ON FRATTINI-LIKE SUBGROUPS

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1. Introduction and results. For any group G, denote by $\varphi_f(G)$ (respectively $L(G)$) the intersection of all maximal subgroups of finite index (respectively finite nonprime index) in G, with the usual provision that the subgroup concerned equals G if no such maximals exist. The subgroup $\varphi_f(G)$ was discussed in [**1**] in connection with a property ν possessed by certain groups: a group G has ν if and only if every nonnilpotent, normal subgroup of G has a finite, nonnilpotent G-image. It was shown there, for instance, that $G/\varphi_f(G)$ has ν for all groups G. The subgroup $L(G)$, in the case where G is finite, was investigated at some length in [**3**], one of the main results being that $L(G)$ is supersoluble. (A published proof of this result appears as Theorem 3 of [**4**]). The present paper is concerned with the role of $L(G)$ in groups G having property ν or a related property σ, the definition of which is obtained by replacing "nonnilpotent" by "nonsupersoluble" in the definition of ν. We also present a result (namely Theorem 4) which displays a close relationship between the subgroups $L(G)$ and $\varphi_f(G)$ in an arbitrary group G. Some of the results for finite groups in [**3**] are found to hold with rather weaker hypotheses and, in fact, remain true for groups with ν or σ. We recall that if a group has σ it also has ν ([**2**], Theorem 2) but not conversely. For example, $G = \langle x, y : y^{-1}xy = x^2 \rangle$ has ν but not σ. It is a well-known result of Gaschütz ([**8**], 5.2.15) that, in a finite group G, if H is a normal subgroup containing $\varphi(G)$ such that $H/\varphi(G)$ is nilpotent than H is nilpotent. This remains true in the case where G is any group with ν [**1**, Proposition 1]. Our first result is in a similar vein and is a generalization of Theorem 9 of [**7**] and Theorem 1.2.9 of [**3**], the latter of which states that, for a finite group G, if $G/L(G)$ is supersoluble, then so is G.

THEOREM 1. *Let G be a group and H a normal subgroup of G such that $H/H \cap L(G)$ is supersoluble. If H is finitely generated and has property σ, then H is supersoluble.*

COROLLARY. *Let G be a group with σ and suppose that H is a normal subgroup of G such that $H/H \cap L(G)$ is supersoluble. Then H is supersoluble. In particular, $L(G)$ is supersoluble.*

In order to see that the corollary is indeed a consequence of Theorem 1, we need only establish that H is finitely generated, since, by Lemma 3 of [**2**], the property σ is inherited by normal subgroups. It suffices, therefore, to show that $L(G)$ is supersoluble. But if N is a G-invariant subgroup of finite index in $L(G)$ then $L(G/N) = L(G)/N$ is supersoluble, by Theorem 1. The result follows since G has σ.

We note that, whereas in the theorem of Gaschütz mentioned above (and in the corresponding theorem on groups with ν) it suffices that H be subnormal in G, it is not clear whether we may replace "normal" by "subnormal" in our corollary to Theorem 1. Unlike the class of nilpotent groups, that of supersoluble (finite) groups is not N_0-closed [**8**, ex. 6, p. 152]. Whether the generalized form of the corollary nevertheless holds does not seem easy to determine. A closely related problem (and one which we have been unable to solve) is: Let G be a finite group and H a subnormal subgroup of G containing $\varphi(G)$ such that $H/\varphi(G)$ is supersoluble. Is H supersoluble?

We mention that Mukherjee and Bhattacharya [**7**, Theorem 9] answer this question in the affirmative in the case where H is normal in G.

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In general, if $L(G)$ is supersoluble, then G need not satisfy σ. For let S be a finite, nonabelian simple group, let M be a direct sum of infinitely many irreducible S-modules M_i, $i \in \mathbb{N}$, and write $G = M]S$ for the natural split extension. It is routine to verify that $L(G) = 1$, while M is a nonsupersoluble normal subgroup of G all of whose finite images are supersoluble. However, with an additional hypothesis, we are able to provide a sufficient condition, as follows.

THEOREM 2. *A group G has σ if and only if $L(G)$ is polycyclic and $G/L(G)$ has σ.*

The hypothesis that $L(G)$ is polycyclic certainly cannot be dispensed with. The group $G = \langle x, y : y^{-1}xy = x^2 \rangle$ does not have σ, although every maximal subgroup of G has prime index and so $L(G) = G$.

In a finite soluble group G, $G' \cap L(G)$ is nilpotent [**3**, Theorem 1.2.4]. Our next result shows that the hypothesis of solubility is not required. Indeed, the following holds.

THEOREM 3. *Let G be a group with ν. Then $G' \cap L(G)$ is finitely generated nilpotent.*

In the special case where G has σ, we know from the corollary to Theorem 1 that $L(G)$ is supersoluble and hence $L'(G)$ is nilpotent. Among other things, Theorem 3 may be viewed as an improvement on this last result.

It was remarked earlier that $G/\varphi_f(G)$ has ν for all groups G. It follows from Theorem 3 that $G' \cap L(G)$ is always finitely generated nilpotent modulo $\varphi_f(G)$. By a result of P. Hall [**6**, Lemma 3] we have $(G' \cap L(G))' \leq \varphi_f(G)$ for all G. In fact, rather more than this is true.

THEOREM 4. *Let G be any group. Then $G'' \cap L(G) \leq \varphi_f(G)$.*

We saw that Theorem 1 was suggested, in part, by a similar result concerning the nilpotency of certain normal subgroups of ν-groups. Our final theorem establishes the nilpotency of certain subgroups H of ν-groups G such that H is nilpotent modulo $L(G)$. (The corresponding nilpotency result for groups with σ is once again an immediate consequence).

THEOREM 5. *Let G be a group with ν and suppose that H is an ascendant subgroup of G'. Then H is nilpotent if and only if $H/H \cap L(G)$ is nilpotent.*

The special case of this theorem where G is finite and $H = G''$ is Corollary 1.2.7 of [**3**]. It is easy to see that we cannot remove from Theorem 5 the hypothesis that H is contained in G'. If G is a finite supersoluble group then $G = L(G)$ but G need not of course be nilpotent.

2. Proofs of the theorems. If N is a normal subgroup of a group G then $L(G)N/N \leq L(G/N)$. This elementary fact will be used henceforth without further mention.

Proof of Theorem 1. Suppose that the hypotheses of the theorem are satisfied but H is not supersoluble. Then there is a normal subgroup W of finite index in H such that H/W is not supersoluble. Since H is finitely generated, we may assume $W \lhd G$. We may further assume that $W = 1$ and H is finite. Let $K = H \cap L(G)$ (which is nontrivial) and let p denote the largest prime dividing the order of K. Let P be a Sylow p-subgroup of K.

Then, by Sylow's Theorem and the Frattini argument, we have $G = KN_G(P)$. If $N_G(P) \neq G$, then there is a maximal subgroup M of finite index in G such that $N_G(P) \leq M$. Since $K \leq L(G)$, the index of M in G is some prime t, say. Again by Sylow's Theorem, the index of $N_K(P)$ in each of the subgroups K and $K \cap M$ is congruent to 1 modulo p. Thus $t = |K : M \cap K| \equiv 1 \pmod p$, contradicting the choice of p. Hence $P \lhd G$. Now let N be a minimal normal subgroup of G contained in P. By induction on the order of H we may assume that H/N is supersoluble. The set Ω of supersoluble projectors of H forms a conjugacy class of supersoluble, self-normalizing subgroups of H [**5**, Satz 7.10, p. 700 and Hilfssatz 7.11, p. 701]. Let $S \in \Omega$. Applying the Frattini argument again, we find that $G = HN_G(S)$ and hence $G = NN_G(S)$. If $N_G(S) \neq G$, then there is a maximal subgroup T of G such that $G = NT$ and $|G : T| = p$. Since N is abelian, $N \cap T$ is normal in G and hence trivial. Thus N has order p and H is supersoluble, a contradiction. Hence $N_G(S) = G$ and so $S = H$, a final contradiction.

Proof of Theorem 2. Suppose that G has σ. By the corollary to Theorem 1, $L(G)$ is supersoluble and hence polycyclic. Let H be a normal subgroup of G containing $L(G)$ such that all finite G-images of $H/L(G)$ are supersoluble. Let W be an arbitrary normal subgroup of finite index in G. By Theorem 1 HW/W is supersoluble and, since G has σ, H is supersoluble, by Lemma 1 of [**2**]. Therefore $H/L(G)$ is supersoluble and $G/L(G)$ has σ. Conversely, suppose that $L(G)$ is polycyclic and that $G/L(G)$ has σ. Let H be a normal subgroup of G all of whose finite G-images are supersoluble. Then $HL(G)/L(G)$ is supersoluble and thus H is polycyclic. By a theorem of Baer [**9**, Lemma 11.11] H is supersoluble. Therefore G has σ.

Proof of Theorem 3. Suppose first that G is a finite group and let N be a minimal normal subgroup of G. By induction on the order of G we may assume that $(G/N)' \cap L(G/N)$ is nilpotent. Put $H = G' \cap L(G)$. Then HN/N is nilpotent. Clearly we may assume that N is the unique minimal normal subgroup of G and that $N \leq H$. By Theorem 1, H is supersoluble and so N is an elementary abelian p-group, for some prime p. Let Ω denote the set of nilpotent projectors of H. Again from [**5**, pp. 700 and 701], Ω is a conjugacy class of nilpotent, self-normalizing subgroups of H (the Carter subgroups of H). Let $C \in \Omega$. Then $H = NC$, $G = HN_G(C)$ and $G = NN_G(C)$. If $C \lhd G$ then $H = C$ and we are finished. Otherwise, let M be a maximal subgroup of G containing $N_G(C)$. Then, arguing as we did towards the end of the proof of Theorem 1, we deduce that N has order p. Since M is core-free, we see that $C_G(N) \cap M = 1$ and thus $N = C_G(N)$. It follows that $G' \leq N$ and hence that H is nilpotent. Now suppose that G is an arbitrary group with ν and let T be a normal subgroup of finite index in G. By the above, $G' \cap L(G)$ is nilpotent modulo T. Hence, by Lemma 1 of [**1**], $G' \cap L(G)$ is nilpotent.

Proof of Theorem 4. Let G be a group and put $H = G'' \cap L(G)$. In order to show $H \leq \varphi_f(G)$ we may suppose $\varphi_f(G) = 1$. Then G is residually (finite with trivial Frattini subgroup), which may be seen by considering the normal cores of the maximal subgroups of finite index. Thus we may assume that G is finite. Suppose, for a contradiction, that G is of minimal order subject to $\varphi(G) = 1$ and $H \neq 1$. By Theorem 1, H is supersoluble and hence contains a nontrivial, G-invariant p-subgroup P, for some prime p. Let M be a maximal subgroup of G with $P \nleq M$. Then $|G : M| = p$. Now let N be the core of M in G. Then $\varphi(G/N) = 1$ and so, if $N \neq 1$, induction gives $H \leq N$ and thus the contradiction

$P \leqslant M$. Hence $N = 1$ and, via its action on the right cosets of M, G embeds in the symmetric group of degree p. It follows that P has order p and that $C_G(P) = P$. Therefore G is metacyclic and $H \leqslant G'' = 1$, the required contradiction.

Proof of Theorem 5. Let G and H be as given. If H is nilpotent, then of course $H/H \cap L(G)$ is nilpotent. Conversely, assume that $H/H \cap L(G)$ is nilpotent. One checks easily that the hypotheses on H are retained (by the images of H) in each finite image of G. If HK/K is nilpotent for all normal subgroups K of finite index in G then H^GK/K is also nilpotent and thus, by property ν, H^G is nilpotent. In order to show that H is nilpotent, therefore, we may assume G to be finite. Further, we may suppose that H is normal in G and, for a contradiction, that H is not nilpotent. For every nontrivial normal subgroup T of G, HT/T is nilpotent. It follows that G has a unique minimal normal subgroup N and $N \leqslant H \cap L(G)$. By Theorem 1, $L(G)$ is supersoluble, and so N is an elementary abelian p-group for some prime p. Also H is soluble. As in the proof of Theorem 3 we may use the Carter subgroups of H to deduce that N has order p and that $N = C_G(N)$. Thus G is metabelian and H is abelian. This contradiction completes the proof.

Addendum (July, 1992) The question concerning a finite group G with a subnormal subgroup H such that $H/\Phi(G)$ is supersoluble has been answered affirmatively by A. Ballester-Bolinches.

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