omega')$. A similar result is true for the lower integral and moreover $\overline{\int} f\geqq\underline{\int} f$. The proof of the theorem is now completed easily, using the Proposition 1.

Lemma 2. *Let δ be a domain contained in $\Omega\times\Omega'$. Let v be an extended real valued function on δ, satisfying* (i) $v>-\infty$ *and* (ii) *v is hyperharmonic in each variable for every fixed value of the other. Then v is either identically $+\infty$ on δ or v is finite on an everywhere dense subset of δ.*

Proof. Let $\omega_1\subset\Omega$ and $\omega_2\subset\Omega'$ be domains of the respective spaces. Assume $v\equiv+\infty$ on a non-empty open subset V of $\omega_1\times\omega_2$. Let $p(V)$ be the projection of V on Ω. Suppose $x_0\in p(V)$. Then, the hyperharmonic function $v(x_0, y)$ on the section of $\omega_1\times\omega_2$ through x_0 (which is homeomorphic to ω_2) is $+\infty$ on a non-void open set, namely $V\cap\{(x, y) : x=x_0\}$. Hence $v(x_0, y)=+\infty$, for every $y\in\omega_2$. This is true for every $x_0\in p(V)$. Now, for any $y\in\omega_2$, the hyperharmonic function $v(x, y)$ is $+\infty$ on the open non-void subset $p(V)\subset\omega_1$; and ω_1 being a domain, we have $v(x, y)\equiv+\infty$ for every $x\in\omega_1$. This is true for every $y\in\omega_2$. Hence $v\equiv+\infty$ on $\omega_1\times\omega_2$.

Define the subset σ of δ as follows:

$$\sigma=\{(x, y) : \exists \text{ a neighbourhood of } (x, y) \text{ contained in } \delta \text{ such that } v\equiv+\infty \text{ on this neighbourhood.}\}$$

Then σ is an open set. Let (x_0, y_0) be any boundary point in δ of σ. Then there exists a rectangular neighbourhood N of (x_0, y_0) (where the sides are connected open sets) such that $\sigma\cap N\neq\emptyset$, and $N\subset\delta$. Hence, v is $+\infty$ on a non-void open subset $N\cap\sigma$ of the rectangular open (domain) N, which implies

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that $v \equiv +\infty$ on N. That is, $(x_0, y_0) \in \sigma$. This being true for all the boundary points of σ in δ, it follows that σ is relatively closed in δ. Hence we have either $\sigma = \phi$ or $\sigma = \delta$. This completes the proof of the lemma.

Definition 1. Let δ be an open subset of $\Omega \times \Omega'$. Define the class $MS(\delta)$ of multiply superharmonic functions in δ, as follows:

$$MS(\delta) = \left\{ v : \begin{array}{l} v > -\infty \text{ and lower semi-continuous on } \delta \\ v \text{ is hyperharmonic in each variable for} \\ \quad \text{every fixed value of the other} \\ v \not\equiv +\infty \text{ on any connected component of } \delta \end{array} \right\}$$

It is easy to see that (i) if $v_1, v_2 \in MS(\delta)$ and $\alpha_1 \geqq 0$ and $\alpha_2 \geqq 0$ then $\alpha_1 v_1 + \alpha_2 v_2 \in MS(\delta)$; (ii) if $v_1, v_2 \in MS(\delta)$, then the function $v = \inf\,(v_1, v_2)$ also belongs to $MS(\delta)$; (iii) $MH(\delta) \subset MS(\delta)$ and (iv) if $\{v_i\}_{i \in I}$ is any increasing directed family contained in $MS(\delta)$, then $v = \sup\limits_{i \in I} v_i$ belongs to $MS(\delta)$ if it is not identically $+\infty$ on any connected component of δ.

Theorem 2. *Let v be an extended real valued function on an open subset δ contained in $\Omega \times \Omega'$, satisfying* (i) $v > -\infty$, (ii) *v is bounded below on every compact subset of δ and* (iii) *v is hyperharmonic in each variable for every fixed value of the other. Then v is lower semi-continuous on δ.*

Proof. Let us first prove the theorem assuming that v is superharmonic in each variable for every fixed value of the other variable.

Let $\omega \subset \Omega$ and $\omega' \subset \Omega'$ be regular domains of the respective spaces such that $\bar{\omega} \times \bar{\omega}' \subset \delta$. Define, for any $x \in \omega$ and $y \in \bar{\omega}'$,

$$\varphi(x, y, \omega) = \int v(\xi, y)\, \rho_x^{\omega}(d\xi).$$

Let k be a real number such that $v \geqq k$ on $\bar{\omega} \times \bar{\omega}'$. For any fixed $y \in \bar{\omega}'$, since $v(x, y)$ is superharmonic in x, the function $\varphi(x, y, \omega)$ is harmonic of $x \in \omega$. Now let x be fixed in ω. Suppose $y_n \in \bar{\omega}'$ and $y_n \to y \in \bar{\omega}'$. Then,

$$\begin{aligned} \liminf_{n \to \infty} \varphi(x, y_n, \omega) &= \liminf_{n \to \infty} \int v(\xi, y_n)\, \rho_x^{\omega}(d\xi) \\ &\geqq \int \liminf_{n \to \infty} v(\xi, y_n)\, \rho_x^{\omega}(d\xi) \\ &\qquad\qquad \text{(Fatou's lemma)} \\ &\geqq \int v(\xi, y)\, \rho_x^{\omega}(d\xi) \\ &\qquad\qquad (v(\xi, .) \text{ is lower semi-continuous}) \end{aligned}$$

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(The measure ρ_x^{ω} is totally finite and $v(\xi, y_n) \geqq k$ for all n and $\xi \in \bar{\omega}$ hence the use of Fatou's lemma is justified.)

Hence $\varphi(x, y, \omega)$ is a lower semi-continuous function of $y \in \bar{\omega}'$, for every $x \in \omega$.

Let ω_1 be any regular domain of Ω such that $\bar{\omega}_1 \subset \omega$. On the compact space $\bar{\omega}_1 \times \bar{\omega}'$, $\varphi(x, y, \omega)$ is continuous in x for every fixed $y \in \bar{\omega}'$ and lower semi-continuous in y for every $x \in \bar{\omega}_1$. Hence $\varphi(x, y, \omega)$ is a borel measurable function on $\bar{\omega}_1 \times \bar{\omega}'$. Moreover,

$$k\int \rho_x^{\omega}(d\xi) \leqq \varphi(x, y, \omega) \leqq v(x, y) \tag{1}$$

for every $(x, y) \in \bar{\omega}_1 \times \bar{\omega}'$.

Now, define for every $y \in \omega'$ and $x \in \omega$,

$$\sigma(x, y, \omega, \omega') = \int \varphi(x, \eta, \omega)\, \rho_y^{\omega'}(d\eta).$$

Once again, by the inequality (1) and the fact that v is superharmonic for every fixed $x \in \omega$, we get that $\sigma(x, y, \omega, \omega')$ is harmonic of $y \in \omega'$ for every $x \in \omega$. Now, we shall show that σ is a continuous function on $\omega \times \omega'$.

Let $\omega_1 \subset \Omega$ be a regular domain such that $\bar{\omega}_1 \subset \omega$. It is clear, from the inequality (1), that $\varphi(x, y, \omega)$ is lower bounded on $\bar{\omega}_1 \times \bar{\omega}'$ and it is moreover measurable. Since the measures $\rho_x^{\omega_1}$ and $\rho_y^{\omega'}$ are finite, by Fubini's theorem, we get

$$\begin{aligned}
\int \sigma(\xi, y, \omega, \omega')\, \rho_x^{\omega_1}(d\xi) &= \int \rho_x^{\omega_1}(d\xi) \int \varphi(\xi, \eta, \omega)\, \rho_y^{\omega'}(d\eta) \\
&= \int \rho_y^{\omega'}(d\eta) \int \varphi(\xi, \eta, \omega)\, \rho_x^{\omega_1}(d\xi) \\
&= \int \varphi(x, \eta, \omega)\, \rho_y^{\omega'}(d\eta) \\
&= \sigma(x, y, \omega, \omega').
\end{aligned}$$

This is true for all the points x of ω_1 and in turn for all such regular domains. Hence, $\sigma(x, y, \omega, \omega')$ is harmonic in x for every $y \in \omega'$. Suppose $h(x, y) = \iint 1\, \rho_x^{\omega}(d\xi)\, \rho_y^{\omega'}(d\eta)$. Then

$$v(x, y) \geqq \sigma(x, y, \omega, \omega') \geqq kh(x, y) \tag{2}$$

for every $(x, y) \in \omega \times \omega'$.

Now, the function $\sigma - kh$ on $\omega \times \omega'$ is $\geqq 0$ and is harmonic in each variable for every fixed value of the other. Hence, by the lemma 1, $\sigma - kh$ is a continuous function on $\omega \times \omega'$. It follows that $\sigma = \sigma - kh + kh$ is also a continuous function on $\omega \times \omega'$.

Let us now take a point $(x_0, y_0) \in \omega \times \omega'$. Let $\omega_1 \subset \Omega$ and $\omega_1' \subset \Omega'$ be regular domains of the respective spaces such that $\bar{\omega}_1 \subset \omega$ and $\bar{\omega}_1' \subset \omega'$ and $(x_0, y_0) \in \omega_1 \times \omega_1'$. Then

$$\begin{aligned}
\sigma(x_0, y_0, \omega, \omega') &= \int \sigma(x, y, \omega, \omega')\, \rho_{x_0}^{\omega_1}(dx)\, \rho_{y_0}^{\omega'_1}(dy) \\
&= \int \rho_{y_0}^{\omega'_1}(dy) \int \sigma(x, y, \omega, \omega')\, \rho_{x_0}^{\omega_1}(dx) \\
&\leqq \int \rho_{y_0}^{\omega'_1}(dy) \int v(x, y)\, \rho_{x_0}^{\omega_1}(dx) \qquad \text{[from the inequality (2)]} \\
&= \int \rho_{y_0}^{\omega'_1}(dy)\, \varphi(x_0, y, \omega_1) \\
&= \sigma(x_0, y_0, \omega_1, \omega_1').
\end{aligned}$$

Hence $\{\sigma(x_0, y_0, U, U')\}$ is an increasing directed family of real numbers (for all the neighbourhoods $U \times U'$ of (x_0, y_0) where U and U' are regular domains such that $\bar{U} \times \bar{U}' \subset \delta$).

Define the function V on δ by setting

$$V(x, y) = \lim_{U, U'} \sigma(x, y, U, U') = \sup_{U, U'} \sigma(x, y, U, U')$$

for every $(x, y) \in \delta$.

We shall show that (i) V is lower semi-continuous on δ and (ii) $V = v$ on δ, and this will prove the theorem for such functions v.

From the definition, $V(x, y) \leqq v(x, y)$ (using inequality (2)). Let $(x_0, y_0) \in \delta$ and $\hat{v}$ the lower semi-continuous regularisation of v. Suppose $\omega \times \omega'$ is a rectangular neighbourhood of (x_0, y_0) (where the sides are regular domains) such that $\bar{\omega} \times \bar{\omega}' \subset \delta$. Then

$$\begin{aligned}
\hat{v}(x_0, y_0) = \liminf_{\substack{(x, y) \in \omega \times \omega' \\ (x, y) \to (x_0, y_0)}} v(x, y) &\geqq \liminf_{\substack{(x, y) \in \omega \times \omega' \\ (x, y) \to (x_0, y_0)}} \sigma(x, y, \omega, \omega') \\
&= \sigma(x_0, y_0, \omega, \omega'),
\end{aligned}$$

since $\sigma(x, y, \omega, \omega')$ is continuous on $\omega \times \omega'$. This inequality

$$\hat{v}(x_0, y_0) \geqq \sigma(x_0, y_0, \omega, \omega')$$

is true for a fundamental system of regular rectangular neighbourhoods of (x_0, y_0). Hence

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$$\hat{v}(x_0, y_0) \geqq V(x_0, y_0) \tag{3}$$

On the other hand, if $\lambda < \hat{v}(x_0, y_0)$, then there exists a neighbourhood N of (x_0, y_0), $N \subset \delta$, such that $v(x, y) > \lambda$ for every $(x, y) \in N$. Now, for any pair of regular domains ω and ω' such that $(x_0, y_0) \in \omega \times \omega' \subset \bar{\omega} \times \bar{\omega}' \subset N$, we have

$$\begin{aligned} V(x_0, y_0) \geqq \sigma(x_0, y_0, \omega, \omega') &= \int \rho_{y_0}^{\omega'}(dy) \int v(x, y) \rho_{x_0}^{\omega}(dx) \\ &\geqq \lambda \left(\int d\rho_{x_0}^{\omega} \right) \left(\int d\rho_{y_0}^{\omega'} \right). \end{aligned}$$

Taking the limit as the regular domains shrink respectively to x_0 and y_0, we get

$$\begin{aligned} V(x_0, y_0) &\geqq \lambda \lim \left(\int d\rho_{x_0}^{\omega} \right) \left(\int d\rho_{y_0}^{\omega'} \right) \\ &= \lambda \end{aligned} \qquad [2]$$

This is true for every $\lambda < \hat{v}(x_0, y_0)$, hence $V \geqq \hat{v}$. This, combined with the inequality (3), gives us that $V = \hat{v}$. Hence V is lower semi-continuous.

Let now $(x_0, y_0) \in \delta$. Let ω and ω' be regular domains such that $(x_0, y_0) \in \omega \times \omega' \subset \bar{\omega} \times \bar{\omega}' \subset \delta$. Choose the following sequences of regular domains.

(1) $\{\omega_n\}$ satisfying: (i) $x_0 \in \omega_{n+1} \subset \bar{\omega}_{n+1} \subset \omega_n \subset \bar{\omega}_n \subset \omega$ for every $n \geqq 1$;

and (ii) $\cap \omega_n = \{x_0\}$.

(2) $\{\omega'_n\}$ satisfying: (i) $y_0 \in \omega'_{n+1} \subset \bar{\omega}'_{n+1} \subset \omega'_n \subset \bar{\omega}'_n \subset \omega'$ for every $n \geqq 1$;

and (ii) $\cap \omega'_n = \{y_0\}$.

Let us fix $y \in \omega'$. Since $v(x, y)$ is superharmonic on ω,

$$\varphi(x_0, y, \omega_n) = \int v(x, y) \rho_{x_0}^{\omega_n}(dx) \nearrow v(x_0, y) \qquad [2]$$

as n tends to infinity. Hence,

$$\begin{aligned} \lim_{n \to \infty} \sigma(x_0, y_0, \omega_n, \omega'_m) &= \lim_{n \to \infty} \int \varphi(x_0, \eta, \omega_n) \rho_{y_0}^{\omega'_m}(d\eta) \\ &= \int v(x_0, \eta) \rho_{y_0}^{\omega'_m}(d\eta) \end{aligned}$$

(monotone conv. theorem).

(Note that the monotone convergence theorem is applicable, as in the proof of the earlier part). From this we get

$$V(x_0, y_0) \geqq \lim_{n\to\infty} \sigma(x_0, y_0, \omega_n, \omega'_m)$$
$$= \int v(x_0, y)\, \rho_{y_0}^{\omega'm}(dy).$$

This is evidently true whatever be the domain ω'_m $(m \geqq 1)$. Hence

$$V(x_0, y_0) \geqq \lim_{m\to\infty} \int v(x_0, y)\, \rho_{y_0}^{\omega'm}(dy)$$
$$= v(x_0, y_0).$$

Thus we get $V = v$ and this shows that v is lower semi-continuous on δ.

To complete the proof of the theorem, let us consider a v as in the hypotheses of the theorem. Let $V > 0$ and $V' > 0$ be continuous potentials on Ω and Ω' respectively. Then clearly, for every positive integer N, the function v_N defined by

$$v_N(x, y) = \operatorname{Inf}\, [v(x, y),\ NV(x)\, V'(y)]$$

is lower semi-continuous on δ. It follows that v which is the increasing limit of v_N is also lower semi-continuous on δ. The proof is complete.

3. *MH*-minorants

Let $\omega \subset \Omega$ be a relatively compact open set and $\partial\omega$ its boundary. Consider the Dirichlet problem with the trace on ω of the family of all neighbourhoods of all points of $\partial\omega$. Then the finite continuous functions are resolutive for this problem [7]. And to each point $x \in \omega$, corresponds a positive Radon measure μ_x^ω on the compact space $\partial\omega$, such that the upper (Perron) solution $\overline{H}_f^\omega$, corresponding to any extended real valued function f on $\partial\omega$ satisfies the equation $\overline{H}_f^\omega(x) = \int f(z)\, \mu_x^\omega(dz)$. In particular, all the borel measurable functions φ on $\partial\omega$, that are μ_x^ω-summable, are resolutive and the solution $H_\varphi^\omega(x) = \int \varphi(z)\, \mu_x^\omega(dz)$.

LEMMA 3. *Let $\omega \subset \Omega$ and $\omega' \subset \Omega'$ be relatively compact open subsets. Let v be a multiply-superharmonic function defined on an open set containing $\bar{\omega} \times \bar{\omega}'$. Let, for every $x \in \omega$ and $y \in \omega'$,*

$$D_v^{\omega, \omega'}(x, y) = \int v(\xi, \eta)\, \mu_x^\omega(d\xi)\, \mu_y^{\omega'}(d\eta).$$

Then $D_v^{\omega, \omega'}$ is multiply-harmonic on $\omega \times \omega'$. Moreover, if ω_1 and ω'_1 are open sets such that $\bar{\omega}_1 \subset \omega$ and $\bar{\omega}'_1 \subset \omega'$, then $D_v^{\omega_1, \omega'_1} \geqq D_v^{\omega, \omega'}$ in $\omega_1 \times \omega'_1$, *Proof.* Let $h(x, y) = \int d\mu_x^{\omega} d\mu_y^{\omega'}$, for every $(x,y) \in \omega \times \omega'$. Then clearly, $h(x,y)$ is harmonic in each variable for every fixed value of the other and also $h \geqq 0$, hence $h \in MH(\omega \times \omega')$. It follows that $\int \alpha d\mu d_x^{\omega} \mu_y^{\omega'} \in MH(\omega \times \omega')$, for every real number α.

Let k be a real number such that $v \geqq k$ on $\partial\omega \times \partial\omega'$. For every $\eta \in \bar{\omega}'$, since $v(\xi, \eta)$ is superharmonic of ξ, we have $v(x, \eta) \geqq \int v(\xi, \eta) \mu_x^{\omega}(d\xi)$, for every $x \in \omega$. The latter function is harmonic in ω. Let $\varphi(x, \eta) = \int v(\xi, \eta) \mu_x^{\omega}(d\xi)$, for every $(x, \eta) \in \omega \times \bar{\omega}'$. Then, by Fatou's lemma, it is easily proved that $\varphi(x, \eta)$ is lower semi-continuous on $\bar{\omega}'$, for every $x \in \omega$. Now, if δ' is any regular domain such that $\bar{\delta}' \subset \omega'$, then

$$\begin{aligned}\int \varphi(x, \eta) \rho_y^{\delta'}(d\eta) &= \int \rho_y^{\delta'}(d\eta) \int v(\xi, \eta) \mu_x^{\omega}(d\xi) \\ &= \int \mu_x^{\omega}(d\xi) \int v(\xi, \eta) \rho_y^{\delta'}(d\eta) \qquad \text{(Fubini's theorem)} \\ &\leqq \int \mu_x^{\omega}(d\xi) v(\xi, y) \\ &= \varphi(x, y).\end{aligned}$$

It follows that $\varphi(x, y)$ is superharmonic on ω', for every $x \in \omega$. From this we deduce that

$$D_v^{\omega, \omega'}(x, y) = \int \varphi(x, \eta) \mu_y^{\omega'}(d\eta) \leqq \varphi(x, y) \leqq v(x, y).$$

Also, $D_v^{\omega, \omega'}$ is harmonic on ω', for every fixed $x \in \omega$. Now, by using Fubini's theorem and the fact that $\varphi(x, .)$ is harmonic on ω, it is proved easily that $D_v^{\omega, \omega'}$ is also harmonic in x for every fixed $y \in \omega'$. Further $D_v^{\omega, \omega'} \geqq kh$, from which we deduce that $D_v^{\omega, \omega'}$ is continuous on $\omega \times \omega'$, using the lemma 1.

Suppose ω_1 and ω_1' are open subsets of ω and ω' respectively such that $\bar{\omega}_1 \subset \omega$ and $\bar{\omega}_1' \subset \omega'$. Then, for any $u \in MH(\omega \times \omega')$, we have evidently

$$u(x, y) = \int u(\xi, \eta) \mu_x^{\omega_1}(d\xi) \mu_y^{\omega'_1}(d\eta).$$

Applying this to $D_v^{\omega, \omega'} \in MH(\omega \times \omega')$, we get

$$D_v^{\omega, \omega'}(x, y) = \int D_v^{\omega, \omega'}(\xi, \eta) \mu_x^{\omega_1}(d\xi) \mu_y^{\omega'_1}(d\eta)$$

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$$\leqq \int v(\xi, \eta)\, \mu_x^{\omega_1}(d\xi)\, \mu_y^{\omega'_1}(d\eta)$$
$$= D_v^{\omega_1, \omega'_1}(x, y).$$

This proves the lemma.

THEOREM 3. *Let δ and δ' be open subsets of Ω and Ω' respectively. For any $v \in MS(\delta \times \delta')$, if there exists a multiply harmonic minorant, then there is a greatest multiply harmonic minorant.*

Proof. Suppose $v \geqq h$ where $h \in MH(\delta \times \delta')$. Let $\{\omega_n \times \omega'_n\}_{n \geqq 1}$ be a sequence of relatively compact open rectangles of $\Omega \times \Omega'$ such that (i) $\bar{\omega}_n \times \bar{\omega}'_n \subset \omega_{n+1} \times \omega'_{n+1} \subset \delta \times \delta'$ for every $n \geqq 1$ and (ii) $\bigcup_n \omega_n \times \omega'_n = \delta \times \delta'$.

Now, consider $D_v^n = D_v^{\omega_n, \omega'_n}$, as defined in the lemma 3. For every $(x, y) \in \delta \times \delta'$, $D_v^n(x, y)$ is defined for all n after a certain stage and is a decreasing sequence of real numbers. Let $D_v(x, y) = \lim_{n \to \infty} D_v^n(x, y)$. Since $D_v^n(x, y) \leqq v(x, y)$ for every $(x, y) \in \omega_n \times \omega'_n$ and all n, we get that $D_v(x, y) \leqq v(x, y)$ for all $(x, y) \in \delta \times \delta'$. For every point $(x, y) \in \delta \times \delta'$, there exists a connected neighbourhood V which is contained with its closure in $\omega_n \times \omega'_n$, for all n after a certain stage. From the lemma 3, we have that D_v^n is a decreasing sequence of multiply harmonic functions on V, for all n after a certain stage. But, since $v \geqq h$ on $\delta \times \delta'$, we get $D_v^n(x, y) \geqq D_h^n(x, y) = h(x, y)$ for every $(x, y) \in \omega_n \times \omega'_n$. This shows *that the limit of $D_v^n(x, y)$ is multiply harmonic in V. It follows that D_v is a* multiply harmonic function on $\delta \times \delta'$ and $D_v \leqq v$.

Suppose $u \in MH(\delta \times \delta')$ and $u \leqq v$. Then $D_v^n(x, y) \geqq D_u^n(x, y) = u(x, y)$ for every $(x, y) \in \omega_n \times \omega'_n$. From which, we deduce easily that $D_v(x, y) \geqq u(x, y)$ for every element of $\delta \times \delta'$. The proof is complete.

An important corollary is the following result.

THEOREM 4. *The set of all non-negative multiply harmonic functions on any open rectangle is a lattice for the natural order. The set of all multiply harmonic functions on an open rectangle is a complete lattice.*

Proof. Let $\delta \times \delta'$ be an open rectangle. Let u_1 and u_2 be any two non-negative multiply harmonic functions on $\delta \times \delta'$. Let $v = \inf(v_1, v_2)$. Then $v \in MS(\delta \times \delta')$ and $v \geqq 0$. Hence by the theorem 3, v has got the greatest multiply harmonic minorant u. It is clear that u is the greatest element in $MH(\delta \times \delta')$ such that $u \leqq u'$ and $u \leqq u_2$. On the other hand, let $w = \inf(-u_1,$

$-u_2)$. Then $w \in MS(\delta\times\delta')$ and $w \geqq -(u_1+u_2) \in MH(\delta\times\delta')$. Let $-u'$ be the greatest MH-minorant of w. Then it is obvious that $u' \geqq u_1$ and u_2 and u' is the smallest element of $MH(\delta\times\delta')$ which majorises both u_1 and u_2. This proves that $(MH)^+(\delta\times\delta')$ is a lattice.

Let $\{u_i\}_{i\in I}$ be any family of multiply harmonic functions on $\delta\times\delta'$ such that there is a $h \in MH(\delta\times\delta')$ with $h \leqq u_i$ for every $i \in I$. Then, $\hat{v}$ (the lower semi-continuous regularisation of $v = \operatorname*{Inf}_{i\in I} u_i$), belongs to $MS(\delta\times\delta')$ and $\hat{v} \geqq h$. Clearly the greatest MH-minorant of $\hat{v}$ is the lower bound of $\{u_i\}_{i\in I}$. Now the proof of the theorem is completed easily.

Definition 2. For any open rectangle $\delta\times\delta'$, *the class* $MP(\delta\times\delta')$ is defined as follows:

$$MP(\delta\times\delta') = \{v \in MS(\delta\times\delta') : v \geqq 0 \text{ and the greatest } MH\text{-minorant of } v \text{ is identically zero.}\}$$

Proposition 3. *For any pair of elements* $v_1,\ v_2 \in MP(\delta\times\delta')$, *and real numbers* $\alpha_1 \geqq 0$ *and* $\alpha_2 \geqq 0$, $\alpha_1 v_1 + \alpha_2 v_2 \in MP(\delta\times\delta')$ *and* $\inf(v_1,\ v_2) \in MP(\delta\times\delta')$.

Proof. The proof would be complete if we show that $v_1 + v_2 \in MP(\delta\times\delta')$. This is easily deduced from the fact that $D_w^{\omega,\omega'}$ is additive in w for any pair of relatively compact open sets ω and ω'.

4. The Integral Representation

Let us recall the integral representation of positive harmonic functions on Ω and Ω'. Let Δ_{x_0} be the class of all positive harmonic functions on Ω, taking the value 1 at x_0. Let Δ_1 be the set of all extremal or minimal harmonic functions contaiued in Δ_{x_0}. To every positive harmonic function u on Ω, there corresponds a unique Radon measure, called the canonical measure associated to u, μ_u on Δ_{x_0}, charging only Δ_1, such that $u = \int h\mu_u(dh)$. Moreover, Δ_{x_0} is compact (by the axiom 3$'$). Let Δ'_{y_0}, Δ'_1 etc. be defined similarly for Ω', relative to a point $y_0 \in \Omega'$.

Lemma 4. *If* $\{u_n\}$ *is a sequence of positive harmonic functions on* Ω *and* u_n *converges locally uniformly on* Ω *to a harmonic function* u, *then the canonical measures* μ_{u_n} *of* u_n *(on* Δ_{x_0}*) converge weakly to the canonical measure* μ_u *of* u *(on* Δ_{x_0}*).*

Proof. Since $u_n(x_0) \to u(x_0)$, it is clear that the measures μ_{u_n} on Δ_{x_0} are strongly bounded $\left[\text{for } \mu_{u_n}(\Delta_{x_0}) = \int h(x_0)\mu_{u_n}(dh) = u_n(x_0)\right]$. Hence from any subsequence of μ_{u_n}, it is possible to choose a weakly convergent subsequence. But the limit of any such weakly convergent subsequence, is evidently μ_u, (by the uniqueness of integral representation), since

$$\lim \int h(x)\,\mu_{n_i}(dh) = \lim u_{n_i}(x) = u(x) = \int h(x)\mu_u(dh)$$
$$\text{for every } x \in \Omega.$$

Hence it follows that $\{\mu_{u_n}\}$ is itself weakly convergent to μ_u. The proof is complete.

LEMMA 5. *Let $u > 0$ be a multiply harmonic function on $\Omega \times \Omega'$. For every $y \in \Omega'$, let ν_y^u be the canonical measure on Δ_{x_0} associated to the harmonic function $u(., y)$ on Ω. Let δ be a regular domain contained in Ω' and $y_1 \in \delta$. Then,*

(1) $$\psi : \Omega' \to \mathfrak{M}^+(\Delta_{x_0}) \text{ defined by } \psi(y) = \nu_y^u$$

is weakly continuous and (2) *for any finite continuous function f on Δ_{x_0}, the function $y \to \int f(h)\nu_y^u(dh)$ is $\rho_{y_1}^\delta$-integrable.*

Proof. Suppose $\{y_n\}$ is a sequence in Ω' converging to $y' \in \Omega'$. Since u is a continuous function, given $\varepsilon > 0$, we can find a neighbourhood V of $x' \in \Omega$ and V' of $y' \in \Omega'$ such that $|u(x, y) - u(x'', y'')| < \varepsilon$ for $(x'', y''), (x, y) \in V \times V'$. If $y_n \in V'$ for $n \geqq N$, then $|u(x, y_n) - u(x, y')| < \varepsilon$ for $n \geqq N$ and all $x \in V$. In other words, $u(x, y_n)$ converges locally uniformly on Ω to the function $u(x, y')$. Hence by the lemma 4, $\nu_{y_n}^u$ coverges weakly to $\nu_{y'}^u$. That is, $y \to \nu_y^u$ is continuous for the weak topology on $\mathfrak{M}^+(\Delta_{x_0})$. Now, the second part follows immediately.

COROLLARY. *In particular, the mapping $y \to \nu_y^u$ is $\rho_{y_1}^\delta$-adequate* [1] *for every regular domain $\delta \subset \Omega'$ and all points y_1 in δ.*

LEMMA 6. *The measures ν_y^u depend harmonically on $y \in \Omega'$. That is, for any regular domain δ,*

$$\text{if } \lambda_y^{\delta, u} = \int \nu_\eta^u \rho_y^\delta(d\eta), \text{ then } \lambda_y^{\delta, u} = \nu_y^u \text{ for every } y \in \delta.$$

Proof. Because of the lemma 5, we have, for any $\lambda_y^{\delta, u}$-summable function f on Δ_{x_0}, f is $\nu_{y'}^u$-summable for ρ_y^δ-almost every $y' \in \partial\delta$ and

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$$\int f(h)\lambda_y^{\delta,u}(dh) = \int \rho_y^{\delta}(d\eta)\int f(h)\,\nu_\eta^u(dh). \tag{1}$$

In particular, if we take for f the characteristic function of the set $\Delta_{x_0} - \Delta_1$, we get $\lambda_y^{\delta,u}(\Delta_{x_0} - \Delta_1) = 0$, since $\nu_{y'}^u(\Delta_{x_0} - \Delta_1) = 0$ for every y'. Now, for any $x \in \Omega$,

$$\begin{aligned}\int h(x)\,\lambda_y^{\delta,u}(dh) &= \int \rho_y^{\delta}(d\eta)\int h(x)\,\nu_\eta^u(dh)\\ &= \int \rho_y^{\delta}(d\eta)\,u(x,\eta)\end{aligned}$$

since ν_η^u is the canonical measure of the harmonic function $u(.,\eta)$. Hence

$$\begin{aligned}\int h(x)\,\lambda_y^{\delta,u}(dh) &= \int \rho_y^{\delta}(d\eta)\,u(x,\eta)\\ &= u(x,y) \qquad (u(x,.) \text{ is harmonic}).\end{aligned}$$

This is true for every $x \in \Omega$. Hence, by the uniqueness of the measure on Δ_{x_0}, charging only Δ_1, corresponding to the harmonic function $u(.,y)$, we conclude that $\lambda_y^{\delta,u} = \nu_y^u$. The same is true whatever be the point $y \in \delta$ and any regular domain $\delta \subset \Omega'$. The lemma is proved.

LEMMA 7. *The ν_y^u-summability and the sets of ν_y^u-measure zero are independent of $y \in \Omega'$. Further, for any ν_y^u-summable function f on Δ_{x_0}, $\int f(h)\,\nu_y^u(dh)$ is harmonic on Ω'.*

Proof. For every finite continuous function f on Δ_{x_0}, consider $\int f(h)\nu_y^u(dh)$. If δ is any regular domain contained in Ω', then

$$\int \rho_{y_1}^{\delta}(dy)\int f(h)\,\nu_y^u(dh) = \int f(h)\,\nu_{y_1}^u(dh)$$

by the lemma 6. This being true for every point in δ_1 and in turn for all the regular domains, we conclude that $\int f(h)\,\nu_y^u(dh)$ is harmonic on Ω'. Now, by standard arguments (using the convergence property of any directed family of harmonic functions on Ω'), we deduce that for any extended real valued function f on Δ_{x_0}, $\overline{\int} f(h)\,\nu_y^u(dh)$ is identically $+\infty$ or $-\infty$ or else a harmonic function on Ω'. A similar result is true for $\underline{\int} f(h)\,\nu_y^u(dh)$. Further $\overline{\int} f(h)\nu_y^u(dh) \geqq \underline{\int} f(h)\,\nu_y^u(dh)$. The proof of the lemma is now completed easily.

COROLLARY. *In particular, if f is any ν_y^u-summable function on Δ_{x_0}, then*

$F(x, y) = \int f(h)\, h(x)\, \nu_y^u(dh)$ belongs to MH.

Proof. It is clear that for any ν_y^u-summable function f on Δ_{x_0}, the function $F(x, y)$ is harmonic in each variable when the other is fixed. If $f \geqq 0$, then $F \geqq 0$ and $F \in MH$. Since any ν_y^u-summable function is the difference of two non-negative functions each one of which is ν_y^u-summable, the corollary is proved.

Remark. When we are concerned with ν_y^u-summation, it is enough to consider the values of functions on Δ_1.

DEFINITION 3. An element $u \in (MH)^+$ is said to be *minimal* if, for any function $h \in (MH)^+$ satisfying $h \leqq u$, there exists a real number α_h such that $0 \leqq \alpha_h \leqq 1$ and $h = \alpha_h u$.

THEOREM 5. *Let $(MH)_0^+ = \{u \in (MH)^+ : u(x_0, y_0) = 1\}$. An element $u \in (MH)_0^+$ is minimal, if and only if, there exists elements h, h' belonging respectively to Δ_1 and Δ_1', such that $u(x, y) = h(x)\, h'(y)$.*

Proof: Sufficient. Let $h \in \Delta_1$ and $h' \in \Delta_1'$. Suppose $v \in (MH)^+$ and $v \leqq hh'$. For any $x \in \Omega$, $v(x, y) \leqq h(x)\, h(y)$, and since h' is a minimal harmonic function on Ω', there is a constant α_x, depending on x, such that $v(x, y) = \alpha_x h(x)\, h'(y)$, for every $y \in \Omega'$, (where $0 \leqq \alpha_x \leqq 1$). Similarly, we can find real numbers β_y, lying between 0 and 1, for every $y \in \Omega'$, such that $v(x, y) = \beta_y h(x)\, h'(y)$ for all $x \in \Omega$. From this we easily see that $\alpha_x = \beta_y = \nu$, where ν is some real number between 0 and 1; and $v(x, y) = \nu h(x)\, h'(y)$ for all $(x, y) \in \Omega \times \Omega'$. That is, hh' is a minimal element.

Necessary. Let $u \in (MH)_0^+$ be a minimal element. Consider the measures ν_y^u on Δ_{x_0}, corresponding to u, as introduced in the lemma 5. Suppose f is any finite continuous function on Δ_{x_0}, such that $0 \leqq f \leqq 1$. Then, by the lemma 7 and its corollary, $F(x, y) = \int f(h)\, h(x)\, \nu_y^u(dh)$ is an element of $(MH)^+$. Also $F(x, y) \leqq \int h(x)\, \nu_y^u(dh) = u(x, y)$. By the minimality of u, we have a constant α_f such that (i) $0 \leqq \alpha_f \leqq 1$ and $F(x, y) = \alpha_f u(x, y)$. But $\alpha_f u(x, y)$, for any fixed $y \in \Omega'$, has the integral representation $\alpha_f u(x, y) = \int \alpha_f h(x) \nu_y^u(dh)$. Hence, by the uniqueness of integral representation, we have, $\alpha_f = f$ ν_y^u-almost everywhere on Δ_{x_0}. Since sets of measure zero are same for all the measures $\nu_y^u(y \in \Omega')$, (lemma 6), there exists a set $E_f \subset \Delta_{x_0}$ of ν_y^u-measure zero, for all $y \in \Omega'$, such that $\alpha_f = f$ except on E_f. The same is true for all such continuous functions f on Δ_{x_0}. From this we deduce that, for each $y \in \Omega'$, ν_y^u is a constant multiple of the Dirac measure at some point of Δ_1. Otherwise, for some $y \in \Omega'$, suppose that the support of ν_y^u contains two distinct elements z and z'. Then, there exist disjoint compact neighbourhoods K of z and K' of z'. The ν_y^u-measure of each one of K and K' is >0. But by Urysohn's lemma, we can find a continuous function $\varphi : \Delta_{x_0} \to [0, 1]$, such that $\varphi \equiv 0$ on K and $\varphi \equiv 1$ on K'. This is a contradiction to the fact that φ is a constant ν_y^u-almost everywhere on Δ_{x_0}. Again, since sets of measure zero are identical for all the measures ν_y^u (for $y \in \Omega'$), we conclude that $\nu_y^u = \varepsilon_{h_0}\beta_y$, for some $h_0 \in \Delta_{x_0}$ and β_y is a real number depending on y. But $\nu_y^u(\Delta_{x_0} - \Delta_1) = 0$, and hence this element h_0 necessarily belongs to Δ_1. Now

$$\begin{aligned} u(x, y) &= \int h(x)\, \nu_y^u(dh) \\ &= \int h(x)\, \beta_y \varepsilon_{h_0}(dh) \\ &= h_0(x)\, \beta_y. \end{aligned}$$

In particular, it follows that, for every fixed $y \in \Omega'$, $x \to u(x, y)$ is a minimal harmonic function. By similarity, we deduce that $y \to u(x, y)$ is also a minimal harmonic function on Ω'. That is, $y \to \beta_y$ is a minimal harmonic function, say h_0' on Ω'. Further, $\dfrac{u(x_0, y_0)}{h_0(x_0)} = 1$. Hence $h_0' \in \Delta_1'$. So we have proved that $u = h_0 h_0'$ where $h_0 \in \Delta_1$ and $h_0' \in \Delta_1'$. The proof of the theorem is complete.

THEOREM 6. *The set of all elements of* $(MH)^+$, *taking the value* 1 *at any point of* $\Omega \times \Omega'$, *is equi-continuous at that point.*

Proof. Consider $(x_0, y_0) \in \Omega \times \Omega'$. Let $(MH)_0^+ = \{u \in (MH)^+ : u(x_0, y_0) = 1\}$.

Let $u \in (MH)_0^+$. Consider the measures ν_y^u on Δ_{x_0}, associated to u, by the lemma 5. Let $F_u(y) = \int \nu_y^u(dh)$. Then, by the lemma 7, $F_u(y)$ is a positive harmonic function on Ω', and further $F_u(y_0) = \int \nu_{y_0}^u(dh) = \int h(x_0)\, \nu_{y_0}^u(dh) = u(x_0, y_0) = 1$. Hence $F_u \in \Delta_{y_0}'$. This is true for every $u \in (MH)_0^+$.

Given $\varepsilon > 0$, we can find a neighbourhood N' of y_0 such that (i) $\overline{N'}$ is compact, and (ii) for every element $w \in \Delta_{y_0}'$, the inequality $|w(y) - w(y_0)| < \varepsilon/2$ is valid for all $y \in N'$. (This is axiom 3′ for harmonic functions on Ω'.) Moreover,

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there are real numbers m and M such that, for all $y \in \bar{N}'$ and $w \in \Delta'_{y_0}$,

$$0 < m \leqq w(y) \leqq M \qquad [3]$$

Again, the Harnack property (axiom 3′) for positive harmonic functions on Ω, assures us of a neighbourhood N of x_0, such that whatever be $x \in N$ and $v \in \Delta_{x_0}$, the inequality

$$|v(x_0) - v(x)| < \frac{\varepsilon}{2M}$$

is valid.

Now, for all $u \in (MH)_0^+$, since $F_u \in \Delta'_{y_0}$, we have

$$|F_u(y) - F_u(y_0)| < \varepsilon/2 \qquad \text{for every } y \in N' \tag{1}$$

and

$$m \leqq F_u(y) \leqq M \qquad \text{for every } y \in \bar{N}' \tag{2}$$

Now

$$\begin{aligned} u(x, y) - u(x_0, y_0) &= \int h(x)\, \nu_y^u(dh) - \int h(x_0)\, \nu_{y_0}^u(dh) \\ &= \int [h(x) - h(x_0)]\, \nu_y^u(dh) \\ &\qquad + \int h(x_0)\, \nu_y^u(dh) - \int h(x_0)\, \nu_{y_0}^u(dh). \end{aligned}$$

Hence

$$\begin{aligned} |u(x, y) - u(x_0, y_0)| &\leqq \int |h(x) - h(x_0)|\, \nu_y^u(dh) + \left|\int \nu_y^u(dh) - \int \nu_{y_0}^u(dh)\right| \\ &\qquad\qquad \text{as } h(x_0) = 1 \\ &= \int |h(x) - h(x_0)|\, \nu_y^u(dh) + |F_u(y) - F_u(y_0)| \\ &\leqq \frac{\varepsilon}{2M} F_u(y) + |F_u(y) - F_u(y_0)| \text{ if } x \in N \\ &\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ if } x \in N \text{ and } y \in N'. \end{aligned}$$

This is true for all the elements $u \in (MH)_0^+$. That is, $(MH)_0^+$ is equicontinuous at (x_0, y_0). It is clear that the same is valid whatever be the point (x_0, y_0) chosen in $\Omega \times \Omega'$. The theorem is proved.

Let us now consider the vector space $X = (MH)^+ - (MH)^+$. X provided with the topology of uniform convergence on compact subsets of $\Omega \times \Omega'$ is a locally convex topological vector space. The positive cone on X, for the natural order is $(MH)^+$. The theorem 4 asserts that $(MH)^+$ is a lattice for the natural order. Moreover, since $\Omega \times \Omega'$ has a countable base for open sets, $(MH)^+$ is metrizable. Let $(x_0, y_0) \in \Omega \times \Omega'$. Consider $(MH)_0^+ = \{u \in (MH)^+ : u(x_0, y_0) = 1\}$. $(MH)_0^+$ is a base for the cone $(MH)^+$. From the above theorem, we deduce that $(MH)_0^+$ is compact, as in [3]. Hence by Choquet's theorem [4], every element u in $(MH)^+$ is the centre of gravity of a uniquely determined measure on $(MH)_0^+$, charging only the extreme elements of this base. But, it can be easily seen that the extreme elements of this base are precisely the minimal elements of $(MH)^+$, belonging to this base.

Let us now consider Δ_{x_0} and Δ'_{y_0} with the topology of uniform convergence on compact subsets of Ω and Ω'. Both Δ_{x_0} and Δ'_{y_0} are compact spaces. Consider the mapping

$$\Delta_{x_0} \times \Delta'_{y_0} \to (MH)_0^+$$

defined by $(u, u') \to uu'$ It is easy to verify that this mapping is one-one. Let us prove the continuity of this mapping. Let $K \subset \Omega \times \Omega'$ be any compact set and $\varepsilon < 0$. Let $(u_n, u'_n) \to (u_0, u'_0)$ in the product topology. Then $u_n \to u'_0$ and $u'_n \to u'_0$. Let K_1 and K'_1 be the projections of K on Ω and Ω' respectively. Then, there exists $M > 0$ and an integer $N \geqq 0$ such that

(i) $u(x) \leqq M$ and $u'(y) \leqq M$ for all $x \in K_1$, $y \in K'_1$ and all $u \in \Delta_{x_0}$ and $u' \in \Delta'_{y_0}$,

(ii) $|u_n(x) - u_0(x)| < \dfrac{\varepsilon}{2M}$ for $n \geqq N$ and uniformly for $x \in K_1$,

and

(iii) $|u'_n(y) - u'_0(y)| < \dfrac{\varepsilon}{2M}$ for $n \geqq N$ and uniformly for $y \in K'_1$.

Hence, if $(x, y) \in K_1 \times K'_1 \supset K$, then

$$|u_n(x) u'_n(y) - u_0(x) u_0(y)| \leqq |u_n(x)|\, |u'_n(y) - u'_0(y)| + u_0(y) |u_n(x) - u_0(x)|$$
$$< M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M} = \varepsilon.$$

That is, for $n \geqq N$, uniformly on the compact set K,

$$|u_n u'_n - u_0 u'_0| < \varepsilon.$$

This is true for every $\varepsilon > 0$ and all the compact sets of $\Omega \times \Omega'$. Hence, $u_n u'_n$ converges to $u_0 u'_0$ in the compact convergence topology. Now $\Delta_{x_0} \times \Delta'_{y_0}$ being a compact space, the above mapping is, in fact, a homeomorphism of $\Delta_{x_0} \times \Delta'_{y_0}$ onto a compact subset of $(MH)_0^+$. The theorem 5 states that the minimal elements of $(MH)^+$, belonging to this base are precisely the elements belonging to the image of $\Delta_1 \times \Delta'_1$ (under the above homeomorphism). We shall identify the minimal elements of $(MH)_0^+$ with $\Delta_1 \times \Delta'_1$. Thus we have proved the following result.

THEOREM 7. *To every* $u \in (MH)^+$ *corresponds a unique measure* ν_u *on* $(MH)_0^+$, *charging only the set* $\Delta_1 \times \Delta_1'$ *such that*

$$u(x, y) = \int h(x) h'(y) \nu_u(dh dh')$$

for every $(x, y) \in \Omega \times \Omega'$.

Remark. The above results (regarding the integral representation) hold good, as well, for multiply harmonic functions on any open set of the form $\delta \times \delta'$, where $\delta \subset \Omega$ and $\delta' \subset \Omega'$ are connected open sets.

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