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TENSOR PRODUCTS OF DIVISIBLE EFFECT ALGEBRAS

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Tensor products of divisible effect algebras and tensor products of the corresponding universal groups are studied. It is shown that the universal group of the tensor product of divisible effect algebras is (isomorphic to) the tensor product of the corresponding universal groups. Moreover, it is shown that the tensor product of two unit intervals $[0,1]$ of real numbers is not a lattice.

1. INTRODUCTION

Effect algebras as partial algebraic structures with a partially defined operation $\oplus$ and constants 0 and 1 have been introduced as an abstraction of the Hilbert-space effects, that is, self-adjoint operators between 0 and I on a Hilbert space ([**7**]). Hilbert-space effects play an important role in the foundations of quantum mechanics and measurement theory ([**11, 4**]). An equivalent algebraic structure, so-called difference posets (or D-posets) with the partial operation $\ominus$ have been introduced in [**10**].

From the structural point of view, effect algebras are a generalisation of boolean algebras, MV-algebras, orthomodular lattices, orthomodular posets, orthoalgebras. For relations among these structures and some other related structures see, for example, [**5**].

The following definition was introduced in [**7**].

DEFINITION 1.1: An effect algebra is a structure $\mathcal{E} = (E; \oplus, 0, 1)$ consisting of a nonempty set E endowed with a partial binary operation $\oplus$ and with two distinguished elements 0 and 1 which satisfies the following conditions for every $a, b, c \in E$:

- (E1) $a \oplus b = b \oplus a$ in the sense that if one side is defined so is the other, and equality holds,
- (E2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in the sense that if one side is defined, so is the other, and equality holds,
- (E2) to every $a \in E$, there is a unique $a' \in E$ such that $a \oplus a' = 1$,
- (E4) if $a \oplus 1$ is defined then $a = 0$.

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Let $\mathcal{E} = (E; \oplus, 0, 1)$ be an effect algebra. We shall write $a \perp b$ and say that a, b are *orthogonal* if $a \oplus b$ is defined. A partial order $\leqslant$ can be defined on E by defining $a \leqslant b$ if there is $c \in E$ such that $a \oplus c = b$. The element c is then uniquely defined, and may be denoted by $c = b \ominus a$. This enables us to introduce a binary relation $\ominus$, called a *difference*, such that $b \ominus a$ is defined if and only if $a \leqslant b$, and $b \ominus a = c$ if and only if $a \oplus c = b$. In particular, we may write $a' = 1 \ominus a$, the element a' is called the *orthosupplement* of a. In the partial order $\leqslant$, 0 is the smallest and 1 is the greatest element. It turns out that $a \oplus b$ is defined if and only if $a \leqslant b'$, and we have $a \leqslant b$ if and only if $b' \leqslant a'$.

Owing to (E2), we may omit parentheses in expressions like $a_1 \oplus a_2 \oplus a_3$, or more generally, $a_1 \oplus a_2 \oplus \cdots \oplus a_n$, where the latter expression is defined by induction. We shall say that a finite sequence $(a_1, a_2, \ldots, a_n)$ of elements of E is *orthogonal* if $\bigoplus_{i=1}^{n} a_i$ is defined. More generally, an indexed family $(a_\alpha : \alpha \in A)$ is called orthogonal, if every finite subfamily of it is orthogonal, and we define $\bigoplus_{\alpha \in A} a_\alpha := \vee_F \Big(\bigoplus_{\alpha \in F} a_\alpha \Big)$, where the supremum goes over all finite subsets of the index set A, if the supremum exists. An effect algebra $\mathcal{E}$ is called *orthocomplete (σ-orthocomplete)* if the $\oplus$-sum is defined for every (every countable) orthogonal family of elements. It is well-known that an effect algebra $\mathcal{E}$ is σ-orthocomplete if and only if for any non-decreasing sequence (b_n) of elements of E we have that $\bigvee b_n$ exists in E. If $a \in E$, define $0.a = 0$, $1.a = a$, and $\forall n \geqslant 2$, $n.a = (n-1).a \oplus a$ if all the involved elements exist. The greatest $n \in \mathbb{N}$ such that $n.a$ exists in E is called the *isotropic index* of a and is denoted by $\iota(a)$. We agree to write na instead of $n.a$, $a \in E$, $n \in \mathbb{N}$.

If E, F are effect algebras, the mapping $\phi : E \to F$ is a *morphism* if

(i) $\phi(1) = 1$,

(ii) $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

A morphism ϕ is a *monomorphism* if $\phi(a) \perp \phi(b)$ implies $a \perp b$. It is easy to check that a monomorphism is injective. An *isomorphism* is a bijective morphism such that the inverse mapping ϕ^{-1} is also a morphism. Notice that a surjective monomorphism is the same thing as an isomorphism.

In what follows, we often write $a \oplus b$ tacitly assuming that the element is defined.

Let (G, G^+) be a partially ordered Abelian group (additively written) with positive cone, and choose an element $a \in G^+$. Consider the interval $G^+[0, a] = \{g \in G : 0 \leqslant g \leqslant a\}$. Define a partial operation $\oplus$ on $G^+[0, a]$ as follows: $x \perp y$ if $x + y \leqslant a$, and then $x \oplus y = x + y$. It is easy to check that with this operation $\oplus$ and with a as a unit element, $G^+[0, a]$ is an effect algebra. A very important class of effect algebras, so called interval effect algebras, arise this way.

An element $u \in G^+$ is called an *order unit* if for every $g \in G$ there is $n \in \mathbb{N}$ such that $g \leqslant nu$. An element $u \in G^+$ is called a *generative unit* if every element $g \in G^+$ is a finite sum of elements in $G^+[0, u]$. Observe that a generative unit is an order unit, and if

G has an order unit, it is upward directed, equivalently, $G = G^+ - G^+$.

An effect algebra E is an *interval effect algebra* if there is a partially ordered Abelian group (G, G^+), an element $a \in G^+$, and an isomorphism $h : E \to G^+[0, a]$. A group (G, G^+) is called an *ambient group* for E if E is isomorphic with the interval $G^+[0, a]$ and a is a generative unit for G.

An ambient group G for E is called *universal* if for every Abelian group K and, every K-valued additive mapping $f : E \to K$ (that is, a K-valued measure on E), there is a unique group homomorphism $f_* : G \to K$ such that $f_* \cdot \gamma_G(a) = f(a)$ for every $a \in E$, where γ_G is the isomorphism between E and $G^+[0, u]$.

By [**1**], every interval effect algebra has a universal (ambient) group. Clearly, the universal group is unique up to isomorphism.

EXAMPLE 1. The interval $[0, 1]$ of the real line $\mathbb{R}$ is an interval effect algebra. More generally, let $[0, 1]^X$ be the set of all functions from a set X to the unit interval $[0, 1]$. As an interval of $\mathbb{R}^X$, it is an interval effect algebra. Notice that the above examples are also examples of MV-algebras.

EXAMPLE 2. Let H be a Hilbert space. Consider the group of all bounded self-adjoint operators $\mathcal{B}_s(H)$ on H. The interval $\mathcal{E}(H) := [\theta, I]$, where θ is the zero and I is the identity operator, is an interval effect algebra. Elements of $\mathcal{E}(H)$ are called *Hilbert space effects.*

2. DIVISIBLE EFFECT ALGEBRAS

An effect algebra E is called *divisible* if for each $a \in E$ and each $n \in \mathbb{N}$ there is a unique $x \in E$ such that $a = nx$. We shall write $x = (1/n)a$. In a divisible effect algebra, the following properties hold ([**12**]).

LEMMA 2.1. *Let $(E; \oplus, 0, 1)$ be a divisible effect algebra.*

(i) *If $m, n \geqslant 1$ and $a \in E$, then*

$$\frac{1}{n}\Big(\frac{1}{m}a\Big) = \frac{1}{mn}a.$$

(ii) *If $m, n \geqslant 2$ and $a \in E$, then $(1/m)a \perp (1/n)a$ and*

$$\frac{1}{m}a \oplus \frac{1}{n}a = (m+n)\Big(\frac{1}{mn}a\Big) = \frac{m+n}{mn}a.$$

(iii) *If $a, b \in E$ and $a \perp b$ then for any $n \in \mathbb{N}$, $(1/n)a \perp (1/n)b$ and*

$$\frac{1}{n}a \oplus \frac{1}{n}b = \frac{1}{n}(a \oplus b).$$

(iv) *If $a \leqslant b$, then for any $n \in \mathbb{N}$,*

$$\frac{1}{n}a \leqslant \frac{1}{n}b.$$

(v) *If $m \leqslant n$, then for any $a \in E$,*

$$\frac{1}{n}a \leqslant \frac{1}{m}a.$$

(vi) *If na is defined for $n \in \mathbb{N}$, $a \in E$, and $m \geqslant n$, then*

$$\frac{1}{m}(na) = n\Big(\frac{1}{m}a\Big) = \frac{n}{m}a.$$

(vii) *If na is defined for some $n \in \mathbb{N}$, then for any $m \in \mathbb{N}$, $n\big((1/m)a\big)$ is defined, and*

$$n\Big(\frac{1}{m}a\Big) = \frac{1}{m}(na).$$

(viii) *If for some $n \in \mathbb{N}$ and $a, b \in E$, $(1/n)a = (1/n)b$, then $a = b$.*

(ix) *If for some $m, n \in \mathbb{N}$ and $0 \neq a \in E$, $(1/m)a = (1/n)a$, then $m = n$.*

(x) *If $m, n \geqslant 2$, then for any $a, b \in E$, $(1/n)a \perp (1/m)b$.*

Let E and F be divisible effect algebras. A mapping $\phi : E \to F$ is a *d-morphism* if ϕ is a morphism of effect algebras and $\phi\big((1/n)a\big) = (1/n)\phi(a)$.

Lemma 2.2. *Every morphism of divisible effect algebras is a d-morphism.*

Proof: Let E, F be effect algebras and let $\phi : E \to F$ be a morphism. Assume that $(1/n)a$ exists in E. That is, $a = n\big((1/n)a\big) = (1/n)a \oplus (1/n)a \oplus \cdots \oplus (1/n)a$ (n-times), which implies $\phi(a) = \phi\big((1/n)a\big) \oplus \cdots \oplus \phi\big((1/n)a\big)$ (n-times), hence $\phi(a) = n\phi\big((1/n)a\big)$, that is, $\phi\big((1/n)a\big) = (1/n)\phi(a)$. ∎

The following theorem shows that divisible effect algebras are interval effect algebras ([**12, 13**]).

Theorem 2.3. *Let $(E; \oplus, 0, 1)$ be a divisible effect algebra. Then there is a partially ordered Abelian group (G, G^+) such that $G = G^+ - G^+$, with an element $u \in G^+$ such that the following properties are satisfied:*

(i) *The effect algebra E is isomorphic with the divisible effect algebra $[0, u]$.*

(ii) *The interval $[0, u]$ generates G^+ in the sense that every element in G^+ is a finite sum of elements in $[0, u]$.*

(iii) *Every K-valued measure $f : E \to K$, where K is an Abelian group, can be uniquely extended to a group homomorphism $f_* : G \to K$.*

Moreover, the group G can be endowed with a structure of an ordered vector space over the field $\mathbb{Q}$ of rational numbers.

In other words, every divisible effect algebra is an interval effect algebra, and the group G from Theorem 2.3 is its universal group. Moreover, G is divisible and unperforated ([**12**]) in the following sense: A partially ordered Abelian group G is *unperforated* if $na \in G^+$ for some $n \in \mathbb{N}$, $a \in G$, implies $a \in G^+$. An Abelian group is *divisible* if for every $x \in G$ and every $n \in \mathbb{N}$ there is a unique element $y \in G$ such that $ny = x$. Observe that divisibility of G implies that G is torsion free, indeed, for every $a \in G$, $n \in \mathbb{N}$, uniqueness of the solution of $ny = a$ implies that for all $y \in G$, $ny = 0$ implies $y = 0$.

Recall that a partially ordered Abelian group G is *perfect* if it has at least one order unit, every nonzero order unit is generative and, if u is a nonzero order unit in G, then G is the universal group for the interval effect algebra $G^+[0, u]$.

It is easy to check that in an unperforated divisible Abelian group, every order unit u generates G^+. Indeed, $0 \leqslant x \leqslant nu$ implies $x = nu - y$ for some $y \geqslant 0$, hence $x = n(u - (1/n)y)$, and $(1/n)y \geqslant 0$, $0 \leqslant u - (1/n)y \leqslant u$. So every positive x can be written as a finite sum of positive elements under u. It follows from the next proposition, that every torsion free ambient group for a divisible effect algebra is perfect.

Proposition 2.4. *Let E be a divisible effect algebra and let G be a partially ordered torsion free Abelian group such that E is isomorphic with he interval $G^+[0, u]$, where u is a generative unit in G. Then G is a universal group for E.*

Proof: We first prove that G is divisible. If $g \in G^+[0, u]$, then owing to the isomorphism between E and $G^+[0, u]$, the element $(1/n)g$ exists and is uniquely defined in $G^+[0, u]$. If $g \in G^+$, then since u is a generative unit, g can be written in the form $g = \sum_{i=1}^{k} a_i$ with $a_i \in G^+[0, u]$. Then we may put $(1/n)g = \sum_{i=1}^{k}(1/n)a_i$. If $g = \sum_{j=1}^{m} b_j$ is another expression of g as a sum of elements in $G^+[0, u]$, $n\sum_{i=1}^{k}(1/n)a_i = g = n\sum_{j=1}^{m}(1/n)b_j$ implies that $\sum_{i=1}^{k}(1/n)a_i = \sum_{j=1}^{m}(1/n)b_j$ owing to the supposition that G is torsion free. Hence $(1/n)g$ is well defined. Moreover, every $g \in G$ is of the form $g = g_1 - g_2$, $g_1, g_2 \in G^+$, so we may put $(1/n)g = (1/n)g_1 - (1/n)g_2$. If $g = h_1 - h_2$, $h_1, h_2 \in G^+$ is another expression for g, then from $g_1 + h_2 = h_1 + g_2$ we obtain $(1/n)(g_1 + h_2) = (1/n)(h_1 + g_2)$, which implies that $(1/n)g$ is well defined.

If $ng \geqslant 0$, $g \in G$, $n \in \mathbb{N}$, then $ng = a_1 + \cdots + a_k$ with $a_i \in G^+[0, u]$, $i = 1, \ldots, k$. Owing to the divisibility of E, $ng = n\big((1/n)a_1 + \cdots + (1/n)a_k\big)$, and since G is torsion free, $g = (1/n)a_1 + \cdots + (1/n)a_k \geqslant 0$. This shows that G is unperforated.

It remains to check the universal property of G. Let $f : E \to K$ be a K-valued measure, where K is an Abelian group. We have, for every $a \in E$, $n \in \mathbb{N}$, $f(n(1/n)a) = f(a)$, so that $nf\big((1/n)a\big) = f(a)$. We extend $f : E \to K$ as follows. First assume that $g \in G^+$, then $g = \sum_{i=1}^{k} a_i$, $a_i \in E$ (we may identify E with $G^+[0, u]$). Define $f_*(g)$

$= \sum_{i=1}^{k} f(a_i)$. Let $g = \sum_{j=1}^{m} b_j, b_j \in E$ be another expression of g. Then we have

$$\frac{1}{mk} g = \frac{1}{k} \sum_{j=1}^{m} \frac{1}{m} b_j$$
$$= \frac{1}{m} \sum_{i=1}^{k} \frac{1}{k} a_j \leqslant u,$$

so that $(1/mk)g \in E$. From this we obtain

$$f_*(g) = mkf_*\Big(\frac{1}{mk}g\Big) = mkf\Big(\frac{1}{mk}g\Big) = mk\frac{1}{k}\bigoplus_{j=1}^{m} f\Big(\frac{1}{m}b_j\Big) = \sum_{j=1}^{m} f(b_j)$$
$$= mk\frac{1}{m}\bigoplus_{i=1}^{k} f\Big(\frac{1}{k}a_i\Big) = \sum_{i=1}^{k} f(a_i).$$

This proves that f_* is well defined, and it is clearly additive on positive elements.

Now let $g = g_1 - g_2$ with $g_1, g_2 \in G^+$. Put $f_*(g) = f_*(g_1) - f_*(g_2)$. If $g = h_1 - h_2, h_1, h_2 \in G^+$ is another form of g, then from $g_1 - g_2 = h_1 - h_2$ we obtain $g_1 + h_2 = g_2 + h_1$, and by the previous part of this proof, $f_*(g_1) + f_*(h_2) = f_*(g_2) + f_*(h_1)$, hence $f_*(g_1) - f_*(g_2) = f_*(h_1) - f_*(h_2)$. This proves that $f_* : G \to K$ is well defined group homomorphism. ∎

3. Tensor products of Abelian groups

Let us consider structures consisting of sets endowed with a commutative, associative operation $+$ with zero element 0 (commutative monoids, Abelian groups), and possibly with a partial ordering $\leqslant$ (partially ordered Abelian groups).

If A, B, C are structures, and $f : A \times B \to C$, we say that f is a *bi-morphism* ([**15**]) when for all $a \in A$ (respectively, $b \in B$) the map $f(a,.)$ (respectively, $f(.,b)$)is a homomorphism of monoids, if $\leqslant$ is defined in A, B, C, we say that f is *positive* when for all positive $a \in A$ and $b \in B$, we have $f(a,b) \geqslant 0$. We say that the [positive] bimorphism f is *universal* (relative to a given category of structures) when for every structure D and every [positive] bimorphism $g : A \times B \to D$, there exists a unique [positive] homomorphism $\overline{g} : C \to D$ such that $\overline{g} \cdot f = g$, in this case the pair (C, f) is unique up to isomorphism and the custom is to call it the *tensor product* of A and B, written $C = A \otimes B$, $f(a,b) = a \otimes b$. This notion is very sensitive to the category of structures under consideration, the latter will be used as a superscript: $\otimes^{cm}$ will denote the tensor product of commutative monoids, $\otimes^{ag}$ will denote the tensor product of Abelian groups and $\otimes^{oag}$ will denote the tensor product of partially ordered Abelian groups. Notice that for all the categories above, the tensor product exists. For relations between $\otimes^{cm}$ and $\otimes^{oag}$ see [**15**].

The tensor product of Abelian groups does not preserve all inner structure of the given groups. In [**15**, Example 1.5], two torsion-free directed interpolation groups A and B are constructed such that $A \otimes^{oag} B$ is not an interpolation group. Even more surprising result is [**15**, Example 1.6]: $\mathbb{R} \otimes^{oag} \mathbb{R}$ is not lattice-ordered group.

Observe that every torsion free partially ordered Abelian group can be made unperforated if we define a new ordering cone by putting $G^+ := \{g \in G : \exists n \in \mathbb{N}, ng \geqslant 0\}$. Indeed, it suffices to prove that G^+ is strict. So assume that $x, y \in G^+$ and $x + y = 0$. Then there are $n, m \in \mathbb{N}$ such that $nx \geqslant 0, my \geqslant 0$ (in the original ordering). Then $0 = mn(x + y) = m(nx) + n(my)$, which implies that $mnx = mny = 0$, and since G is torsion free, $x = 0 = y$.

Theorem 3.1. *Let G and H be partially ordered Abelian groups. Then*

(i) *the tensor product $G \otimes^{ag} H$ of G and H is a partially ordered Abelian group,*

(ii) *if u_G and u_H are (generative) order units in G and H respectively, then $u_G \otimes u_H$ is a (generative) order unit in $G \otimes^{ag} H$.*

Proof: (i) Let $G \otimes^{ag} H$ be a tensor product of G and H as Abelian groups. Define $(G \otimes^{ag} H)^+$ as the set of all finite sums of pure tensors $a \otimes b$ where $a \in G^+$, $b \in H^+$. To see that $G \otimes^{ag} H$ is partially ordered, it suffices to prove that whenever $x_1, \ldots, x_n$ in G and $y_1, \ldots, y_n$ in H are strictly positive, then

$$(x_1 \otimes y_1) + \cdots + (x_n \otimes y_n) \neq 0.$$

In this part, we follow the proof of [**8**, Proposition 2.1]. Let G' be the subgroup of G generated by $x_1, \ldots, x_n$, then the element $v = x_1 + \cdots + x_n$ is an order unit in G'. Since $v > 0$, there is a state s on (G', v), and since $s(v) = s(x_1) + \cdots + s(x_n) = 1$, there must be $s(x_j) > 0$ for at least one j. Since $\mathbb{R}$ is divisible, s extends to a homomorphism $g : G \to \mathbb{R}$ (not necessarily positive). Similarly, there is a homomorphism $h : H \to \mathbb{R}$ such that $h(y_i) \geqslant 0$ for $i = 1, \ldots, k$ and $h(y_j) > 0$ for at least one $j \in \{1, \ldots, k\}$. Let us renumber $\{1, \ldots, n\}$ such that $s(x_i) > 0$ for $i = 1, \ldots, k$ and $s(x_i) = 0$ for $i = k+1, \ldots, n$.

There is a homomorphism $f : G \otimes^{ag} H \to \mathbb{R}$ such that

$$f(x \otimes y) = g(x)h(y), x \in G, y \in H$$

and from

$$f(x_1 \otimes y_1 + \cdots + x_n \otimes y_n) = s(x_1)h(y_1) + \cdots + s(x_n)h(y_n) > 0,$$

we see that $x_1 \otimes y_1 + \cdots + x_n \otimes y_n \neq 0$.

Let $x_1, y_1 \in G$, $x_2, y_2 \in H$, $0 \leqslant x_i \leqslant y_i$, $i = 1, 2$. We have

$$y_1 \otimes y_2 - x_1 \otimes x_2 = y_1 \otimes y_2 - y_1 \otimes x_2 + y_1 \otimes x_2 - x_1 \otimes x_2 = y_1 \otimes (y_2 - x_2) + (y_1 - x_1) \otimes x_2,$$

and since both summands on the right are in $(G \otimes^{ag} H)^+$, we have

$$x_1 \otimes x_2 \leqslant y_1 \otimes y_2.$$

(ii) Let u_G and u_H be order units in G and H, respectively, and $u = u_G \otimes u_H$. Let $x \in G \otimes^{ag} H$. Then x is a finite sum of pure tensors, and we have to prove that every of these pure tensors is bounded by a positive multiple of u. Thus we may assume that $x = x_1 \otimes x_2$, $x_1 \in G$, $x_2 \in H$. Hence x is the sum of terms of the form $\pm(y_1 \otimes y_2)$, $y_1 \in G^+, y_2 \in H^+$. So it suffices to prove that $x = x_1 \otimes x_2$, $x_1 \in G^+, x_2 \in H^+$ is bounded by nu, $n \in \mathbb{N}$. Since $x_1 \leqslant k_1 u_G$, $x_2 \leqslant k_2 u_H$ for some $k_1, k_2 \in \mathbb{N}$, we have $x_1 \otimes x_2 \leqslant k^2 u$, where $k = \max(k_1, k_2)$. If u_G and u_H are generative, then for every $v \in (G \otimes^{ag} H)^+$, we have $v = x_1 \otimes y_1 + \dots + x_n \otimes y_n$, $x_i \in G^+, y_i \in H^+, i = 1, \dots, n$, and $x_i = x_{i1} + \dots + x_{im_i}$, $i = 1, \dots, n$, $y_j = y_{j1} + \dots + y_{jk_j}$, $j = 1, \dots, n$ with $0 \leqslant x_{ij} \leqslant u_G$, $0 \leqslant y_{rs} \leqslant u_H$. We then get that v is the sum of pure tensors of the form $x_{ij} \otimes y_{rs}$, which all are under $u_G \otimes u_H$, and hence $u_G \otimes u_H$ generates $(G \otimes^{ag} H)^+$. ∎

THEOREM 3.2. *Let G and H be divisible unperforated partially ordered groups. The tensor product $G \otimes^d H$ in the category of divisible unperforated partially ordered Abelian groups exists, and equals to $G \otimes^{oag} H$. More precisely, $G \otimes^{oag} H$ has the following universal property: to every positive d-bi-morphism $\beta : G \times H \to K$, where K is an unperforated partially ordered Abelian group, there is a unique positive d-homomorphism $\beta_* : G \otimes^{oag} H \to K$ such that $\beta(a, b) = \beta_*(a \otimes b)$.*

PROOF: Let F be the free group on the set $G \times H$ and $\eta : G \times H \to F$ the natural embedding. Every element of F can be expressed uniquely by $x = \sum_{i=1}^{n} \eta(x_i, y_i)$ where $x_i \in G, y_i \in H, i = 1, \dots, n$. We define $0 \neq x \in F$ to be positive if for each i, $0 \leqslant x_i \in G$, $0 \leqslant y_i \in H$. Then F becomes an unperforated partially ordered group. Let M be the subgroup of F generated by the elements of the form

$$\eta(a_1 + a_2, b) - \eta(a_1, b) - \eta(a_2, b) \text{ and}$$
$$\eta(a, b_1 + b_2) - \eta(a, b_1) - \eta(a, b_2).$$

It was proved in **[14]** that M is convex in F. The order on F then induces the desired order on F/M, and $F/M = G \otimes^{oag} H$.

Let $\tau : G \times H \to F/M$ be the canonical mapping, so $\tau(g, h) = \eta(g, h) + M$. The mapping τ is clearly positive bilinear with respect to rational numbers. Indeed,

$$\tau(na, b) = n\tau(a, b) = \tau(a, nb) \text{ and}$$

$$\tau(a, b) = n\tau\left(\frac{1}{n}a, b\right) = n\tau\left(a, \frac{1}{n}b\right)$$

implies that

$$\frac{m}{n}\tau(a, b) = \tau\left(\frac{m}{n}a, b\right) = \tau\left(a, \frac{m}{n}b\right),$$

since F/M is torsion free.

We need to verify that F/M is unperforated. Let $x \in F/M$ and suppose that $nx > 0$ for some $n \in \mathbb{N}$. Then $nx = \sum_{i=1}^{k} \tau(a_i, b_i)$, where $0 < a_i \in G$ and $0 < b_i \in H$ $(1 \leqslant i \leqslant k)$. For each i let $c_i \in G$ be the (unique) solution to the equation $nz = a_i$; necessarily $c_i > 0$ since G is unperforated. Then

$$nx = \sum_{i=1}^{k} \tau(a_i, b_i) = \sum_{i=1}^{k} \tau(nc_i, b_i) = n \sum_{i=1}^{k} \tau(c_i, b_i),$$

F/M is torsion free, so that $x = \sum_{i=1}^{k} \tau(c_i, b_i) \geqslant 0$ by the definition of $(F/M)^+$.

Now let β be any positive d-bi-morphism of $G \times H$ into an unperforated, partially ordered group L. Define $\overline{\beta}\big(\tau(g,h)\big) = \beta(g,h)$, since β and τ are positive d-bi-morphisms, so is $\overline{\beta}$. So $\overline{\beta}$ can be extended to the whole F/M.

Hence F/M is the tensor product of G and H in the category of unperforated, divisible, partially ordered Abelian groups, and we may write $G \otimes^d H = F/M$. ∎

4. Tensor product of divisible effect algebras

Let E, F and T be effect algebras. We recall that a mapping $\beta : E \times F \to T$ is a *bi-morphism* if the following conditions are satisfied.

(i) For every $a \in E$, $\beta(a, b \oplus c) = \beta(a,b) \oplus \beta(a,c)$ whenever $b \oplus c$ exists in F.

(ii) For every $c \in F$, $\beta(a \oplus b, c) = \beta(a,c) \oplus \beta(b,c)$ whenever $a \oplus b$ exists in E.

(iii) $\beta(1,1) = 1$.

Definition 4.1: Let E, F and T be effect algebras. A bi-morphism $\beta : E \times F \to T$ is called a d-bi-morphism if the following conditions hold.

(i) For all $n \in \mathbb{N}$, $a, b \in E$, $\beta\big((1/n)a, b\big) = (1/n)\beta(a,b)$ whenever $(1/n)a$ exists in E,

(ii) For all $n \in \mathbb{N}$, $a, b \in E$, $\beta(a, (1/n)b) = (1/n)\beta(a,b)$ whenever $(1/n)b$ exists in F.

Proof of the following lemma is straightforward.

Lemma 4.2. *Let E, F and L be divisible effect algebras. Then every bi-morphism $\beta : E \times F \to L$ is a d-bi-morphism.*

Definition 4.3: ([**2, 3**]) Let E and F be effect algebras. An effect algebra T is a tensor product of E and F in the category of effect algebras if the following statements hold.

(i) There is a bi-morphism $\tau : E \times F \to T$, such that for any bi-morphism $\beta : E \times F \to L$ into an effect algebra L there is a morphism $\phi : T \to L$ such that $\phi \circ \beta = \tau$.

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(ii) Every element $d \in T$ is a finite $\oplus$-sum of elements of the form $\tau(a,b)$, that is, $d = \bigoplus_{i=1}^{n} \tau(a_i, b_i)$.

If the tensor product exists, it is unique up to isomorphism It has been proved in [**2**] that the tensor product exists if and only if there is at least one bi-morphism $\beta : E \times F \to L$, where L is any effect algebra. In particular, the tensor product exists if each of E and F has at least one state (that is, a morphism into the unit interval of reals) ([**6**]).

The following definition is a natural reformulation of the definition of tensor products of effect algebras.

DEFINITION 4.4: Let E and F be divisible effect algebras. A divisible effect algebra T is a tensor product of E and F in the category of divisible effect algebras if the following statements hold.

(i) There is a d-bi-morphism $\tau : E \times F \to T$, such that for any d-bi-morphism $\beta : E \times F \to L$ into a divisible effect algebra L there is a d-morphism $\phi : T \to L$ such that $\phi \circ \beta = \tau$.

(ii) Every element $d \in T$ is a finite $\oplus$-sum of elements of the form $\tau(a,b)$, that is, $d = \bigoplus_{i=1}^{n} \tau(a_i, b_i)$.

It is usual to denote by $E \otimes F$ the tensor product of effect algebras E and F in the category of effect algebras. We shall write $T = E \otimes^d F$ if T is a tensor product of E and F in the category of divisible effect algebras. Clearly, if $E \otimes^d F$ exists, it is unique up to a d-isomorphism.

We shall say that an effect algebra has the property (d) if for all $x, y \in E$, $n \in \mathbb{N}$ such that nx and ny exist, $nx = ny$ implies $x = y$. Clearly, every divisible effect algebra has this property.

THEOREM 4.5. *Let E, F be divisible effect algebras. The tensor product $E \otimes^d F$ in the category of divisible effect algebras exists and coincides with the tensor product $E \otimes F$ in the category of effect algebras with property (d).*

PROOF: Let (G_E, u), (G_F, v) be universal groups for E and F, respectively, and let $\gamma_E : E \to G_E^+[0,u]$, $\gamma_F : F \to G_F^+[0,v]$ the corresponding isomorphisms. Let $T = G_E \otimes^{oag} G_F$ be the tensor product of universal groups. Define $\tau : E \times F \to T^+[0, u \otimes v]$ by $\tau(a,b) = a \otimes b$. Then τ is a d-bi-morphism into a divisible effect algebra $T^+[0, u \otimes v]$. Similarly as in the case of tensor product of effect algebras, we can prove that tensor product in the category of divisible effect algebras exists. Denote it by K. Similarly, tensor product in the category of effect algebras with property (d) exists, and denote it by L. Let $\kappa : E \times F \to K$ and $\lambda : E \times F \to L$ be the corresponding tensor product bi-morphisms. Since K has property (d), there is a morphism $\phi : L \to K$ which extends κ. If we prove that L is divisible, then there is a morphism $\psi : K \to L$ which extends λ.

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Since $\kappa(E \times F)$ generates K, and $\lambda(E \times F)$ generates L, the isomorphism of K and L follows.

Let $u \in L$, then $u = a_1 \otimes b_1 \oplus \cdots \oplus a_n \otimes b_n$ with $a_i \in E, B_i \in F$, $i = 1, \ldots, n$. Define $(1/n)u = (1/n)a_1 \otimes b_1 \oplus \cdots \oplus (1/n)a_n \otimes b_n$. Since L has property (d), $(1/n)u$ is uniquely defined. It is easy to check that the properties of a divisible effect algebra are satisfied. $\square$

THEOREM 4.6. *Let (G_E, u) and (G_F, v) be the universal groups for divisible effect algebras E and F, respectively. Then $(G_E \otimes^{oag} G_F, u \otimes v)$ is the universal group for $E \otimes^d F$.*

PROOF: Put $H = G_E \otimes^{oag} G_F$, and let $\tau : G_E \times G_F \to H$ be the tensor product bimorphism. We can identify E with an interval $G_E^+[0, u]$, and F with an interval $G_F^+[0, v]$, where $u \in G_E$, $v \in G_F$ are the corresponding generative units. Now $\forall x, y \in G_E \times G_F$, $x \in E$ if and only if $0 \leqslant x \leqslant u$, $y \in F$ if and only if $0 \leqslant y \leqslant v$, so that $0 \leqslant \tau(x, y) \leqslant \tau(u, v) = u \otimes v$. On the other hand, every element $g \in H^+$ is a finite sum of pure tensors $g = \sum_{i=1}^{n} \tau(x_i, y_i)$ with $x_i \in G_E^+$, $y_i \in G_F^+$, and every x_i is a finite sum of elements from E, every y_j is a finite sum of elements from F. Consequently, g can be written as a finite sum $\sum_{i=1}^{m} \tau(a_i, b_i)$ with $a_i \in G_E^+[0, u]$, $b_i \in G_F^+[0, v]$ for all i, and $a_i, b_i > 0$ gives $\tau(a_i, b_i) > 0$. Therefore, we may identify $a \otimes b$ in $E \otimes F$ with $\tau(a, b) \in H^+[0, u \otimes v]$, and since the corresponding pure tensors generate $E \otimes F$ and $H^+[0, u \otimes v]$, respectively, we may identify $E \otimes F$ with $H^+[0, u \otimes v]$. Using the fact that $E \otimes F$ is divisible, and applying Proposition 2.4, we obtain the desired result. $\square$

5. THE TENSOR PRODUCT $\mathbb{R}^+[0, 1] \otimes \mathbb{R}^+[0, 1]$

It was an open problem whether $\mathbb{R}^+[0, 1] \otimes \mathbb{R}^+[0, 1] = \mathbb{R}^+[0, 1]$ (in the category of effect algebras), see for example, [**9**, Section 5]. Using results in [**15**] and [**9**], we prove that it is not the case. We note that in [**9**], there was proved that $\mathbb{R}^+[0, 1] \otimes_\sigma \mathbb{R}^+[0, 1] = \mathbb{R}^+[0, 1]$ (in the category of σ-orthocomplete-effect algebras). We shall need the following results.

LEMMA 5.1. [**15**, Claim 1] *Let A and B be torsion free Abelian groups and let $a, a' \in A \setminus \{0\}$, $b, b' \in B \setminus \{0\}$ such that $a \otimes^{ag} b = a' \otimes^{ag} b'$. Then both (a, a') and (b, b') are not independent over $\mathbb{Z}$, that is, there are nonzero pairs of integers (m, m') and (n, n') such that $ma = m'a'$ and $nb = n'b'$.*

The next result collets some useful results about bi-morphisms on $\mathbb{R}^+[0, 1] \times \mathbb{R}^+[0, 1]$ [**9**, Theorem 4.8].

LEMMA 5.2. *Let $\beta : E \times E \to P$ be a bimorphism, where $E = \mathbb{R}^+[0, 1]$ and P is an effect algebra.*

(i) *For every $a \in E$, $r \in \mathbb{Q}^+[0, 1]$ we have $\beta(a, r) = \beta(ar, 1)$.*

(ii) *If $\beta(a, b) \geqslant \beta(c, d)$, then $ab \geqslant cd$.*

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(iii) *If $ab > cd$, then $\beta(a,b) > \beta(c,d)$.*

Theorem 5.3. *Put $E = \mathbb{R}^+[0,1]$. Then $E \otimes E$ is not lattice ordered.*

Proof: Tensor product $E \otimes E$ exists, since there is a state on E, [5].

Put $\alpha = \sqrt{2}/2$, and let $(p_n/q_n) \to \alpha$ be a sequence of rational numbers such that $1/2 \leqslant p_{2n}/q_{2n} \leqslant \alpha \leqslant p_{2n+1}/q_{2n+1} \leqslant 1$. We may assume that $\rho_n = (p_{2n+1}/q_{2n+1}) \ominus (p_{2n}/q_{2n}) \leqslant 1/n$.

Put $a = 1 \otimes (1/2)$, $b = \alpha \otimes (1/2\alpha)$.

For all $n \geqslant 2$, we have

$$\begin{aligned} b = \alpha \otimes \frac{1}{2\alpha} &\leqslant \frac{p_{2n+1}}{q_{2n+1}} \otimes \frac{1}{2}\frac{q_{2n}}{p_{2n}} \\ &= \left(\frac{p_{2n}}{q_{2n}} \oplus \rho_n\right) \otimes \frac{1}{2}\frac{q_{2n}}{p_{2n}} \\ &= a \oplus \left(\rho_n \otimes \frac{1}{2}\frac{q_{2n}}{p_{2n}}\right) \\ &\leqslant a \oplus \frac{2}{n} a \end{aligned}$$

Similarly,

$$\begin{aligned} b = \alpha \otimes \frac{1}{2\alpha} &\geqslant \frac{p_{2n}}{q_{2n}} \otimes \frac{1}{2}\frac{q_{2n+1}}{p_{2n+1}} \\ &= \left(\frac{p_{2n+1}}{q_{2n+1}} \ominus \rho_n\right) \otimes \frac{1}{2}\frac{q_{2n+1}}{p_{2n+1}} \\ &= 1 \otimes \frac{1}{2} \ominus \rho_n \otimes \frac{1}{2}\frac{q_{2n+1}}{p_{2n+1}} \\ &= a \ominus \rho_n \otimes \frac{1}{2}\frac{q_{2n+1}}{p_{2n+1}} \\ &\geqslant a \ominus \frac{2}{n} a. \end{aligned}$$

This yields that for every $n \geqslant 2$,

$$a \ominus \frac{2}{n} a \leqslant b \leqslant a \oplus \frac{2}{n} a.$$

For sufficiently great n, $b \oplus (2/n)a \leqslant \alpha \otimes 1 \oplus (1/n) \otimes 1 = \big(\alpha \oplus (1/n)\big) \otimes 1 \leqslant 1 \otimes 1$. If $E \otimes E$ is lattice ordered, then $\forall n \geqslant n_0$,

$$c := \big((a \vee b) \ominus a\big) \vee \big((a \vee b) \ominus b\big) \leqslant \frac{2}{n} a$$

Let $\beta : E \times E \to E$, $\beta(a,b) = ab$, then β is a bi-morphism, and it uniquely extends to a morphism $\phi : E \otimes E \to E$. By Lemma 5.2, $ab > 0$ implies $a \otimes b > 0$, and since every element in $E \otimes E$ is a finite $\oplus$-sum of pure tensors, it follows that $\ker \phi = \{0\}$.

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Now

$$\phi(c) \leqslant \phi(\frac{2}{n}a) = \frac{1}{n}$$

for all $n \geqslant n_0$ implies $\phi(c) = 0$, hence $c = 0$. That is, $1 \otimes (1/2) = \alpha \otimes (1/2\alpha)$.

Let $T = \mathbb{R} \otimes^{oag} \mathbb{R}$, and let $\tau : \mathbb{R} \times \mathbb{R} \to T$ be the universal bi-morphism. The restriction of τ to $E \times E$ is a bi-morphism to the effect algebra $T^+[0, \tau(1,1)]$, which by the properties of tensor products uniquely extends to a morphism $\psi : E \otimes E \to T^+[0, \tau(1,1)]$ such that $\psi(a \otimes b) = \tau(a, b)$. Now $1 \otimes (1/2) = \alpha \otimes (1/2\alpha)$ implies $\psi(1 \otimes (1/2)) = \psi(\alpha \otimes (1/2\alpha))$, hence $\tau(1, (1/2)) = \tau(\alpha, (1/2\alpha))$, and we may use Lemma 5.1, which implies $1 = (m/n)\alpha$ for some integers m, n, which is impossible since α is irrational. It follows that $E \otimes E$ is not a lattice, hence $E \otimes E \neq E$. ∎

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