LOCAL UNIQUENESS IN BOUNDARY PROBLEMS†

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1. Introduction

The study of periodic, irrotational waves of finite amplitude in an incompressible fluid of infinite depth was reduced by Levi-Civita (1) to the determination of a function

$$\omega(\zeta) = \theta + i\tau, \quad \omega(0) = 0, \quad \zeta = \rho e^{i\sigma},$$

regular analytic in the interior of the unit circle $\rho = 1$ and which satisfies the condition

$$\frac{d\tau}{d\sigma} = pe^{-3\tau} \sin\theta, \quad p = \text{const.}$$

on the boundary. Here θ is the angle of inclination of the flow, $q = ce^{\tau}$ is its speed and c is the velocity of wave propagation.

The uniqueness of the solution to the problem is a question of long standing, raised by Levi-Civita himself. Both Levi-Civita and Lichtenstein (2) demonstrated existence and uniqueness if p is sufficiently close to a positive integer. Levi-Civita used Cauchy's method of " Calcul des Limites " and Lichtenstein the method of non-linear integral equations. In 1957 Stoker (3) returned to the question using the methods of modern functional analysis; in particular an iteration procedure devised by Littman and Nirenberg. Recently Dunninger and the author (4), using elementary methods, obtained the following result on uniqueness.

Given a solution $\omega_1(\zeta) = \theta_1 + i\tau_1$, no other solution $\omega_2(\zeta) = \theta_2 + i\tau_2$ exists for which

$$\begin{aligned} \left(\frac{\sin\frac{1}{2}\theta_2}{\sin\frac{1}{2}\theta_1}\right)^{\frac{4}{3}} < \frac{q_2}{q_1} < \left(\frac{\cos\frac{1}{2}\theta_2}{\cos\frac{1}{2}\theta_1}\right)^{\frac{4}{3}} \text{ if } |\theta_2| < |\theta_1|, \\ \left(\frac{\cos\frac{1}{2}\theta_2}{\cos\frac{1}{2}\theta_1}\right)^{\frac{4}{3}} < \frac{q_2}{q_1} < \left(\frac{\sin\frac{1}{2}\theta_2}{\sin\frac{1}{2}\theta_1}\right)^{\frac{4}{3}} \text{ if } |\theta_2| > |\theta_1|, \end{aligned} \tag{1.1}$$

provided λ^{-1}, $\lambda = \sin\theta_2/\sin\theta_1$ *are regular and* $|\theta_1| + |\theta_2| < \pi$ *in* $\rho \leqq 1$.

The inequalities (1.1) are portrayed in the hodograph plane (the plane with polar coordinates q, θ) by the shaded areas in Fig. 1a. Under the conditions stated no second solution exists for which the hodograph point (q_2, θ_2) lies in

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the shaded region attached to the hodograph point (q_1, θ_1), both points (q_1, θ_1), (q_2, θ_2) corresponding to the same point ζ in the unit circle $\rho<1$. One would prefer a uniqueness theorem without the inequalities (1.1), i.e., one in which the shaded region attached to (q_1, θ_1) is the entire plane, or, if this is not possible, a full neighbourhood of (q_1, θ_1); for example, the circular region in Fig. 1b. One arrives in this way at a concept of local uniqueness to which we shall return in a moment.

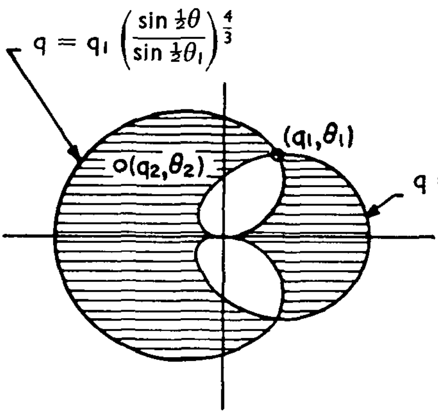

FIG. 1a

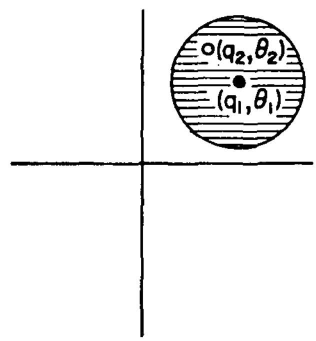

FIG. 1b

The general problem treated in the paper which includes the Levi-Civita problem is the following.

Determine a function $w = u+iv$ of the complex variable $z = x+iy$ regular analytic in a region S of the z-plane bounded by an analytic curve C upon which

$$u_n = h(s)f(u, v), \tag{1.2}$$

where u_n denotes the external normal derivative of u on C, and h, f are given analytic functions of their arguments, with s denoting some parameter on C, e.g., the arc length.

Prompted by the above results on the Levi-Civita problem, we seek those functions $f(u, v)$ in (1.2) for which a solution w is locally ρ-unique in the following sense, as defined by Cushing (**5**).

A solution $w_1 = u_1+iv_1$ of (1.2) is *locally ρ-unique* if a non-negative function † $\rho = \rho(u_1, v_1)$ can be found such that there is no second solution

$$w_2 = u_2+iv_2$$

for which

$$0 \leqq |w_1(z)-w_2(z)| \leqq \rho, \quad z \in S. \tag{1.3}$$

† Cushing (**5**) takes $\rho = \rho(x, y)$, but the choice $\rho = \rho(u_1, v_1)$ is more convenient for our purposes.

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Obviously if the solution to (1.2) is unique in the ordinary sense, it is locally ρ-unique, but the converse need not be true.

To avoid difficulties on the boundary C we assume that any solution w is regular analytic in a region $R \supset S+C$.

2. Preliminary results

In this section we collect a number of results for later use.

The integral identity

$$\int_C \tau\left(f_2\frac{\partial u_1}{\partial n}-f_1\frac{\partial u_2}{\partial n}\right)ds = \int_S Q\,dS, \tag{2.1}$$

a ready consequence of Gauss' theorem, is basic to our considerations. Here $\tau = \tau(u_1, u_2, u_3, u_4)$ is an arbitrary function of the real and imaginary parts of two analytic functions

$$w_1 = u_1+iu_3, \quad w_2 = u_2+iu_4,$$

and

$$f_1 = f_1(u_1, u_3), \quad f_2 = f_2(u_2, u_4),$$

are assumed given in advance. Q is a quadratic form

$$\begin{aligned} Q &= ap_1^2+2bp_1p_2+cp_2^2+2d(p_2p_3-p_1p_4)+ap_3^2+2bp_3p_4+cp_4^2, \\ p_k &= \frac{\partial u_k}{\partial x}, \end{aligned} \tag{2.2}$$

in the partial derivatives p_k with coefficients

$$\begin{aligned} a &= f_2\tau_{u_1}, & 2b &= (f_2\tau)_{u_2}-(f_1\tau)_{u_1}, \\ c &= -f_1\tau_{u_2}, & 2d &= -(f_1\tau)_{u_3}-(f_2\tau)_{u_4}, \end{aligned} \tag{2.3}$$

which are determined, once the function τ has been selected.

If w_1, w_2 are two solutions to (1.2), i.e., if

$$\frac{\partial u_1}{\partial n} = h(s)f(u_1, u_3), \quad \frac{\partial u_2}{\partial n} = h(s)f(u_2, u_4) \tag{2.4}$$

hold on C and if we take

$$f_1 = f(u_1, u_3), \quad f_2 = f(u_2, u_4), \tag{2.5}$$

the identity (2.1) implies $Q \equiv 0$ in S, provided Q is definite, or at least semi-definite. If Q is definite (semi-definite) there is no loss in generality in assuming Q positive definite (positive semi-definite), since Q changes its sign if τ is replaced by $-\tau$.

If E_4 denotes the four dimensional space of the variables u_1, u_2, u_3, u_4, the domain D in which Q is positive definite obviously depends on the function τ and one seeks a function τ that will make D as large as possible in view of the following lemma (4).

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Lemma 2.1. *If* $w_1 = u_1 + iu_3$ *is a non-constant solution of the boundary problem* (1.2), *there is no other solution* $w_2 = u_2 + iu_4$ *for which the integrals in* (2.1) *exist and the manifold*

$$M_2\colon\ u_1 = u_1(x, y),\quad u_2 = u_2(x, y),$$

$$u_3 = u_3(x, y),\quad u_4 = u_4(x, y),\quad (x, y) \in S,$$

lies in D.

To prove the lemma, assume that a second solution w_2 exists. On substituting from (2.4) into the integral identity (2.1), the integral over S vanishes. Since Q is positive definite, this implies $Q = 0$ and therefore that

$$p_1 = p_2 = p_3 = p_4 = 0, \tag{2.6}$$

hold in S. In view of the Cauchy-Riemann equations

$$p_1 = q_3, \qquad p_2 = q_4, \qquad q_k = \frac{\partial u_k}{\partial y},$$

$$q_1 = -p_3, \quad q_2 = -p_4,$$

$w_1 =$ constant, contrary to hypothesis.

To illustrate the lemma, consider the Levi-Civita problem, for which, in present notation

$$f = e^{-3v} \sin u.$$

Let us take

$$\tau = \frac{\cos u_2 - \cos u_1}{\sin u_1 \sin u_2} e^{3(u_3+u_4)} = \frac{(\lambda^{-1} - \lambda)e^{3(u_3+u_4)}}{\cos u_1 + \cos u_2}, \quad \lambda = \frac{\sin u_2}{\sin u_1}.$$

One finds

$$a = \frac{\lambda^2 + \cos^2 u_2}{1 + \cos u_1 \cos u_2} r_1^2, \quad 2b = -(\lambda r_1^2 + \lambda^{-1} r_2^2),$$

$$c = \frac{\lambda^{-2} + \cos^2 u_1}{1 + \cos u_1 \cos u_2} r_2^2, \quad d = 0,$$

where

$$u_1 = \theta_1, \quad u_2 = \theta_2, \quad r_1^2 = e^{3u_3} = (q_1/c)^3, \quad r_2^2 = e^{3u_4} = (q_2/c)^3, \tag{2.7}$$

from which it is clear that the integrals in (2.1) will exist if λ^{-1}, λ are regular and $|u_1| + |u_2| < \pi$. The domain D is defined by the inequalities

$$a > 0, \quad \Delta = b^2 - ac < 0.$$

The first is obviously satisfied. In view of

$$\Delta = \tfrac{1}{4} \csc^2 u_1 \csc^2 u_2 [(r_1 + r_2)^2 - (r_1 \cos u_2 + r_2 \cos u_1)^2]$$

$$\cdot \left(r_1 \sin^2 \frac{u_2}{2} - r_2 \sin^2 \frac{u_1}{2}\right)\left(r_1 \cos^2 \frac{u_2}{2} - r_2 \cos^2 \frac{u_1}{2}\right),$$

the second, on returning to the hodograph variables (2.7), amounts to prescribing the inequalities (1.1) pictured by the shaded regions in Figure 1a.

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In section 3 we shall need the reversed Cauchy inequality (6), which in its simplest form states that if

$$(x, y) = x_1y_1+x_2y_2-x_3y_3,$$

then $(x, x)<0$ implies that

$$(x, y)^2 \geqq (x, x)(y, y), \tag{2.8}$$

with equality prevailing if and only if the vectors

$$x = x_1, x_2, x_3; \; y = y_1, y_2, y_3$$

are linearly dependent.

3. The quadratic form Q

A simple calculation verifies that

$$aQ = (ap_1+bp_2-dp_4)^2+(ap_3+bp_4+dp_2)^2-\Delta(p_2^2+p_4^2),$$
$$\Delta = b^2+d^2-ac. \tag{3.1}$$

Consequently if $a>0$, the quadratic form Q is positive definite if $\Delta<0$, positive semi-definite if $\Delta = 0$ and indefinite if $\Delta>0$. Δ is known as the *discriminant* of Q.

The domain D of E_4 in which Q is positive definite is accordingly defined by the inequalities

$$a>0, \quad \Delta = b^2+d^2-ac<0, \tag{3.2}$$

the coefficients a, b, c, d being determined by τ in accordance with (2.3).

The points in E_4 which correspond to equal functions w_1, w_2 form a two dimensional subspace

$$S_2\colon\; u_2 = u_1,\; u_4 = u_3.$$

Likewise the points in E_4 which correspond to functions w_1, w_2 for which

$$0<|w_1-w_2| \leqq \rho, \quad \rho = \rho(u_1, u_3), \tag{3.3}$$

form a " neighbourhood "

$$D_\rho\colon\; 0<(u_1-u_2)^2+(u_4-u_3)^2 \leqq \rho^2,$$

of S_2.

If $D_\rho \subset D$ and w_1 is a non-constant solution of (1.2), there is no other solution w_2 for which the integrals in (2.1) exist and the manifold M_2 in Lemma 2.1 lies in D_ρ or even in $\bar{D}_\rho$, i.e., in

$$\bar{D}_\rho\colon\; 0 \leqq |w_1-w_2| \leqq \rho, \quad \rho = \rho(u_1, u_3), \tag{3.3'}$$

since $w_1 = w_2$ can occur for at most finitely many points of S, and the partial derivatives p_k in (2.6) will continue to vanish at these points by continuity. Thus we see that if a function τ can be found so that $D_\rho \subset D$, i.e., Q is positive definite in D_ρ, a non-constant solution w_1 is locally ρ-unique, at least among solutions w_2 for which the integrals in (2.1) exist.

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We now take up the problem of determining the function τ, so that Q will be positive definite in D_ρ for some $\rho = \rho(u_1, u_3)$, i.e., so that the inequalities (3.2) hold in D_ρ.

Let us denote evaluation in S_2 by a bar, for example

$$\bar{f}_2 = f_1, \quad \bar{f}_2' = f_1', \quad \bar{\dot{f}}_2 = \dot{f}_1,$$

where

$$f_1' = \frac{\partial f_1}{\partial u_1}, \quad f_2' = \frac{\partial f_2}{\partial u_2}, \quad \dot{f}_1 = \frac{\partial f_1}{\partial u_3}, \quad \dot{f}_2 = \frac{\partial f_2}{\partial u_4}.$$

As our first condition on τ, from (2.3) we are led to require

$$\bar{a} = f_1 \bar{\tau}_{u_1} > 0. \tag{3.4}$$

On substituting from (2.3), one finds that

$$\Delta = U^2 + V^2 + W,$$

where

$$U = \tfrac{1}{2}[f_1\tau_{u_1} + f_2\tau_{u_2} - (f_1' - f_2')\tau],$$

$$V = \tfrac{1}{2}[f_1\tau_{u_3} + f_2\tau_{u_4} + (\dot{f}_1 + \dot{f}_2)\tau],$$

$$W = f_1(f_1' - f_2')\tau\tau_{u_1}. \tag{3.5}$$

Clearly

$$\bar{W} = 0, \quad \bar{\Delta} = \bar{U}^2 + \bar{V}^2 \geqq 0$$

and we can secure $\bar{\Delta} = 0$, by choosing τ subject to the conditions

$$\bar{\tau}_{u_1} + \bar{\tau}_{u_2} = 0, \quad f_1[\bar{\tau}_{u_3} + \bar{\tau}_{u_4}] + 2\dot{f}_1\bar{\tau} = 0,$$

where by $\bar{\tau}_{u_1}$, for example, we mean the value of τ_{u_1} on S_2 and not the partial derivative of $\bar{\tau}$ with respect to u_1 which equals $\bar{\tau}_{u_1} + \bar{\tau}_{u_2}$. We shall prove that $\bar{a} > 0$, $\Delta < 0$ in D_ρ imply $\bar{\tau} = 0$, provided $f_1'' \neq 0$. Indeed Δ is positive whenever W is positive and W can be written in the form

$$W = \frac{a}{\lambda}(f_1' - f_2')\tau, \quad \lambda = f_2/f_1.$$

If $\bar{\tau} \neq 0$, the quantities a, λ, τ maintain fixed signs in the neighbourhood of S_2, but $f_1' - f_2'$ will change sign, in as much as $f_1'' \neq 0$. Thus W, and Δ also, would become positive in a neighbourhood of S_2, contrary to the requirement that Δ be negative in D_ρ. Consequently we lay down the following " initial conditions "

$$\bar{\tau} = 0, \quad \bar{\tau}_{u_1} + \bar{\tau}_{u_2} = 0, \quad \bar{\tau}_{u_3} + \bar{\tau}_{u_4} = 0,$$

for τ, the latter two being obvious consequences of the first.

For u_1, u_3 fixed Δ becomes a function of u_2, u_4. We seek a function τ so that Δ will have zero for a relative maximum at a point $P_0(u_1, u_3)$ of the (u_2, u_4)-plane. If this can be done for each point P_0, we shall have $\Delta < 0$ in a domain D_ρ as desired. Sufficient conditions for zero to be a relative maximum for Δ

at P_0 are of course

$$\bar{\Delta} = \bar{\Delta}_{u_2} = \bar{\Delta}_{u_4} = 0, \quad \bar{\Delta}_{u_2u_2} < 0, \quad \bar{\Delta}^2_{u_2u_4} - \bar{\Delta}_{u_2u_2}\bar{\Delta}_{u_4u_4} < 0. \tag{3.6}$$

From the definition of Δ one finds

$$\bar{\Delta}_{u_2} = \bar{W}_{u_2} = -f_1 f_1'' \bar{\tau}\bar{\tau}_{u_1} = 0, \quad \bar{\Delta}_{u_4} = \bar{W}_{u_4} = -f_1 \dot{f}_1' \bar{\tau}\bar{\tau}_{u_1} = 0,$$

$$\bar{\Delta}_{u_2u_2} = 2(\bar{U}^2_{u_2} + \bar{V}^2_{u_2}) + \bar{W}_{u_2u_2}, \quad \bar{\Delta}_{u_2u_4} = 2(\bar{U}_{u_2}\bar{U}_{u_4} + \bar{V}_{u_2}\bar{V}_{u_4}) + \bar{W}_{u_2u_4},$$

$$\bar{\Delta}_{u_4u_4} = 2(\bar{U}^2_{u_4} + \bar{V}^2_{u_4}) + \bar{W}_{u_4u_4},$$

in which

$$\bar{W}_{u_2u_2} = 2f_1 f_1'' \bar{\tau}^2_{u_2}, \quad \bar{W}_{u_2u_4} = f_1 \bar{\tau}_{u_2}(\dot{f}_1' \bar{\tau}_{u_2} + f_1'' \bar{\tau}_{u_4}),$$

$$\bar{W}_{u_4u_4} = 2f_1 \dot{f}_1' \bar{\tau}_{u_2}\bar{\tau}_{u_4}.$$

Clearly $\bar{\Delta}_{u_2u_2} < 0$ implies $\bar{W}_{u_2u_2} < 0$, and therefore

$$f_1 f_1'' < 0, \quad \bar{\tau}_{u_2} \neq 0.$$

One also observes that

$$\bar{W}^2_{u_2u_4} - \bar{W}_{u_2u_2}\bar{W}_{u_4u_4} = f_1^2 \bar{\tau}^2_{u_2}(\dot{f}_1' \bar{\tau}_{u_2} - f_1'' \bar{\tau}_{u_4})^2 \geqq 0.$$

If inequality holds, P_0 is a saddle point of W, so that W, and Δ also, would be positive in a neighbourhood of P_0. Therefore equality holds, and we have the additional " initial condition "

$$\dot{f}_1' \bar{\tau}_{u_2} - f_1'' \bar{\tau}_{u_4} = 0 \tag{3.7}$$

on τ. Assuming that this condition is fulfilled, we write

$$\bar{W}_{u_2u_2} = -2R^2, \quad \bar{W}_{u_2u_4} = -2RS, \quad \bar{W}_{u_4u_4} = -2S^2$$

so that

$$\bar{\Delta}_{u_2u_2} = 2(\bar{U}^2_{u_2} + \bar{V}^2_{u_2} - R^2), \quad \bar{\Delta}_{u_2u_4} = 2(\bar{U}_{u_2}\bar{U}_{u_4} + \bar{V}_{u_2}\bar{V}_{u_4} - RS),$$

$$\bar{\Delta}_{u_4u_4} = 2(\bar{U}^2_{u_4} + \bar{V}^2_{u_4} - S^2).$$

In view of the reversed Cauchy inequality (2.8)

$$\bar{\Delta}^2_{u_2u_4} - \bar{\Delta}_{u_2u_2}\bar{\Delta}_{u_4u_4} \geqq 0,$$

with equality prevailing if and only if

$$\frac{\bar{U}_{u_2}}{\bar{U}_{u_4}} = \frac{\bar{V}_{u_2}}{\bar{V}_{u_4}} = \frac{R}{S}. \tag{3.8}$$

Thus the inequalities in (3.6) cannot be fulfilled simultaneously. In place of the last inequality, we are forced to consider the ambiguous case

$$\bar{\Delta}^2_{u_2u_4} - \bar{\Delta}_{u_2u_2}\bar{\Delta}_{u_4u_4} = 0 \tag{3.9}$$

at the critical point P_0 in our attempt to insure that Δ has a relative maximum at P_0.

One readily verifies that

$$\frac{R}{S} = \frac{\bar{W}_{u_2u_2}}{\bar{W}_{u_2u_4}} = \frac{f_1''}{\dot{f}_1'}.$$

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If we introduce

$$\omega = -\frac{f_1''}{\dot{f}_1'},$$

under the assumption that

$$f_1'' \not\equiv 0, \quad \dot{f}_1' \not\equiv 0,$$

i.e., f_1 is neither a linear function of u_1, nor an additively separable function of u_1, u_3, equations (3.7), (3.8) become

$$\bar{\tau}_{u_2} + \omega\bar{\tau}_{u_4} = 0, \quad \bar{U}_{u_2} + \omega\bar{U}_{u_4} = 0, \quad \bar{V}_{u_2} + \omega\bar{V}_{u_4} = 0.$$

From the first of these equations we see that

$$\bar{\tau}_{u_2u_1} + \bar{\tau}_{u_2u_2} + \omega(\bar{\tau}_{u_4u_1} + \bar{\tau}_{u_4u_2}) + \omega'\bar{\tau}_{u_4} = 0,$$

$$\bar{\tau}_{u_2u_3} + \bar{\tau}_{u_2u_4} + \omega(\bar{\tau}_{u_4u_3} + \bar{\tau}_{u_4u_4}) + \dot{\omega}\bar{\tau}_{u_4} = 0,$$

and when these equations are used to eliminate the partial derivatives of the first two orders of τ on S_2 from the last two equations, we obtain an over-determined system

$$f_1\omega' + f_1'\omega + \dot{f}_1\omega^2 = 0, \quad f_1\dot{\omega} - f_1' - \dot{f}_1\omega = 0,$$

of partial differential equations for f_1. From the lemma in the appendix it follows that the function f_1 must be one of the following types

$$\text{(i) } f_1 = g(mu_1 + nu_3), \quad \text{(ii) } f_1 = \bar{f}_1 + g\left(\frac{u_3 - \bar{u}_3}{u_1 - \bar{u}_1}\right), \tag{3.10}$$

where $g = g(\xi)$ is an arbitrary function and m, n, $\bar{u}_1$, $\bar{u}_3$, $\bar{f}_1$ denote arbitrary constants.

Summing up our results, we see that the conditions (3.6) for Δ to have zero for a relative maximum at P_0 cannot be satisfied simultaneously. If the first four are satisfied, the ambiguous case (3.9) arises at the critical point P_0 and if f_1 is neither a linear function of u_1, nor an additively separable function of u_1, u_3, it must be one of the types (3.10). Boundary problems of the latter two types have been studied by Cushing (5) in his thesis and Dunninger (7) has obtained uniqueness theorems if f_1 is of type (i) in (3.10) in the special cases

$$f_1 = e^{mu_1 + nu_3}, \quad f_1 = (mu_1 + nu_3)^{1+p}, \quad m \neq 0, \quad (p = 1, 2, \ldots).$$

4. The case $f = f(mu + nv)$

This section will be devoted to the special case of the boundary problem (1.2)

$$u_n = h(s)f(\xi), \quad \xi = mu + nv, \quad m, n = \text{const.}, \tag{4.1}$$

to which we have been led by the considerations of the previous section.

Given a solution $w_1 = u_1 + iu_3$ of (4.1), a one-parameter family of solutions

$$w_2 = w_1 + i(m + in)c, \quad c = \text{const. (real)}, \tag{4.2}$$

is generated by it. Under what conditions are these the only solutions of (4.1)?

The mapping

$$\xi_1 = mu_1+nu_3, \quad \xi_2 = mu_2+nu_4, \tag{4.3}$$

carries the space E_4 of the variables u_1, u_2, u_3, u_4 into the plane E_2 of the variables ξ_1, ξ_2. This mapping carries the hyperplane

$$S_3: \; m(u_1-u_2)+n(u_3-u_4) = 0,$$

in E_4 into the straight line

$$S_1: \; \xi_1 = \xi_2$$

of E_2. In addition to containing the subspace S_2 of equal solutions, S_3 also contains the one-parameter family of solutions (4.2) generated by a given solution w_1. If w_1, w_2 are two solutions of (4.1), not related by (4.2), the mapping (4.3) carries the region S of the z-plane into a set Σ of the plane E_2. Clearly

$$|\xi_1-\xi_2| \leqq |w_1-w_2|\sqrt{(m^2+n^2)}.$$

Consequently if w_1, w_2 satisfy (3.3′) the set Σ will be confined to the band

$$0 \leqq |\xi_1-\xi_2| < \rho\sqrt{(m^2+n^2)} \tag{4.4}$$

about the straight line S_1, the level lines $\xi = 0$ of the harmonic function $\xi = \xi_1-\xi_2$, mapping into segments of S_1.

If we assume that

$$\tau = \tau(\xi_1, \xi_2), \quad \xi_1 = mu_1+nu_3, \quad \xi_2 = mu_2+nu_4,$$

U, V, W and Δ become functions of ξ_1, ξ_2. For ξ_1 fixed Δ becomes a function of ξ_2, and in place of (3.6) we now seek a function τ for which

$$\bar{\Delta} = \bar{\Delta}_{\xi_2} = 0, \quad \bar{\Delta}_{\xi_2\xi_2} < 0, \tag{4.5}$$

the $\bar{}$ denoting evaluation on S_1.

From the definition of U, V, W in (3.5) we now have

$$U = \tfrac{1}{2}m[f_1\tau_{\xi_1}+f_2\tau_{\xi_2}-(f_1'-f_2')\tau],$$
$$V = \tfrac{1}{2}n[f_1\tau_{\xi_1}+f_2\tau_{\xi_2}+(f_1'+f_2')\tau],$$
$$W = m^2 f_1(f_1'-f_2')\tau\tau_{\xi_1},$$

where

$$f_1 = f(\xi_1), \quad f_2 = f(\xi_2), \quad f_1' = f'(\xi_1), \quad f_2' = f'(\xi_2).$$

One readily verifies that, in as much as $\bar{\Delta} = \bar{W} = 0$,

$$\bar{U} = \tfrac{1}{2}mf_1(\bar{\tau}_{\xi_1}+\bar{\tau}_{\xi_2}) = 0, \quad \bar{V} = \tfrac{1}{2}n[f_1(\bar{\tau}_{\xi_1}+\bar{\tau}_{\xi_2})+2f_1'\bar{\tau}] = 0,$$

and therefore that

$$\bar{\tau} = \bar{\tau}_{\xi_1}+\bar{\tau}_{\xi_2} = 0,$$

in consequence of which

$$\bar{\Delta}_{\xi_2} = 0, \quad \bar{\Delta}_{\xi_2\xi_2} = 2[\bar{U}_{\xi_2}^2+\bar{V}_{\xi_2}^2]+\bar{W}_{\xi_2\xi_2}.$$

If we assume that τ is regular analytic in the neighbourhood of S_1, in particular permits the expansion

$$\tau = \alpha_1(\xi_2-\xi_1)+\tfrac{1}{2}\alpha_2(\xi_2-\xi_1)^2+\dots, \quad \alpha_k = \alpha_k(\xi_1),$$

one has

$$\bar{\tau} = 0, \quad \bar{\tau}_{\xi_1} = -\alpha_1, \quad \bar{\tau}_{\xi_2} = \alpha_1, \quad \bar{\tau}_{\xi_1\xi_2} = \alpha_1' - \alpha_2, \quad \bar{\tau}_{\xi_2\xi_2} = \alpha_2,$$

and the formulas for $\bar{U}_{\xi_2}$, $\bar{V}_{\xi_2}$, $\bar{W}_{\xi_2\xi_2}$ simplify to

$$\bar{U}_{\xi_2} = \tfrac{1}{2}m(f_1\alpha_1' + f_1'\alpha_1), \quad \bar{V}_{\xi_2} = \tfrac{1}{2}n(f_1\alpha_1' + 3f_1'\alpha_1),$$

$$\bar{W}_{\xi_2\xi_2} = 2m^2 f_1 f_1''\alpha_1^2.$$

When these are inserted in the expression for $\bar{\Delta}_{\xi_2\xi_2}$ above, one finds

$$\bar{\Delta}_{\xi_2\xi_2} = \tfrac{1}{2}(m^2+n^2)[(f_1\alpha_1' + (1+2l)f_1'\alpha_1)^2 + 4(1-l)(f_1 f_1'' + l f_1'^2)\alpha_1^2]$$

where

$$l = n^2/(m^2+n^2), \quad 0 \leqq l < 1, \quad m \neq 0.$$

To make $\bar{\Delta}_{\xi_2\xi_2}$ negative, we set

$$f_1\alpha_1' + (1+2l)f_1'\alpha_1 = 0.$$

This implies

$$\alpha_1 = k f_1^{-1-2l}, \quad k = \text{const.} \tag{4.6}$$

Under these circumstances $\bar{\Delta}_{\xi_2\xi_2}$ will be negative if and only if f_1 satisfies the inequality

$$f_1 f_1'' + l f_1'^2 < 0.$$

A simple calculation verifies that, if $k = -1/m$ in (4.6), $\bar{a} = f_1^{-2l}$, so that $\bar{a}$ will be positive as long as f_1 is positive.

Summing up our results, we have the following theorem.

Theorem 4.1. *If the function $f(u, v)$ in the boundary problem* (1.2) *has the special form*

$$f = f(\xi), \quad \xi = mu + nv, \quad m, n = \text{const.}, \quad m \neq 0,$$

and $f(\xi)$ fulfils the inequality

$$ff'' + lf'^2 < 0, \quad l = n^2/(m^2+n^2), \quad 0 \leqq l < 1, \tag{4.7}$$

apart from the solutions (4.2) *any non-constant solution $w_1 = u_1 + iu_3$ of* (4.1) *for which*

$$f_1 = f(\xi_1) = f_1(mu_1 + nu_3) > 0 \quad \text{in} \quad S + C$$

is locally ρ-unique for some function $\rho = \rho(\xi_1) > 0$.

Under the conditions of the theorem τ may be chosen so that Q is positive semi-definite on S_1 and positive definite in a neighbourhood

$$0 < |\xi_1 - \xi_2| < \delta, \quad \delta = \delta(\xi_1), \tag{4.8}$$

of S_1 for some function $\delta(\xi_1)$. If we choose $\rho = \delta/\sqrt{(m^2+n^2)}$ in (3.3′), it follows from (4.4) that the set Σ into which S is carried by the mapping (4.3) will be confined to the strip

$$0 \leqq |\xi_1 - \xi_2| < \delta. \tag{4.9}$$

If there is a second solution w_2 of (4.1) for which (4.8) holds, the fundamental identity (2.1) implies $p_1 = p_2 = p_3 = p_4 = 0$, and $w_1 = \text{const.}$, contrary to hypothesis. Consequently such a solution w_2 cannot exist. This result can be extended to include equality as in (4.9), for, the solutions w_2 in (4.2) being excluded, equality can occur only on the level lines $\xi_1 - \xi_2 = 0$ in S and the partial derivatives p_k will continue to vanish on these level lines by continuity.

That functions $f(\xi)$ exist for which the inequality (4.7) holds can be seen by simple examples. If $f = \xi^{1/p}$, one readily verifies that

$$ff'' + lf'^2 = \frac{1+l-p}{p^2\xi^{2(1-q)}}, \quad q = 1/p,$$

and consequently (4.7) holds for $\xi > 0$ provided $p > 1 + l$. More generally if

$$f = g^{1/p}, \quad g = g(\xi),$$

we have

$$ff'' + lf'^2 = \frac{(1+l-p)g'^2 + pgg''}{p^2 g^{2(1-q)}}, \quad q = 1/p.$$

If p is an odd integer greater than $1 + l$ and $gg'' \leqq 0$, inequality (4.7) holds, provided g' does not vanish simultaneously with gg'', e.g., $g = \sin \xi$.

5. The case $f = f(u)$

The restriction $f_1 > 0$ in Theorem 4.1 is an unhappy one. Clearly it is not met in the Levi-Civita problem, in which $\sin \theta = 0$ at the crests and troughs of the waves. In this section we show how the restriction can be removed in the special case in which f in (1.2) does not depend on v. This question has also been treated with other methods by Cushing (**5**).

For the function τ we take

$$\tau = \alpha(u_1)\phi(\lambda), \quad \lambda = f_2/f_1, \quad f_1 = f(u_1), \quad f_2 = f(u_2), \tag{5.1}$$

where $\alpha(u_1)$, $\phi(\lambda)$ are two analytic functions at our disposal. We assume that $\bar{\phi} = \phi(1) = 0$, to insure that τ vanishes on the straight line S_1: $u_1 = u_2$. Using (2.3) one verifies that

$$a = \alpha' f_2 \phi - \alpha f_1' \lambda^2 \phi',$$

$$2b = [\alpha(f_2' - f_1') - \alpha' f_1]\phi + \alpha\lambda(f_1' + f_2')\phi',$$

$$c = -\alpha f_2' \phi', \quad d = 0.$$

Consequently the integrals in the fundamental identity (2.1) exist, provided the ratio λ is regular analytic in $S + C$.

For u_1 fixed Δ becomes a function of u_2 and in place of (4.5) we seek τ so that

$$\bar{\Delta} = \bar{\Delta}_{u_2} = 0, \quad \bar{\Delta}_{u_2 u_2} < 0. \tag{5.2}$$

where, from (3.5)

$$\Delta = U^2 + W, \quad U = \tfrac{1}{2}[f_1 \tau_{u_1} + f_2 \tau_{u_2} - (f_1' - f_2')\tau],$$
$$W = f_1(f_1' - f_2')\tau\tau_{u_1}.$$

One readily verifies that $\bar{\tau} = 0$ implies the first two equalities in (5.2). In addition one has

$$\bar{\Delta}_{u_2 u_2} = 2\bar{U}_{u_2}^2 + \bar{W}_{u_2 u_2}, \quad \bar{U}_{u_2} = \tfrac{1}{2}\bar{\Phi}'(\alpha f_1')',$$
$$\bar{W}_{u_2 u_2} = 2\bar{\Phi}'^2 \frac{f_1'^2 f_1''}{f_1} \alpha^2,$$

so that if one chooses

$$\bar{\Phi}' = \phi'(1) = -1, \quad \alpha = 1/f_1'$$

one obtains

$$\bar{a} = 1, \quad \bar{\Delta}_{u_2 u_2} = 2f_1''/f_1.$$

The case $f = f(mu + nv)$ considered in the previous section reduces to the case $f = f(u)$ if $m = 1$ and $n = 0$. Theorem 4.1 still applies, but in addition we have the following theorem.

Theorem 5.1. *If the function $f(u, v)$ in the boundary problem* (1.2) *has the special form*

$$f = f(u),$$

and $f(u)$ fulfils the inequality

$$f''/f < 0$$

apart from the solutions

$$w_2 = w_1 + ic, \quad c = \text{const. (real)},$$

any non-constant solution $w_1 = u_1 + iu_3$ for which

$$f_1' = f'(u_1) \neq 0 \quad in \quad S + C$$

is locally ρ-unique for some function $\rho = \rho(u_1) > 0$ among the analytic functions $w_2 = u_2 + iu_4$ for which $\lambda = f_2/f_1$ is regular analytic in $S + C$.

If, for example, $f = \sin u$, the hypotheses of the theorem are met for a non-constant solution $w_1 = u_1 + iu_3$ provided $0 \leqq |u_1| < \pi/2$ on $S + C$.

Appendix

This section contains a proof of the following lemma

Lemma. *The only solutions of the over-determined system*

$$zw_x + z_x w + z_y w^2 = 0, \quad zw_y - z_x - z_y w = 0, \quad w = -z_{xx}/z_{xy},$$

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are developable surfaces $z = z(x, y)$, and are either those with parallel, horizontal rulings

$$z = g(mx+ny), \quad m, n = \text{const.},$$

or are cones

$$z = z_0+(x-x_0)g\left(\frac{y-y_0}{x-x_0}\right), \quad x_0, y_0, z_0 = \text{const.}$$

From the integrability condition

$$w_{xy}-w_{yx} = w(z_{xx}z_{yy}-z_{xy}^2)/zz_{xy} = 0$$

we see that any solution $z = z(x, y)$ is a developable surface. Taking a developable surface to be the envelope of a one-parameter family of planes

$$z = ax+by+c,$$

where a, b, c are functions of a parameter t, the Cartesian equation of the envelope is obtained by eliminating t between this equation and

$$\dot{a}x+\dot{b}y+\dot{c} = 0$$

(where the dots denote differentiation with respect to t), an equation which defines $t = t(x, y)$ implicitly. One readily verifies that

$$z_x = a, \quad z_y = b, \quad z_{xx} = \dot{a}t_x, \quad z_{xy} = \dot{a}t_y = \dot{b}t_x, \quad z_{yy} = \dot{b}t_y,$$

from which it follows that the developable surface will be a solution of the system if and only if the Wronskian of a, b, c vanishes, i.e., the functions a, b, c are linearly dependent.

If a, b are linearly dependent, say $na = mb$ for constant m, n, the envelope is a ruled surface

$$z = g(mx+ny)$$

with parallel, horizontal rulings. If a, b are linearly independent, we can write $ax_0+by_0+c = 0$, with x_0, y_0 constant, and find that the envelope is a cone

$$z = z_0+(x-x_0)g\left(\frac{y-y_0}{x-x_0}\right).$$

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