

SEMIGROUPS WHOSE RIGHT IDEALS ARE FINITELY GENERATED

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Abstract We call a semigroup S *weakly right noetherian* if every right ideal of S is finitely generated; equivalently, S satisfies the ascending chain condition on right ideals. We provide an equivalent formulation of the property of being weakly right noetherian in terms of principal right ideals, and we also characterize weakly right noetherian monoids in terms of their acts. We investigate the behaviour of the property of being weakly right noetherian under quotients, subsemigroups and various semigroup-theoretic constructions. In particular, we find necessary and sufficient conditions for the direct product of two semigroups to be weakly right noetherian. We characterize weakly right noetherian regular semigroups in terms of their idempotents. We also find necessary and sufficient conditions for a strong semilattice of completely simple semigroups to be weakly right noetherian. Finally, we prove that a commutative semigroup S with finitely many archimedean components is weakly (right) noetherian if and only if S/\mathcal{H} is finitely generated.

Keywords: semigroup; right ideal; finite generation; ascending chain condition

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1. Introduction

A *finiteness condition* for a class of universal algebras is a property that is satisfied by at least all finite members of that class. Some of the most important finiteness conditions are ascending chain conditions. The study of ascending chain conditions on ideals of rings, initiated by Noether in the early part of the last century, has been instrumental in the development of the structure theory of rings. A ring is *right* (respectively *left*) *Noetherian* if it satisfies the ascending chain condition on right (respectively left) ideals, and *Noetherian* if it is both right Noetherian and left Noetherian. Noetherian rings play a key role in many major ring-theoretic results, such as Hilbert's basis theorem and Krull's intersection theorem.

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We call a semigroup S weakly right noetherian if every right ideal of S is finitely generated.* Similarly, a semigroup is *weakly left noetherian* if every left ideal is finitely generated. We call a semigroup *weakly noetherian* if it is both weakly right noetherian and weakly left noetherian. It is clear that each of these properties is a finiteness condition. In this paper, we will focus on weakly *right* noetherian semigroups. Such semigroups have received a significant amount of attention; see for instance [1, 6, 14, 26].

Related to the property of being weakly right noetherian is the stronger condition that every right congruence is finitely generated; we call semigroups satisfying this condition *right noetherian*. Such semigroups were studied systematically in [21] and had previously been considered in [12, 17, 18]. Another related notion is that of the universal right congruence being finitely generated, which was first considered in [5]. The stronger condition that every right congruence of finite index is finitely generated (where index means the number of classes) was introduced and studied in [20].

This paper is structured as follows. In §2, we provide the necessary preliminary material. In §3, we present some equivalent formulations of the property of being weakly right noetherian. In §4 and 5, we explore how a semigroup and its substructures and quotients relate to one another with regard to the property of being weakly right noetherian. We then investigate how this property behaves under various semigroup-theoretic constructions in §6. Specifically, we consider direct products, free products, semilattices of semigroups, Rees matrix semigroups, Brandt extensions and Bruck–Reilly extensions. Section 7 is concerned with regular semigroups. We first consider regular semigroups in general, and then focus on the important subclasses of inverse semigroups and completely regular semigroups. Finally, in §8, we consider commutative semigroups. The main result of that section is a necessary and sufficient condition for a commutative semigroup with finitely many archimedean components to be weakly noetherian.

2. Preliminaries

In this section, we establish some basic definitions and notation. We begin by providing some set-theoretic definitions.

A relation \leq on a set P is said to be *preorder* if it is both reflexive and transitive. If a preorder is also symmetric, then it is an *equivalence relation*. On the other hand, if a preorder is antisymmetric then it is a *partial order*.

A *poset* is a set P together with a partial order \leq . Given any set X , a collection \mathcal{P} of subsets of X forms a poset under the partial order of inclusion. In particular, the set of all right ideals of a semigroup is a poset (under \subseteq).

Let (P, \leq_P) and (Q, \leq_Q) be two posets. A map $\theta : P \rightarrow Q$ is said to be *order-preserving* if $x \leq_P y$ implies $x\theta \leq_Q y\theta$ for all $x, y \in P$. A map $\theta : P \rightarrow Q$ is an *isomorphism* if both θ and θ^{-1} are order-preserving (i.e. $x \leq_P y$ if and only if $x\theta \leq_Q y\theta$ for all $x, y \in P$) and θ is a bijection. We say that P and Q are *isomorphic* if there exists an isomorphism between them. Note that to show that a map $\theta : P \rightarrow Q$ is an isomorphism, it suffices to prove that θ is *surjective* and that $x \leq_P y$ if and only if $x\theta \leq_Q y\theta$ for all $x, y \in P$.

* Weakly right noetherian semigroups are also known in the literature as *right noetherian*. However, we use the term ‘right noetherian’ to denote semigroups whose right congruences are all finitely generated.

Two elements a and b of a poset P are said to be comparable if either $a \leq b$ or $b \leq a$; otherwise, a and b are *incomparable*. A subset of P in which any two elements are comparable is called a *chain*. An *antichain* of P is a subset consisting of pairwise incomparable elements.

We now turn our attention to semigroups. We refer the reader to [13] for a more comprehensive introduction to semigroup theory. Throughout the remainder of the section, S will denote a semigroup.

We denote by S^1 the monoid obtained from S by adjoining an identity if necessary (if S is already a monoid, then $S^1 = S$). Similarly, we denote by S^0 the semigroup with zero obtained from S by adjoining a zero if necessary.

Let M be a monoid with identity 1. An element $a \in M$ is said to be right invertible if there exists $b \in M$ such that $ab = 1$. Left invertible elements are defined dually. An element of M is called a *unit* if it is both right invertible and left invertible. The units of M form a group, called the *group of units* of M , which we denote by $U(M)$.

We denote the set of idempotents of S by $E(S)$. If $S = E(S)$, it is called a *band*. A *semilattice* is a commutative band. The multiplication in a semilattice E induces the following partial order:

$$e \geq f \iff ef = f.$$

In this way, we may view E as a meet-semilattice in the order-theoretic sense. Conversely, any order-theoretic meet-semilattice may be viewed as a commutative band with meet taken as the binary operation.

If $a, b \in S$ are such that $a = aba$ and $b = bab$, then b is called an *inverse* of a . The semigroup S is said to be regular if every element of S has an inverse. If, additionally, the inverse of each element of S is unique, then S is an inverse semigroup. It is well known that a semigroup is inverse if and only if it is regular and its idempotents form a semilattice [13, Theorem 5.1.1].

A subset $I \subseteq S$ is said to be a *right ideal* of S if $IS \subseteq I$. Left ideals are defined dually, and an *ideal* of S is a subset that it is both a right ideal and a left ideal.

Given a subset $X \subseteq S$, the *right ideal generated by X* is the set XS^1 . A right ideal I of S is said to be *finitely generated* if it can be generated by a finite set.

Note that a right ideal of S can be generated by a set *as a right ideal* or *as a semigroup*. For proper right ideals, we will always use the term ‘generate’ in the former sense. When we say that S is generated by a set X , we mean ‘generated as a semigroup’, unless stated otherwise, and we write $S = \langle X \rangle$. We note that a right ideal can be finitely generated as a right ideal but not as a semigroup; e.g. any non-finitely group G is certainly finitely generated as a right ideal.

A *right congruence* on S is an equivalence relation ρ on S such that $(a, b) \in \rho$ implies $(ac, bc) \in \rho$ for all $a, b, c \in S$; *left congruences* are defined analogously. A congruence is a relation that is both a right congruence and left congruence. For a congruence ρ on S , we denote the congruence class of an element $a \in S$ by $[a]_\rho$.

Recall that a semigroup is *right noetherian* if every right congruence is finitely generated. (A right congruence ρ on S is finitely generated if there exists a finite set $X \subseteq \rho$ such that ρ is the smallest right congruence on S containing X .) Right noetherian semigroups are weakly right noetherian [21, Lemma 2.7], but the converse certainly does not hold. Indeed, unlike the situation for rings, the lattice of right congruences on a semigroup

is not, in general, isomorphic to the lattice of right ideals. For example, groups have no proper right ideals, but the lattice of right congruences on a group is isomorphic to its lattice of subgroups. Consequently, groups are trivially weakly right noetherian, but a group is right noetherian if and only if all its subgroups are finitely generated [21, Proposition 2.14].

The most essential tools for understanding the structure of a semigroup are its Green's relations \mathcal{L} , \mathcal{R} , \mathcal{H} , \mathcal{D} and \mathcal{J} . They are defined as follows.

Two elements $a, b \in S$ are \mathcal{L} -related if they generate the same principal left ideal, i.e. $S^1a = S^1b$. Similarly, two elements $a, b \in S$ are \mathcal{R} -related if they generate the same principal right ideal. Green's relation \mathcal{H} is defined as $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. Two elements $a, b \in S$ are \mathcal{D} -related if there exists $c \in S$ such that $a \mathcal{L} c$ and $c \mathcal{R} b$. Finally, if two elements $a, b \in S$ generate the same principal ideal (i.e. $S^1aS^1 = S^1bS^1$), then they are said to be \mathcal{J} -related.

It is obvious from the definitions that \mathcal{L} , \mathcal{R} , \mathcal{H} and \mathcal{J} are equivalence relations on S , and it turns out that \mathcal{D} is also an equivalence relation. Moreover, Green's relation \mathcal{L} is a right congruence on S and \mathcal{R} is a left congruence on S .

Green's relation \mathcal{R} defines a preorder $\leq_{\mathcal{R}}$ on S , given by

$$a \leq_{\mathcal{R}} b \iff aS^1 \subseteq bS^1.$$

The preorder \mathcal{R} induces a partial order on the set of \mathcal{R} -classes of S : $R_a \leq R_b$ if and only if $a \leq_{\mathcal{R}} b$. It is easy to see that the poset of \mathcal{R} -classes of S is isomorphic to the poset of principal right ideals of S , via the isomorphism $R_a \mapsto aS^1$. Similarly, one can define preorders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$, leading to partial orders on the sets of \mathcal{L} -classes and \mathcal{J} -classes, respectively.

Note that when we need to distinguish between Green's relations on different semigroups, we will write them with the semigroup as a subscript; i.e. \mathcal{K}_S stands for \mathcal{K} , where \mathcal{K} is any of Green's relations on S .

It is easy to see that the following inclusions between Green's relations hold:

$$\mathcal{H} \subseteq \mathcal{L}, \quad \mathcal{H} \subseteq \mathcal{R}, \quad \mathcal{L} \subseteq \mathcal{D}, \quad \mathcal{R} \subseteq \mathcal{D}, \quad \mathcal{D} \subseteq \mathcal{J}.$$

It can be easily shown that every right (respectively left) ideal is a union of \mathcal{R} -classes (respectively \mathcal{L} -classes), and every ideal is a union of \mathcal{J} -classes. A semigroup with no proper (right) ideals is called (*right*) *simple*. A simple semigroup has a single \mathcal{J} -class; if it is right simple, then it has a single \mathcal{R} -class. If S has a zero 0 , $S^2 \neq \{0\}$, and $\{0\}$ is the only proper ideal of S , then it is called *0-simple*.

Given an ideal I of S , the Rees quotient of S by I , denoted by S/I , is the set $(S \setminus I) \cup \{0\}$ with multiplication given by

$$a \cdot b = \begin{cases} ab & \text{if } a, b, ab \in S \setminus I, \\ 0 & \text{otherwise.} \end{cases}$$

Let J be a \mathcal{J} -class of S . The *principal factor* of J is defined as follows. If J is the unique minimal ideal of S , called the *kernel* of S , its principal factor is itself. Otherwise, the principal factor of J is the Rees quotient of the subsemigroup S^1xS^1 , where x is any element of J , by the ideal $(S^1xS^1) \setminus J$.

The *principal factors* of S are the principal factors of its \mathcal{J} -classes. The kernel of S , if it exists, is simple; all other principal factors are either 0-simple or *null* (every product of two elements equals zero).

A *principal series* of a semigroup S is a finite chain of ideals

$$K(S) = I_1 \subset I_2 \subset \dots \subset I_n = S,$$

where $K(S)$ is the kernel of S , and I_k is maximal in I_{k+1} for each $i \in \{1, \dots, n - 1\}$. The kernel $K(S)$ and the Rees quotients I_{k+1}/I_k are the principal factors of S .

It is folklore that a semigroup has a principal series of length n if and only if it has exactly n \mathcal{J} -classes.

Closely related to the notion of one-sided ideals is that of semigroup acts. We provide some basic definitions about acts; one should consult [16] for more information.

A (*right*) S -act is a non-empty set A together with a map

$$A \times S \rightarrow A, (a, s) \mapsto as$$

such that $a(st) = (as)t$ for all $a \in A$ and $s, t \in S$. (If S is a monoid, we also require that $a1 = a$ for all $a \in A$.) For instance, S itself is an S -act via right multiplication.

A subset B of an S -act A is a *subact* of A if $bs \in B$ for all $b \in B$ and $s \in S$. Note that the right ideals of S are precisely the subacts of the S -act S .

Given an S -act A and a subact B of A , the *Rees quotient* of A by B , denoted by A/B , is the set $(A \setminus B) \cup \{0\}$ with action given by

$$a \cdot s = \begin{cases} as & \text{if } as \in A \setminus B \\ 0 & \text{otherwise,} \end{cases}$$

and $0 \cdot s = 0$, for all $a \in A \setminus B$ and $s \in S$. It can be easily verified that A/B is an S -act via the above action.

A subset U of an S -act A is a *generating set* for A if $A = US^1$, and A is said to be *finitely generated* if it has a finite generating set.

We call an S -act A *noetherian* if every subact of A is finitely generated; equivalently, A satisfies the ascending chain condition on its subacts. In particular, the S -act S being noetherian is equivalent to S being a weakly right noetherian semigroup.

3. Equivalent formulations and elementary facts

We begin this section by presenting equivalent characterizations of weakly right noetherian semigroups in terms of the ascending chain condition and maximal condition on right ideals. The proof of this result is essentially the same as that of the analogue for rings and is omitted.

Proposition 3.1. *The following are equivalent for a semigroup S :*

- (1) S is weakly right noetherian;
- (2) S satisfies the ascending chain condition on right ideals; that is, every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of right ideals of S eventually terminates;
- (3) every non-empty set of right ideals of S has a maximal element.

We now provide characterizations of weakly right noetherian semigroups in terms of their principal right ideals and also in terms of their \mathcal{R} -class structure.

Theorem 3.2. *The following are equivalent for a semigroup S :*

- (1) S is weakly right noetherian;
- (2) S satisfies the ascending chain condition on principal right ideals and contains no infinite antichain of principal right ideals (under \subseteq).
- (3) the poset of \mathcal{R} -classes of S contains no infinite strictly ascending chain or infinite antichain.

Proof. (1) \Rightarrow (2). By Proposition 3.1, S certainly satisfies the ascending chain condition on principal right ideals. The fact that S contains no infinite antichain of principal right ideal was proven in [12, Lemma 1.6], but we provide a proof for completeness.

Suppose for a contradiction that there exists an infinite antichain $\{a_i S^1 : i \in \mathbb{N}\}$ of principal right ideals of S . For each $n \in \mathbb{N}$, let $I_n = \{a_1, \dots, a_n\} S^1$. Suppose that $I_m = I_n$ for some $m \leq n$. Then $a_n \in a_i S^1$ for some $i \leq m$. It must be the case that $i = m = n$, for otherwise the incomparability of $a_i S^1$ and $a_n S^1$ would be contradicted. But then we have an infinite strictly ascending chain

$$I_1 \subsetneq I_2 \subsetneq \dots$$

of right ideals of S , contradicting Proposition 3.1.

(2) \Rightarrow (1). Suppose that S is not weakly right noetherian yet the poset of principal right ideals of S does satisfy the ascending chain condition. We need to construct an infinite antichain of principal right ideals of S .

By Proposition 3.1, there exists an infinite strictly ascending chain

$$I_1 \subsetneq I_2 \subsetneq \dots$$

of right ideals of S . Choose elements $a_1 \in I_1$ and $a_k \in I_k \setminus I_{k-1}$ for $k \geq 2$. Then certainly $a_k S^1$ is not contained in any $a_j S^1$, $j < k$, since $a_j S^1 \subseteq I_j$ and $a_k \in I_k \setminus I_j$.

Consider the infinite set $\{a_i S^1 : i \in \mathbb{N}\}$ of principal right ideals of S . This set contains a maximal element, say $a_{k_1} S^1$; that is, $a_{k_1} S^1$ is not contained in any $a_j S^1$, $j \neq k_1$. Indeed, if this were not the case, then there would exist an infinite strictly ascending chain of principal right ideals of S , contradicting the assumption.

Now consider the infinite set $\{a_i S^1 : i \geq k_1 + 1\}$. Again, this set contains a maximal element, say $a_{k_2} S^1$. Thus $a_{k_2} S^1$ is not contained in $a_j S^1$ for any $j > k_1$, $j \neq k_2$. In fact, $a_{k_2} S^1$ is not contained in $a_j S^1$ for any $j \in \mathbb{N} \setminus \{k_2\}$, since, as observed above, $a_{k_2} S^1$ is not contained in any $a_j S^1$, $j < k$.

Continuing this process ad infinitum, we obtain an infinite antichain $\{a_{k_i} S^1 : i \in \mathbb{N}\}$ of principal right ideals of S , as required.

(2) \Leftrightarrow (3). This follows from the fact, established in §2, that the poset of \mathcal{R} -classes of S is isomorphic to the poset of principal right ideals of S . \square

Corollary 3.3. *Any semigroup with finitely many \mathcal{R} -classes is weakly right noetherian. In particular, all finite semigroups and all right simple semigroups (which include groups) are weakly right noetherian.*

Remark 3.4. The condition that every (two-sided) ideal of a semigroup S is finitely generated has been considered in [1, 15]. By an argument essentially the same as the proof of Theorem 3.2, this condition is equivalent to S satisfying the ascending chain condition on principal ideals and containing no infinite antichain of principal ideals, and also to S containing no infinite strictly ascending chain or infinite antichain of \mathcal{J} -classes. Any weakly right noetherian semigroup satisfies this condition (since every ideal is a one-sided ideal), but the converse does not hold. Indeed, any simple semigroup trivially satisfies the condition that every ideal is finitely generated, but there exist simple semigroups that are not right noetherian; e.g. any completely simple semigroup with infinitely many \mathcal{R} -classes (see Corollary 6.21).

The following result shows that there exist semigroups that satisfy the ascending chain condition on principal right ideals but are not weakly right noetherian.

Proposition 3.5. *Let X be a non-empty set. Then the free semigroup F_X on X satisfies the ascending chain condition on principal right ideals. However, F_X is weakly right noetherian if and only if $|X| = 1$.*

Proof. Consider two elements $u, v \in F_X$. Clearly, $uF_X^1 \subsetneq vF_X^1$ if and only if v is a proper prefix of u , in which case $|u| > |v|$. It follows that there cannot exist an infinite strictly ascending chain of principal right ideals of F_X .

If $|X| = 1$, then clearly $F_X \cong \mathbb{N}$ contains no incomparable elements, so it is weakly right noetherian by Theorem 3.2.

Suppose $|X| \geq 2$, and choose distinct elements $x, y \in X$. For $i \neq j$, the element $x^i y$ is not a prefix of $x^j y$, so F_X contains an infinite antichain $\{(x^i y)F_X^1 : i \in \mathbb{N}\}$ of principal right ideals. Hence, F_X is not weakly right noetherian by Theorem 3.2. \square

Remark 3.6. We can readily deduce from Proposition 3.5 that the property of being weakly right noetherian is not closed under subsemigroups. Indeed, the free semigroup F_X is a subsemigroup of the free group on X , which is certainly weakly right noetherian.

Monoid acts play the analogous role in the theory of monoids to that of modules in the theory of rings. It is well known that a ring R is right Noetherian if and only if every finitely generated right R -module is *Noetherian* (i.e. it satisfies the ascending chain condition on its submodules) [8, Corollary 1.4]. We now present the analogue of this result for monoid acts.

Proposition 3.7. *The following are equivalent for a monoid M :*

- (1) M is weakly right noetherian;
- (2) every finitely generated right M -act is noetherian.

Proof. (1) \Rightarrow (2). Let A be a right M -act with a finite generating set X , and let B be a subact of A . For each $x \in X$, we define a set

$$I_x = \{m \in M : xm \in B\}.$$

Let X' be the set of elements in X such that $I_x \neq \emptyset$. Since B is a subact of A , we have that I_x is a right ideal of S for each $x \in X'$. Since M is weakly right noetherian, for each $x \in X'$ there exists a finite set $U_x \subseteq I_x$ such that $I_x = U_x M$. We claim that B is generated by the set

$$U = \bigcup_{x \in X'} xU_x.$$

Indeed, let $b \in B$. Since $b \in A$, we have that $b = xm$ for some $x \in X$ and $m \in M$. Now $m \in I_x$, so $m = un$ for some $u \in U_x$ and $n \in M$, and hence $b = (xu)n \in UM$.

(2) \Rightarrow (1). This follows from the fact that the right ideals of M are subacts of the cyclic right M -act M . \square

For commutative semigroups, clearly the properties of being weakly right noetherian and being weakly left noetherian coincide. It is a well-known result, due to Rédei [24], that every congruence on a finitely generated commutative semigroup is finitely generated; that is:

Theorem 3.8 (Rédei [24]). *Every finitely generated commutative semigroup is noetherian (and hence weakly noetherian).*

In the remainder of this section, we state some useful facts about weakly right noetherian semigroups. The following lemma is well known and will be used repeatedly throughout the remainder of the paper, usually without explicit mention.

Lemma 3.9. *Let S be a semigroup and let X be a subset of S . If the right ideal XS^1 of S is finitely generated, then there exists a finite subset $Y \subseteq X$ such that $XS^1 = YS^1$.*

Let S be a semigroup. An element $s \in S$ is said to be decomposable if $s \in S^2$. An element is *indecomposable* if it is not decomposable.

Lemma 3.10. *Let S be a weakly right noetherian semigroup. Then S has only finitely many indecomposable elements.*

Proof. Since S is weakly right noetherian, S is finitely generated as a right ideal; that is, there exists a finite set $X \subseteq S$ such that $S = XS^1$. Therefore, we have that $S \setminus X \subseteq S^2$, so S has at most $|X|$ indecomposable elements. \square

We now show that a semigroup composed of a finite union of weakly right noetherian subsemigroups is also weakly right noetherian.

Lemma 3.11. *Let S be a semigroup, and suppose that S is a union of subsemigroups S_1, \dots, S_n . If each S_i is weakly right noetherian, then S is also weakly right noetherian.*

Proof. Let I be a right ideal of S . For $j \in \{1, \dots, n\}$, let I_j be the restriction of I to S_j . Then I_j is a right ideal of S_j . Since S_j is weakly right noetherian, I_j is generated by

some finite set X_j . We claim that I is generated by the finite set $X = \bigcup_{i=1}^n X_i$. Indeed, if $a \in I$, then $a \in I_j = X_j S_j^1$ for some $j \in \{1, \dots, n\}$. □

4. Quotients and ideals

In this section, we consider the relationship between a semigroup and its quotients and ideals with regard to the property of being weakly right noetherian. We first show that this property is closed under quotients (or, equivalently, homomorphic images).

Lemma 4.1. *Let S be a semigroup and let ρ be a congruence on S . If S is weakly right noetherian, then so is S/ρ .*

Proof. Let I be a right ideal of S/ρ , and define a set

$$J = \{a \in S : [a]_\rho \in I\}.$$

It is clear that J is a right ideal of S . Since S is weakly right noetherian, J is generated by a finite set X . We claim that I is generated by the finite set $Y = \{[x]_\rho : x \in X\}$. Indeed, let $u \in I$. Select $a \in S$ such that $[a]_\rho = u$. Then $a = xs$ for some $x \in X$ and $s \in S^1$. If $s = 1$, then $u = [x]_\rho \in Y$. Otherwise, we have $u = [x]_\rho [s]_\rho \in Y(S/\rho)$, as required. □

Remark 4.2. The converse of Lemma 4.1 does not hold. Indeed, the free semigroup F_X with $|X| \geq 2$ is not weakly right noetherian by Proposition 3.5, but there certainly exist quotients of F_X that are weakly right noetherian.

If ρ is a congruence contained in \mathcal{R} , then the converse of Lemma 4.1 holds. In fact, we have:

Lemma 4.3. *Let S be a semigroup and let $\rho \subseteq \mathcal{R}$ be a congruence on S . Then the poset of \mathcal{R} -classes of S is isomorphic to the poset of \mathcal{R} -classes of S/ρ . In particular, S is weakly right noetherian if and only if S/ρ is weakly right noetherian.*

Proof. Let P be the poset of principal right ideals of S , and let Q be the poset of principal right ideals of S/ρ . Since the poset of principal right ideals of a semigroup is isomorphic to the poset of \mathcal{R} -classes, it suffices to prove that P and Q are isomorphic. It then follows from Theorem 3.2 that S is weakly right noetherian if and only if S/ρ is weakly right noetherian.

Define a map $\theta : P \rightarrow Q$, $aS^1 \mapsto [a]_\rho(S/\rho)^1$. Clearly, θ is surjective, so we just need to show that $aS^1 \subseteq bS^1$ if and only if $[a]_\rho(S/\rho)^1 \subseteq [b]_\rho(S/\rho)^1$. It is clear that the forward direction holds. Conversely, if $[a]_\rho(S/\rho)^1 \subseteq [b]_\rho(S/\rho)^1$ then $[a]_\rho = [b]_\rho u$ for some $u \in (S/\rho)^1$. If $u = 1$, then $(a, b) \in \rho \subseteq \mathcal{R}$, so $aS^1 = bS^1$. If $u \in S/\rho$, then $u = [s]_\rho$ for some $s \in S$. Then $(a, bs) \in \rho \subseteq \mathcal{R}$, so $aS^1 = (bs)S^1 \subseteq bS^1$, as required. □

The following question naturally arises from Lemma 4.3.

Open Problem 4.4. Given a semigroup S and congruence ρ on S , what is the relationship between the poset of \mathcal{R} -classes of S and the poset of \mathcal{R} -classes of S/ρ ?

Remark 4.5. It is natural to wonder whether certain results about weakly right noetherian semigroups, such as Theorem 3.2 and Lemma 4.1, can be derived from more general results about posets. In particular, inspired by Theorem 3.2, one could investigate the finiteness condition that a poset contains no infinite ascending chain or infinite antichain. For instance, is this condition closed under homomorphic images?

The next result states that if a right ideal I of a semigroup S is weakly right noetherian and the Rees quotient of the S -act S by I (where I is regarded as a subact of S) is noetherian, then S is weakly right noetherian.

Proposition 4.6. *Let S be a semigroup and let I be a right ideal of S . If I is weakly right noetherian and S/I is noetherian (as an S -act), then S is weakly right noetherian.*

Proof. Let J be a right ideal of S . Suppose first that $I \cap J = \emptyset$. Then $J \subseteq S \setminus I$ and J may be viewed as a subact of S/I . Since S/I is noetherian, there exists a finite set $U \subseteq I$ such that $J = US^1$. Thus J is finitely generated as a right ideal of S .

Now suppose that $I \cap J \neq \emptyset$. Then $I \cap J$ is a right ideal of I . Since I is weakly right noetherian, $I \cap J$ is generated by some finite set X . Considering J as an S -act, the Rees quotient $A = J/(I \cap J)$ is a subact of S/I . Since S/I is noetherian, there exists a finite set $Y \subseteq A$ such that $A = YS^1$. Let $Y' = Y \setminus \{0\}$. We claim that $J = (X \cup Y')S^1$.

Indeed, let $a \in J$. If $a \in I$, then $a \in X(I \cap J)^1$. If $a \in J \setminus I$, then $a \in A$, so $a = ys$ for some $y \in Y$ and $s \in S^1$. Since $a \in S \setminus I$, we must have $y \in Y'$. In either case, we have $a \in (X \cup Y')S^1$, as required. \square

Corollary 4.7. *Let S be a semigroup and let I be an ideal of S . If both I and the Rees quotient S/I are weakly right noetherian, then S is weakly right noetherian.*

Proof. We shall prove that S/I , viewed as the Rees quotient of the S -act S by the subact I , is a noetherian S -act. It then follows from Proposition 4.6 that S is weakly right noetherian.

So, let A be a subact of S/I . We need to prove that A is finitely generated. Fix an element $z \in I$. We claim that A is a right ideal of S/I (considered as a semigroup). Indeed, let $a \in A$ and $u \in S/I$. If $a, u \in S \setminus I$, then $au \in A$ since A is a subact of S/I . Otherwise, we have $a = 0$ or $u = 0$, in which case $au = 0 \in A$. (Since A is a subact of S/I , we have $0 = a \cdot z \in A$.) Since S is weakly right noetherian, there exists a finite set $X \subseteq A$ such that $A = X(S/I)^1$. We claim that $A = XS^1$. Indeed, if $a \in A$, then $a = xu$ for some $x \in X$ and $u \in (S/I)^1$. If $a \in S^1 \setminus I$, then $u \in S^1 \setminus I$. Otherwise, $a = 0 = xz$. This completes the proof. \square

Recall that in a semigroup S with a principal series $K(S) = I_1 \subseteq \dots \subseteq I_n = S$, the kernel $K(S)$ and the Rees quotients I_{k+1}/I_k are the principal factors of S . Therefore, if all the principal factors are weakly right noetherian, then by successively applying Corollary 4.7, we deduce that S is weakly right noetherian.

Corollary 4.8. *Let S be a semigroup with a principal series. If all the principal factors of S are weakly right noetherian, then S is weakly right noetherian.*

Although ideals of weakly right noetherian semigroup are not in general weakly right noetherian (see, for instance, [Example 8.3](#)), they do satisfy the ascending chain condition on principal right ideals. In fact, we prove a stronger statement:

Proposition 4.9. *Let S be a semigroup and let I be an ideal of S . If S satisfies the ascending chain condition on principal right ideals, then so does I .*

Proof. Consider an infinite ascending chain

$$a_1I^1 \subseteq a_2I^1 \subseteq \dots$$

of principal right ideals of I . Then for each $i, j \in \mathbb{N}$ with $i < j$, there exists $u_{i,j} \in I^1$ such that $a_i = a_j u_{i,j}$. Clearly, we have an infinite ascending chain

$$a_1S^1 \subseteq a_2S^1 \subseteq \dots$$

of principal right ideals of S . By assumption, there exists $n \in \mathbb{N}$ such that $a_mS^1 = a_nS^1$ for all $m \geq n$. Let $m > n$. If $u_{n,m} = 1$, then $a_n = a_m$. Now suppose that $u_{n,m} \in I$. There exists $s_m \in S^1$ such that $a_m = a_n s_m$, and hence

$$a_m = a_n s_m = a_m u_{n,m} s_m = a_n (s_m u_{n,m} s_m).$$

We have that $s_m u_{n,m} s_m \in I$ since I is an ideal, so $a_m \in a_n I$. Since $a_n I^1 \subseteq a_m I^1$, we conclude that $a_n I^1 = a_m I^1$. Since m was chosen arbitrarily, we have shown that the above ascending chain of principal right ideals of I terminates at $a_n I^1$. □

5. Subsemigroups

As mentioned previously, subsemigroups of weakly right noetherian semigroups need not be weakly right noetherian themselves. In this section, we explore various situations in which the property of being weakly right noetherian passes from a semigroup S to a subsemigroup T , and vice versa.

Lemma 5.1. *Let S be a semigroup, and let T be a subsemigroup of S such that $S \setminus T$ is contained in a finite union of \mathcal{R} -classes. If T is weakly right noetherian, then S is weakly right noetherian.*

Proof. Let I be a right ideal of S . Then $I \cap T$ is a right ideal of T . Since T is weakly right noetherian, there exists a finite set $X \subseteq I \cap T$ such that $I \cap T = XT^1$.

Let R_1, \dots, R_n be the \mathcal{R} -classes that intersect with $S \setminus T$. For each $i \in \{1, \dots, n\}$, fix $r_i \in R_i$. It is easy to see that if I intersects an \mathcal{R} -class R , then $R \subseteq I$. We claim that I is generated by the finite set

$$Y = X \cup \{r_i : R_i \subseteq I, 1 \leq i \leq n\}.$$

Indeed, let $a \in I$. If $a \in T$, then $a \in I \cap T = XT^1$. If $a \in S \setminus T$, then $a = r_i s$ for some $i \in \{1, \dots, n\}$ and $s \in S^1$. Hence, in either case, we have that $a \in YS^1$. □

Remark 5.2. The converse of Lemma 5.1 does not hold. Indeed, in Remark 6.10, an example is provided of a weakly right noetherian semigroup S with a subsemigroup T such that $S \setminus T$ is a group and T is not weakly right noetherian.

In [27], Wallace introduced the idea of Greens' relations taken relative to a subsemigroup. Let S be a semigroup and let T be a subsemigroup of S . The T -relative Green's relation \mathcal{R}^T on S is given by

$$a\mathcal{R}^T b \iff aT^1 = bT^1.$$

The relation \mathcal{L}^T is defined dually, and $\mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T$. The relations \mathcal{D}^T and \mathcal{J}^T are defined in a similar way. All of these relations are equivalence relations on S , and they respect T in the sense that each class lies entirely in T or entirely in $S \setminus T$. The subsemigroup T is said to have *finite Green index* in S if there are only finitely many \mathcal{H}^T -classes in $S \setminus T$. The notion of Green index for subsemigroups was introduced in [10].

Proposition 5.3. *Let S be a semigroup, and let T be a subsemigroup of S such that $S \setminus T$ is a finite union of \mathcal{R}^T -classes. Then S is weakly right noetherian if and only if T is weakly right noetherian.*

Proof. (\Rightarrow) Let I be a right ideal of T . Let $I' = IS^1$; that is, the right ideal of S generated by I . Since S is weakly right noetherian, we have that I' is generated by some finite subset $X \subseteq I$ by Lemma 3.9. Let R_1, \dots, R_n be the \mathcal{R}^T -classes in $S \setminus T$, and for each $i \in \{1, \dots, n\}$ fix $r_i \in R_i$. We claim that I is generated by the finite set

$$Y = X \cup \{xr_i \in I : x \in X, 1 \leq i \leq n\}.$$

Indeed, let $a \in I$. Then $a \in I'$, so $a = xs$ for some $x \in X$ and $s \in S^1$. If $s \in T^1$, then $a \in YT^1$. If $s \in S \setminus T$, then $s \in R_i$ for some $i \in \{1, \dots, n\}$, so there exist $t, u \in T^1$ such that $s = r_i t$ and $r_i = su$. Then $xr_i = (xs)u = au \in I$, and hence $a = xs = (xr_i)t \in YT^1$, as required.

(\Leftarrow) Each \mathcal{R}^S -class is a union of \mathcal{R}^T -classes by [10, Proposition 9], so $S \setminus T$ is contained in a finite union of \mathcal{R}^S -classes, and hence S is weakly right noetherian by Lemma 5.1. \square

Corollary 5.4. *Let S be a semigroup and let T be subsemigroup of S with finite Green index. Then S is weakly right noetherian if and only if T is weakly right noetherian.*

Corollary 5.5. *Let S be a semigroup, and let T be a subsemigroup of S such that $S \setminus T$ is finite. Then S is weakly right noetherian if and only if T is weakly right noetherian.*

In particular, S is weakly right noetherian if and only if S^1 is weakly right noetherian if and only if S^0 is weakly right noetherian.

Let S be a semigroup and T a subsemigroup of S . It is easy to see that Green's \mathcal{R} -preorder on T is contained in the restriction of Green's \mathcal{R} -preorder on S to T ; that is,

$$\leq_{\mathcal{R}_T} \subseteq \leq_{\mathcal{R}_S} \cap (T \times T).$$

We say that T *preserves* \mathcal{R} (in S), or is \mathcal{R} -preserving, if

$$\leq_{\mathcal{R}_T} = \leq_{\mathcal{R}_S} \cap (T \times T).$$

It can be easily shown that if T preserves \mathcal{R} , then \mathcal{R}_T is the restriction of \mathcal{R}_S to T . The next result states that the property of being weakly right noetherian is inherited by \mathcal{R} -preserving subsemigroups.

Proposition 5.6. *Let S be a semigroup and let T be an \mathcal{R} -preserving subsemigroup of S . If S is weakly right noetherian, then so is T .*

Proof. Let I be a right ideal of T , and let $I' = IS^1$. Since S is weakly right noetherian, $I' = XS^1$ for some finite subset $X \subseteq I$. If $a \in I$, then $a \in xS^1$ for some $x \in X$, so $a \leq_{\mathcal{R}_S} x$. By assumption, we have $a \leq_{\mathcal{R}_T} x$, so $a \in xT^1$. Thus $I = XT^1$ is finitely generated. \square

For a regular subsemigroup T of a semigroup S , Green’s relation \mathcal{R}_T is the restriction of \mathcal{R}_S to T (likewise, \mathcal{L}_T and \mathcal{H}_T are the restrictions to T of \mathcal{L}_S and \mathcal{H}_S , respectively) [13, Proposition 2.4.2]. In fact, T preserves \mathcal{R} . Indeed, if $a, b \in T$ and $a \leq_{\mathcal{R}_S} b$, then there exists $s \in S^1$ such that $a = bs$. Letting b' be any inverse of b , we have that $a = bb'bs = b(b'a) \in bT$, so $a \leq_{\mathcal{R}_T} b$, as required. Thus, by Proposition 5.6, we have:

Corollary 5.7. *Let S be a semigroup with a regular subsemigroup T . If S is weakly right noetherian, then so is T .*

We say that a semigroup S has *local right identities* if $a \in aS$ for every $a \in S$. It is easy to show that the class of semigroups with local right identities includes all regular semigroups, right simple semigroups and monoids. The notion of having local right identities will be crucial in § 6.1.

Corollary 5.8. *Let S be a semigroup, and let I be a right ideal of S with local right identities. If S is weakly right noetherian, then so is I .*

Proof. We prove that I preserves \mathcal{R} in S . We just need to show that $\leq_{\mathcal{R}_S} \cap (I \times I) \subseteq \leq_{\mathcal{R}_I}$. So, let $(a, b) \in I \times I$ and $a \leq_{\mathcal{R}_S} b$. Then $a \in bS^1$. Since I is a right ideal with local right identities, we have

$$a \in bS^1 \subseteq (bI)S^1 = b(IS^1) \subseteq bI,$$

so $a \leq_{\mathcal{R}_I} b$, as required. \square

A subsemigroup T of a semigroup S is called *right unitary* (in S) if it satisfies the following condition: for all $a \in T$ and $b \in S$, if $ab \in T$ then $b \in T$.

Clearly, a right unitary subsemigroup is \mathcal{R} -preserving, so we deduce the following corollary, first proven in [14], from Proposition 5.6.

Corollary 5.9 (Jespers and Okniński [14, Lemma 1.1(1)]). *Let S be a semigroup and let T be a right unitary subsemigroup of S . If S is weakly right noetherian, then so is T .*

If the complement of a subsemigroup is a left ideal, then the subsemigroup is right unitary, so we have:

Corollary 5.10. *Let S be a semigroup with a subsemigroup T such that $S \setminus T$ is a left ideal of S . If S is weakly right noetherian, then so is T .*

Corollary 5.11. *Let S be a semigroup with a subsemigroup T such that $S \setminus T$ is a weakly right noetherian ideal of S . Then S is weakly right noetherian if and only if T is weakly right noetherian.*

Proof. The direct implication follows from Corollary 5.10. For the converse, let $I = S \setminus T$. Since T is weakly right noetherian, so is $S/I \cong T \cup \{0\}$ by Corollary 5.5. It now follows from Corollary 4.7 that S is weakly right noetherian. \square

The final part of this section concerns semigroups with a kernel.

Proposition 5.12. *Let S be a semigroup with a minimal right ideal. If S is weakly right noetherian, then the kernel $K = K(S)$ is a finite union of pairwise incomparable \mathcal{R}_K -classes and is hence weakly right noetherian.*

Proof. The kernel K is the union of all the minimal right ideals of S [3, Theorem 2.1]. Due to their minimality, these minimal right ideals are single \mathcal{R}_S -classes and are pairwise incomparable. By Theorem 3.2, there are only finitely many of them. By [3, Theorem 2.4], each of these minimal right ideals is a single \mathcal{R}_K -class (so K is \mathcal{R} -preserving). It now follows from Corollary 3.3 that K is weakly right noetherian. \square

Let S be a semigroup with a zero 0 . The *right socle* of S , denoted by $\Sigma_r(S)$, is the union of $\{0\}$ and all the 0 -minimal right ideals of S . It turns out that $\Sigma_r(S)$ is an ideal of S , as noted in [4, Section 6.3]. A similar argument to the one in the proof of Proposition 5.12 yields:

Proposition 5.13. *Let S be a semigroup with zero. If S is weakly right noetherian, then the right socle $\Sigma_r(S)$ consists of $\{0\}$ and a finite union of incomparable \mathcal{R} -classes, and hence $\Sigma_r(S)$ is weakly right noetherian.*

Open Problem 5.14. Let S be a weakly right noetherian semigroup with a kernel K . Is K also weakly right noetherian? If $K = \{0\}$ and S has a 0 -minimal ideal M , is M weakly right noetherian?

6. Constructions

In this section, we investigate the behaviour of the property of being weakly right noetherian under the following semigroup-theoretic constructions: direct products, free products, semilattices of semigroups, Rees matrix semigroups, Brandt extensions and Bruck–Reilly extensions.

6.1. Direct products

The problem of whether the property of being weakly right noetherian is preserved under direct products was previously considered in [6], where it was shown that the direct product of two weakly right noetherian commutative monoids is weakly right noetherian [6, Theorem 3.8].

The purpose of this subsection is to provide necessary and sufficient conditions for the direct product of two semigroups to be weakly right noetherian.

Recall that a semigroup S has local right identities if $a \in aS$ for every $a \in S$.

Theorem 6.1. *Let S and T be two semigroups with S infinite.*

- (1) *Suppose T is infinite. Then $S \times T$ is weakly right noetherian if and only if both S and T are weakly right noetherian and have local right identities.*
- (2) *Suppose T is finite. Then $S \times T$ is weakly right noetherian if and only if S is weakly right noetherian and T has local right identities.*

In order to prove Theorem 6.1, we first present a couple of preliminary results.

Lemma 6.2. *Let S and T be two semigroups with S infinite. If $S \times T$ is weakly right noetherian, then T has local right identities.*

Proof. Let $t \in T$, and let I be the right ideal of $S \times T$ generated by the set $\{(s, t) : s \in S\}$. Since $S \times T$ is weakly right noetherian, there exists a finite set $X \subseteq S$ such that I is generated by the set $\{(x, t) : x \in X\}$.

Choose $s \in S \setminus X$. Then $(s, t) = (x, t)w$ for some $x \in X$ and $w \in (S \times T)^1$. Since $s \neq x$, we conclude that $w \in S \times T$, so $w = (u, v)$ for some $u \in S$ and $v \in T$. It follows that $t = tv \in tT$. Since t was chosen arbitrarily, T has local right identities. □

Proposition 6.3. *Let S and T be two semigroups with local right identities. If both S and T are weakly right noetherian, then $S \times T$ is weakly right noetherian.*

Proof. Let I be a right ideal of $S \times T$. For each $a \in S$, define a set

$$I_a^T = \{t \in T : (a, t) \in I\}.$$

We claim that I_a^T is a right ideal of T . Indeed, let $t \in I_a^T$ and $u \in T$. Since S has local right identities, there exists $s \in S$ such that $as = a$. Since I is a right ideal of $S \times T$, we have that $(a, tu) = (a, t)(s, u) \in I$, so $tu \in I_a^T$.

Similarly, for each $b \in T$ we define a right ideal

$$I_b^S = \{s \in S : (s, b) \in I\}$$

of S . We now make the following claim.

- (1) There exists a finite set $X \subseteq S$ with the following property: for each $a \in S$, there exists $x \in X$ such that $a \in xS$ and $I_a^T = I_x^T$.
- (2) There exists a finite set $Y \subseteq T$ with the following property: for each $b \in T$, there exists $y \in Y$ such that $b \in yT$ and $I_b^S = I_y^S$. □

Proof of claim. Clearly, it is enough to prove (1). We shall just write I_a for I_a^T . Suppose there are infinitely many right ideals of the form I_a . Note that for any $a, s \in S$, we have $I_a \subseteq I_{as}$. Write $S = J_1$. Since S is weakly right noetherian, there exists a finite set $X_1 \subseteq J_1$ such that $J_1 = X_1S^1$. In fact, we have $J_1 = X_1S$, since S having local right

identities implies that $X_1 \subseteq X_1S$. By our assumption, there exists $x_1 \in X_1$ such that there are infinitely many $a \in x_1S$ with $I_a \neq I_{x_1}$. Consider the set

$$J_2 = \{a \in x_1S : I_a \neq I_{x_1}\}.$$

If $a \in J_2$ and $s \in S$, then $I_{x_1} \subsetneq I_a \subseteq I_{as}$, so $as \in J_2$, and hence J_2 is a right ideal of S . Since S is weakly right noetherian, there exists a finite set $X_2 \subseteq J_2$ such that $J_2 = X_2S$, and there exists $x_2 \in X_2$ such that there are infinitely many $a \in x_2S$ with $I_a \neq I_{x_2}$. Continuing in this way, we obtain an infinite ascending chain

$$I_{x_1} \subset I_{x_2} \subset \dots$$

of right ideals of T , but this contradicts the fact that T is weakly right noetherian. Hence, there exists a finite set $U \subseteq S$ such that $I_a \in \{I_u : u \in U\}$ for every $a \in S$. For each $u \in U$, let K_u be the right ideal of S generated by the set

$$H_u = \{a \in S : I_a = I_u\}.$$

Since S is weakly right noetherian, there exists finite set $X_u \subseteq H_u$ such that $H_u = X_uS$. Now set $X = \bigcup_{u \in U} X_u$. It is clear that X satisfies the condition in the statement of the claim.

Returning to the proof of Proposition 6.3, we claim that I is generated by the finite set $Z = I \cap (X \times Y)$. Indeed, let $(a, b) \in I$. Then $a \in I_b^S$ and $b \in I_a^T$. By the above claim, there exist $x \in X$ and $s \in S$ such that $a = xs$ and $I_a^T = I_x^T$, and there exist $y \in Y$ and $t \in T$ such that $b = yt$ and $I_b^S = I_y^S$. We have that

$$a \in I_b^S = I_y^S \implies (a, y) \in I \implies y \in I_a^T = I_x^T \implies (x, y) \in I.$$

We conclude that

$$(a, b) = (x, y)(s, t) \in Z(S \times T),$$

as required. □

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Is $S \times T$ is weakly right noetherian, then both S and T , being homomorphic images of $S \times T$, are weakly right noetherian by Lemma 4.1, and T has local right identities by Lemma 6.2. If T is infinite, then S also has local right identities by Lemma 6.2.

For the case that T is finite, the converse follows immediately from Proposition 6.3. Now assume that T is finite, and suppose that S is weakly right noetherian and T has local right identities. We have that S^1 is weakly right noetherian by Corollary 5.5, and clearly S^1 has local right identities. Therefore, by Proposition 6.3, we have that $S^1 \times T$ is weakly right noetherian. Since $(S^1 \times T) \setminus (S \times T)$ is finite, it follows from Corollary 5.5 that $S \times T$ is weakly right noetherian. □

6.2. Free products

We shall define the free product of two semigroups (respectively monoids) in terms of semigroup (respectively monoid) presentations. For more information about semigroup and monoid presentations, we refer the reader to [3, Section 9.1].

Given two semigroups S and T defined by presentations $\langle X \mid Q \rangle$ and $\langle Y \mid R \rangle$, respectively, the semigroup free product of S and T , denoted by $S * T$, is the semigroup defined by the presentation $\langle X, Y \mid Q, R \rangle$. If S and T are monoids, then the *monoid free product* of S and T , denoted by $S *_1 T$, is the monoid defined by the presentation $\langle X, Y \mid Q, R, 1_S = 1_T \rangle$, where 1_S is a fixed word over X representing the identity of S and 1_T is a fixed word over Y representing the identity of T . In the case that S and T are groups, the monoid free product $S *_1 T$ coincides with the group free product of S and T ; this fact is noted in [13, Section 8.2, p. 266].

In the following, we provide necessary and sufficient conditions for the semigroup (respectively monoid) free product of two semigroups (respectively monoids) to be weakly right noetherian.

Theorem 6.4. *Let S and T be two semigroups. Then $S * T$ is weakly right noetherian if and only if both S and T are trivial.*

Proof. We denote $S * T$ by U .

(\Rightarrow). Suppose that T is non-trivial, and choose $a \in S$ and distinct elements $b, c \in T$. For $i \in \mathbb{N}$, let $u_i = (ab)^i aca$. Let I be the right ideal of U generated by the set $X = \{u_i : i \in \mathbb{N}\}$. For any $i \in \mathbb{N}$, the element u_i cannot be written as $u_j v$ for any $j \neq i$ and $v \in U$, so X is a minimal generating set for I and hence I is not finitely generated. Therefore, U is not weakly right noetherian.

(\Leftarrow). The semigroup U is defined by the presentation

$$\langle e, f \mid e^2 = e, f^2 = f \rangle.$$

Then U is the disjoint union of the following subsemigroups:

$$\langle ef \rangle \cong \mathbb{N}, \langle fe \rangle \cong \mathbb{N}, \{(ef)^i e : i \geq 0\} \cong \mathbb{N}_0, \{(fe)^i f : i \geq 0\} \cong \mathbb{N}_0.$$

Since \mathbb{N} and \mathbb{N}_0 are weakly right noetherian, it follows from Lemma 3.11 that U is weakly right noetherian. □

Before stating our next result, we first make some definitions.

Let M and N be two disjoint monoids. A *reduced sequence* over M and N is a sequence (u_1, \dots, u_n) such that: $u_i \in (M \cup N) \setminus \{1_M, 1_N\}$ for each $i \in \{1, \dots, n\}$; $(u_i, u_{i+1}) \in M \times N$ or $(u_i, u_{i+1}) \in N \times M$ for each $i \in \{1, \dots, n - 1\}$.

Every non-identity element of $M *_1 N$ can be uniquely written as $u_1 \cdots u_n$ for some reduced sequence (u_1, \dots, u_n) over M and N [4, Section 9.4]; the elements $u_i, 1 \leq i \leq n$, are called the *free factors* of $u_1 \cdots u_n$.

Theorem 6.5. *Let M and N be two monoids. Then $M *_1 N$ is weakly right noetherian if and only if one of the following holds:*

- (1) M is weakly right noetherian and N is trivial, or vice versa;

- (2) both M and N contain precisely two elements;
 (3) both M and N are groups.

Proof. We denote $M *_1 N$ by U . If N is trivial, then M is isomorphic to U , so we may assume that both M and N are non-trivial.

(\Rightarrow). Suppose for a contradiction that $|N| \geq 3$ and at least one of M and N is not a group. A monoid in which every element is right invertible is a group. Therefore, we can choose $a \in M \setminus \{1\}$ and distinct elements $b, c \in N \setminus \{1\}$ such that at least one of a, b, c is not right invertible. Let $u_i = (ab)^i acab$ for $i \in \mathbb{N}$, and let I be the right ideal of U generated by $\{u_i : i \in \mathbb{N}\}$. Suppose that I is finitely generated. Then it can be generated by a finite set

$$X = \{u_i : 1 \leq i \leq k\}.$$

Then $u_{k+1} = u_i v$ for some $i \in \{1, \dots, k\}$ and $v \in U$. Since at least one of a, b, c is not right invertible, the first $2i + 2$ free factors of $u_i v$ are a, b, \dots, a, b, a, c . But the free factor in position $2i + 2$ of u_{k+1} is b , so we have a contradiction. Hence, I is not finitely generated and U is not weakly right noetherian.

(\Leftarrow). If M and N are both groups, then U is also a group and hence weakly right noetherian.

Now suppose that both M and N contain precisely two elements and that N is not a group. Then N is isomorphic to the two-element semilattice $\{1, 0\}$, and M is isomorphic to either $\{1, 0\}$ or \mathbb{Z}_2 .

If $M \cong \{1, 0\}$, then U is isomorphic to V^1 where V is the free product of two trivial semigroups. It follows from Theorem 6.4 and Corollary 5.5 that U is weakly right noetherian.

If $M \cong \mathbb{Z}_2$, then U is defined by the monoid presentation

$$\langle a, b \mid a^2 = 1, b^2 = b \rangle.$$

Let $u_1 = ab$, $u_2 = ba$ and $u_3 = bab$, and let $U_i = \{u_i^n : n \geq 0\}$ for $i = 1, 2, 3$. Also, let $U_4 = \{u_1^n (aba) : n \geq 0\}$. Then each U_i is isomorphic to the free monogenic monoid \mathbb{N}_0 , and $U = \bigcup_{i=1}^4 U_i$. Since \mathbb{N}_0 is weakly right noetherian, it follows from Lemma 3.11 that U is weakly right noetherian. \square

6.3. Semilattices of semigroups

Let Y be a semilattice and let $(S_\alpha)_{\alpha \in Y}$ be a family of disjoint semigroups, indexed by Y . If $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semigroup such that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$, then S is called a *semilattice of semigroups*, and we denote it by $S = \mathcal{S}(Y, S_\alpha)$.

Now let $S = \bigcup_{\alpha \in Y} S_\alpha$, and suppose that for each $\alpha, \beta \in Y$ with $\alpha \geq \beta$ there exists a homomorphism $\phi_{\alpha,\beta} : S_\alpha \rightarrow S_\beta$. Furthermore, assume that:

- for each $\alpha \in Y$, the homomorphism $\phi_{\alpha,\alpha}$ is the identity map on S_α ;
- for each $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, we have $\phi_{\alpha,\beta} \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$.

For $a \in S_\alpha$ and $b \in S_\beta$, we define

$$ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}).$$

With this multiplication, S is a semilattice of semigroups. In this case, we call S a *strong semilattice of semigroups* and denote it by $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$.

In the remainder of this section, we investigate under what conditions a (strong) semilattice of semigroups is weakly right noetherian.

The following characterization of weakly right noetherian semilattices follows immediately from Theorem 3.2.

Proposition 6.6 (Gould et al. [9, Proposition 3.1]). *Let Y be a (meet-)semilattice. Then Y is weakly noetherian if and only if it contains no infinite strictly ascending chain or infinite antichain of elements.*

Since the semilattice Y is a homomorphic image of $S = \mathcal{S}(Y, S_\alpha)$, by Lemma 4.1 we have:

Lemma 6.7. *Let $S = \mathcal{S}(Y, S_\alpha)$ be a semilattice of semigroups. If S is weakly right noetherian, then Y is weakly noetherian.*

For a semilattice of semigroups $\mathcal{S}(Y, S_\alpha)$ to be weakly right noetherian, it is not required that all the S_α be weakly right noetherian. In order to show this, we consider the following construction, which will be used again later in the paper.

Construction 6.8. Let S and T be two semigroups with homomorphisms $\theta, \phi : S \rightarrow T$. Let $N_T = \{x_t : t \in T\} \cup \{0\}$ be a null semigroup disjoint from S . We define a multiplication on $S \cup N_T$, extending those on S and N_T , as follows:

$$s \cdot x_t = x_{(s\theta)t}, \quad x_t \cdot s = x_{t(s\phi)}.$$

With this multiplication, $S \cup N_T$ is a semigroup. We denote it by $\mathcal{U}(S, T; \theta, \phi)$. We simplify this expression in the case that $S = T$ by only writing S once, and similarly if $\theta = \phi$.

We may view $\mathcal{U}(S, T; \theta, \phi)$ as a semilattice of semigroups, where the structure semilattice is $\{\alpha, 0\}$ and the corresponding subsemigroups are S and N_T , respectively. Every non-zero element of a null semigroup is indecomposable, so infinite null semigroups are not weakly right noetherian by Lemma 3.10. Therefore, the following result yields the desired counterexample.

Proposition 6.9. *Let S and T be two semigroups with homomorphisms $\theta, \phi : S \rightarrow T$ where ϕ is surjective, and let $U = \mathcal{U}(S, T; \theta, \phi)$. Then U is weakly right noetherian if and only if S is weakly right noetherian.*

Proof. If U is weakly right noetherian, then since $U \setminus S = N_T$ is an ideal of U , we have that S is weakly right noetherian by Corollary 5.10.

Conversely, suppose S is weakly right noetherian, and let I be a right ideal of U . Now $I \cap S$ is either empty or a right ideal of S ; in the latter case it is generated by a finite

set Y since S is weakly right noetherian. We have that T is weakly right noetherian by Lemma 4.1. In particular, $T = AT^1$ for some finite set $A \subseteq T$. For each $a \in A$, define a set

$$I_a = \{s \in S : x_a s \in I\}.$$

Let A' be the set of elements in A such that $I_a \neq \emptyset$. For each $a \in A'$, we have that I_a is a right ideal of S , so it is generated by some finite set U_a since S is weakly right noetherian. We claim that I is generated by the finite set

$$Z = Y \cup \left(\bigcup_{a \in A'} x_a U_a^1 \right).$$

Let $u \in I$. If $u \in S$, then $u \in I \cap S = YS^1$. Clearly, $0 \in ZU$, so we just need to consider the case that $u = x_t$ for some $t \in T$. Then $t = av$ for some $a \in A$ and $v \in T^1$. If $v = 1$, then $u = x_a \in Z$. Otherwise, let $s \in S$ be such that $s\theta = v$, so $s \in I_a = U_a S^1$. It follows that $u = x_a s \in (x_a U_a)S^1$, as required. \square

Remark 6.10. In general, principal factors of weakly right noetherian semigroups need not be weakly right noetherian. Indeed, let G be an infinite group. Then $U = \mathcal{U}(G, \text{id})$ is weakly right noetherian. It has three $\mathcal{H} = \mathcal{J}$ -classes: G , $J = \{x_g : g \in G\}$ and $\{0\}$. The principal factor of J is isomorphic to N_G , which is not weakly right noetherian.

In the case that a semigroup S_β , $\beta \in Y$, has local right identities, it is a necessary condition for $S = \mathcal{S}(Y, S_\alpha)$ to be weakly right noetherian that S_β be weakly right noetherian.

Lemma 6.11. *Let $S = \mathcal{S}(Y, S_\alpha)$ be a semilattice of semigroups, let $\beta \in Y$, and suppose that S_β has local right identities. If S is weakly right noetherian, then so is S_β .*

Proof. Let $Y' = \{\alpha \in Y : \alpha \not\leq \beta\}$, and let $I = \bigcup_{\alpha \in Y'} S_\alpha$. Now, I is an ideal and $T = S \setminus I$ is a subsemigroup of S , so T is weakly right noetherian by Corollary 5.10. Since S_β is an ideal of T with local right identities, it is weakly right noetherian by Corollary 5.8. \square

The following corollary follows from Lemmas 6.11 and 3.11.

Corollary 6.12. *Let $S = \mathcal{S}(Y, S_\alpha)$ be a semilattice of semigroups where Y is finite and each S_α has local right identities. Then S is weakly right noetherian if and only if each S_α is weakly right noetherian.*

We now consider the situation for strong semilattices of semigroups.

Proposition 6.13. *Let $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of semigroups. If S is weakly right noetherian, then Y is weakly noetherian and each S_α is weakly right noetherian.*

Proof. The semilattice Y is weakly noetherian by Lemma 6.7. Now let $\alpha \in Y$. We prove that S_α preserves \mathcal{R} in S , and hence S_α is weakly right noetherian by Proposition 5.6. We write $\mathcal{R} = \mathcal{R}_S$ and $\mathcal{R}_\alpha = \mathcal{R}_{S_\alpha}$. We just need to show that $\leq_{\mathcal{R}} \cap (S_\alpha \times S_\alpha) \subseteq \leq_{\mathcal{R}_\alpha}$. So, let $a, b \in S_\alpha$ and $a \leq_{\mathcal{R}} b$. Then $a = bs$ for some $s \in S^1$. If $s = 1$ then $a = b$, so assume

that $s \in S$. Then $s \in S_\beta$ for some $\beta \in Y$. Since $a = bs \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$, we conclude that $\alpha\beta = \alpha$. It follows that

$$a = (b\phi_{\alpha,\alpha})(s\phi_{\beta,\alpha}) = b(s\phi_{\beta,\alpha}) \in bS_\alpha,$$

so $a \leq_{\mathcal{R}_\alpha} b$, as required. □

Proposition 6.13 and Lemma 3.11 together yield:

Corollary 6.14. *Let $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of semigroups where Y is finite. Then S is weakly right noetherian if and only if each S_α is weakly right noetherian.*

Example 7.17 below shows that the converse of Proposition 6.13 does not hold, even in the case that each S_α is finite.

Open Problem 6.15. Find necessary and sufficient conditions for a strong semilattice of semigroups to be weakly right noetherian.

6.4. Rees matrix semigroups and Brandt extensions

Let S be a semigroup, let I and J be two non-empty index sets, and let $P = (p_{ji})$ be a $J \times I$ matrix with entries from S . The set $I \times S \times J$ becomes a semigroup under the multiplication given by

$$(i, s, j)(k, t, l) = (i, sp_{jkt}, l),$$

and is called the *Rees matrix semigroup over S with respect to P* . We denote this semigroup by $\mathcal{M}(S; I, J; P)$.

We now modify the Rees matrix construction as follows. Let the matrix P have entries from S^0 . The set $(I \times S \times J) \cup \{0\}$ with multiplication given by

$$(i, s, j)(k, t, l) = \begin{cases} (i, sp_{jkt}, l) & \text{if } p_{jk} \in S \setminus \{0\} \\ 0 & \text{if } p_{jk} = 0, \end{cases}$$

and $0(i, s, j) = (i, s, j)0 = 0^2 = 0$, is a semigroup. It is called the *Rees matrix semigroup with zero over S with respect to P* , and is denoted by $\mathcal{M}^0(G; I, J; P)$.

A semigroup is said to be *completely simple* if it is simple and contains minimal left and right ideals. A semigroup with zero is said to be *completely 0-simple* if it is 0-simple and contains 0-minimal left and right ideals. Rees [25] proved that a semigroup is completely 0-simple if and only if it is isomorphic to a Rees matrix semigroup with zero $\mathcal{M}^0(G; I, J; P)$ over a group G such that every row and column of P contains at least one element in G . Consequently, a semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup $\mathcal{M}(G; I, J; P)$ over a group G .

We shall consider conditions under which a Rees matrix semigroup with zero $T = \mathcal{M}^0(S; I, J; P)$ is weakly right noetherian. We begin by considering what affect T being weakly right noetherian has on S and the index sets I and J .

Lemma 6.16. *Let $T = \mathcal{M}^0(S; I, J; P)$ be weakly right noetherian.*

- (1) *The semigroup S is weakly right noetherian and I is finite.*

- (2) Let U be the ideal of S^0 generated by the entries of P . If the set $S \setminus U$ is non-empty, then both $S \setminus U$ and J are finite.

Proof. (1) Let A be a right ideal of S . Then $B = (I \times A \times J) \cup \{0\}$ is a right ideal of T . Since T is weakly right noetherian, there exists a finite set $U \subseteq B$ such that $B = UB^1$. We may assume that $U = I_0 \times X \times J_0$ for some finite sets $I_0 \subseteq I$, $X \subseteq A$, $J_0 \subseteq J$. We claim that $I = I_0$ and $A = XS^1$. Indeed, let $i \in I$, $a \in A$, and pick any $j \in J$. Then $(i, a, j) = (i_0, x, j_0)t$ for some $(i_0, x, j_0) \in U$ and $t \in T^1$. It follows that $i = i_0 \in I_0$ and $a \in xS^1 \subseteq XS^1$, as required.

- (2) Notice that the set $I \times (S \setminus U) \times J$ consists of indecomposable elements of T ; it is hence finite by Lemma 3.10. In particular, both $S \setminus U$ and J are finite. \square

The converse of Lemma 6.16(1) does not hold in general. In order to show this, we first present the following lemma.

Lemma 6.17. Let $T = \mathcal{M}^0(S; I, J; P)$, let J' be a subset of J , let $P' = (p_{ji})_{j \in J', i \in I}$, and let $T' = \mathcal{M}^0(S; I, J'; P')$. If T is weakly right noetherian, then so is T' .

Proof. It is easy to see that T' is a right unitary subsemigroup of T , so it is weakly right noetherian by Corollary 5.9. \square

Example 6.18. Let $T = \mathcal{M}^0(S; I, J; P)$ with S infinite, and suppose there exists $j \in J$ such that $p_{ji} = 0$ for all $i \in I$. Then $T' = \mathcal{M}^0(S; I, \{j\}; P')$, where $P' = (p_{ji})_{i \in I}$, is not weakly right noetherian by Lemma 6.16(2) (in fact, T' is an infinite null semigroup). It follows from Lemma 6.17 that T is not weakly right noetherian.

Remark 6.19. Let $T = \mathcal{M}(S; I, J; P)$ be a Rees matrix semigroup over an infinite semigroup S with a zero 0 adjoined, where $0 \notin S$. If there exists $j \in J$ such that $p_{ji} = 0$ for all $i \in I$, then $T' = \mathcal{M}^0(S; I, J; P)$ is not weakly right noetherian by Example 6.18. Since T' is isomorphic to the Rees quotient of T by the ideal $I \times \{0\} \times J$, we deduce from Lemma 4.1 that T is not weakly right noetherian.

The converse of Lemma 6.16(1) holds in the case that every row of the matrix contains a unit.

Proposition 6.20. Let $S = \mathcal{M}^0(M; I, J; P)$ be a Rees matrix semigroup over a monoid M , and suppose that for every $j \in J$ there exists $i \in I$ such that $p_{ji} \in U(M)$. Then S is weakly right noetherian if and only if M is weakly right noetherian and I is finite.

Proof. The direct implication follows from Lemma 6.16.

For the converse, let A be a right ideal of S . Note that if $(i, u, j) \in A$, then $(i, u, l) \in A$ for all $l \in J$. Indeed, there exists $k \in I$ such that $p_{jk} \in U(M)$, so $(i, u, l) = (i, u, j)(k, p_{jk}^{-1}, l) \in A$. Now fix $j_0 \in J$, and choose $i_0 \in I$ such that $p_{j_0 i_0} \in U(M)$. Let I'

be the set of elements of I that appear in A . For each $i \in I'$, define a set

$$A_i = \{u \in M : (i, u, j_0) \in A\}.$$

We claim that A_i is a right ideal of S . Indeed, if $u \in A_i$ and $m \in M$, then

$$(i, um, j_0) = (i, u, j_0)(i_0, p_{j_0 i_0}^{-1} m, j_0) \in A_i.$$

Since M is weakly right noetherian, there exist finite sets $X_i \subseteq A_i$, $i \in I'$, such that $A_i = X_i M$. We claim that A is generated by the finite set

$$Y = \{(i, x, j_0) : i \in I', x \in X_i\}.$$

Indeed, if $(i, u, j) \in A$ then $(i, u, j_0) \in A$, so $u = xm$ for some $x \in X_i$ and $m \in M$. Hence, we have $(i, u, j) = (i, x, j_0)(i_0, p_{j_0 i_0}^{-1} m, j) \in YM$, as required. □

Corollary 6.21. *Let S be a completely (0-)simple semigroup. Then S is weakly right noetherian if and only if it has finitely many \mathcal{R} -classes.*

Proof. The semigroup S^0 is completely 0-simple. It follows from Proposition 6.20 and Corollary 5.5 that S is weakly right noetherian if and only if it has finitely many \mathcal{R} -classes. □

We now consider Brandt extensions. Let S be a semigroup and let I be a non-empty set. The set $(I \times S \times I) \cup \{0\}$ becomes a semigroup under the multiplication given by

$$(i, s, j)(k, t, l) = \begin{cases} (i, st, l) & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

and $0x = x0 = 0$ for all $x \in (I \times S \times I) \cup \{0\}$. It is called the *Brandt extension* of S by I , and we denote it by $\mathcal{B}(S, I)$.

Notice that if S is a monoid, then $\mathcal{B}(S, I)$ is isomorphic to $\mathcal{M}^0(S; I, I; P)$ where P is the $I \times I$ identity matrix. Brandt extensions of groups are precisely the completely 0-simple inverse semigroups [23, Theorem V.5.1].

Proposition 6.22. *Let S be a semigroup and let I be a non-empty set. Then the Brandt extension $\mathcal{B}(S, I)$ is weakly right noetherian if and only if S is weakly right noetherian and I is finite.*

Proof. Let $T = \mathcal{B}(S, I)$. We just need to prove that if T is weakly right noetherian, then I is finite. Indeed, if I is finite, then $\mathcal{B}(S^1, I) \setminus T$ is finite. It then follows from Corollary 5.5 and Proposition 6.20 that T is weakly right noetherian if and only if S is weakly right noetherian.

So, suppose that T is weakly right noetherian. Then there exists a finite set $U \subseteq T$ such that $T = UT^1$. Let I_0 be the elements of I appearing in U . Let $i \in I$, and pick any $s \in S$. Then $(i, s, i) = (i_1, x, i_2)t$ for some $(i_1, x, i_2) \in U$ and $t \in T^1$. It follows that $i = i_1 = i_2 \in I_0$, and hence $I = I_0$ is finite. □

6.5. Bruck–Reilly extensions

Let M be a monoid with identity 1_M and let $\theta : M \rightarrow M$ be an endomorphism. We define a binary operation on the set $\mathbb{N}_0 \times M \times \mathbb{N}_0$ by

$$(j, a, k)(p, b, q) = (j - k + t, (a\theta^{t-k})(b\theta^{t-p}), q - p + t),$$

where $t = \max(k, p)$ and θ^0 denotes the identity map on M . With this operation the set $\mathbb{N}_0 \times M \times \mathbb{N}_0$ is a monoid with identity $(0, 1_M, 0)$. It is denoted by $BR(M, \theta)$ and is called the *Bruck–Reilly extension of M determined by θ* .

The *bicyclic monoid* is the set $\mathbb{N}_0 \times \mathbb{N}_0$ with multiplication given by

$$(j, k)(p, q) = (j - k + t, q - p + t),$$

where $t = \max(k, p)$. Clearly, the bicyclic monoid is a homomorphic image of $BR(M, \theta)$. We note that the bicyclic monoid is a simple inverse monoid and is defined by the presentation $\langle b, c | bc = 1 \rangle$. It is well known that every one-sided ideal of the bicyclic monoid is principal, so we certainly have:

Lemma 6.23. *The bicyclic monoid is weakly noetherian.*

We shall provide necessary and sufficient conditions for a Bruck–Reilly extension to be weakly right noetherian. In order to do so, we first make the following definition.

Let M be a monoid and let $\theta : M \rightarrow M$ be an endomorphism. We call a sequence $(I_j)_{j \in \mathbb{N}_0}$ of right ideals of M a θ -sequence if $I_j\theta \subseteq I_{j+1}$ for every $j \geq 0$.

Theorem 6.24. *Let M be a monoid and let $\theta : M \rightarrow M$ be an endomorphism. Then $BR(M, \theta)$ is weakly right noetherian if and only if the following conditions hold:*

- (1) M is weakly right noetherian;
- (2) for any θ -sequence $(I_j)_{j \in \mathbb{N}_0}$ of right ideals of M , there exists some $n \in \mathbb{N}_0$ with a finite set $Y \subseteq I_n$ such that $I_{n+r} = (Y\theta^r)M$ for all $r \geq 0$.

Proof. We denote $BR(M, \theta)$ by N . Note that for any right ideal I of N , we have $(j, a, k) \in I$ if and only if $(j, a, 0) \in I$.

(\Rightarrow) The monoid M is isomorphic to the submonoid $M_0 = \{0\} \times M \times \{0\}$ of N . It can easily be shown M_0 is right unitary in N , so $M \cong M_0$ is weakly right noetherian by Corollary 5.9.

Now let $(I_j)_{j \in \mathbb{N}_0}$ be a θ -sequence of right ideals of M . We define a set

$$I = \bigcup_{j \in \mathbb{N}_0} \{(j, a, k) : a \in I_j, l \in \mathbb{N}_0\}.$$

We claim that I is a right ideal of N . Indeed, let $(j, a, k) \in I$ and $(p, m, q) \in N$. Then $A \in I_j$. Let u denote the element

$$(j, a, k)(p, m, q) = (j - k + t, (a\theta^{t-k})(m\theta^{t-p}), q - p + t),$$

where $t = \max(k, p)$. If $t = k$, then $u = (j, a(m\theta^{k-p}), q - p + k) \in I$ since I_j is a right ideal of M . If $t = p$, then $u = (j - k + p, (a\theta^{p-k})m, q) \in I$, since $(a\theta^{p-k}) \in I_{j-k+p}$ and I_{j-k+p} is a right ideal of M .

Since N is weakly right noetherian, there exists a finite set $X \subseteq I$ such that $I = XN$. By the note given at the beginning of the proof, we may assume that the third coordinate of each element of X is 0. Set $n = \max\{j : (j, a, 0) \in X\}$, and let

$$Y = \{a\theta^{n-j} : (j, a, 0) \in X\} \subseteq I_n.$$

Let $r \geq 0$ and let $b \in I_{n+r}$. Then $(n+r, b, 0) \in I$, so there exist $(j, a, 0) \in X$ and $(p, m, q) \in N$ such that $(n+r, b, 0) = (j, a, 0)(p, m, q)$. It follows that $n+r = j+p$, $b = (a\theta^p)m$ and $q = 0$. Hence, we have that

$$b = (a\theta^{n+r-j})m = ((a\theta^{n-j})\theta^r)m \in (Y\theta^r)M,$$

so $I_{n+r} = (Y\theta^r)M$, as required.

(\Leftarrow) Let I be a right ideal of N . For each $j \in \mathbb{N}_0$ define a set

$$I_j = \{a \in M : (j, a, 0) \in I\}.$$

Clearly, I_j is either empty or a right ideal of N . Let n_0 be minimal such that I_{n_0} is non-empty. For any $j \geq n_0$, we have

$$a \in I_j \implies (j, a, 0) \in I \implies (j+1, a\theta, 0) = (j, a, 0)(1, 1_M, 0) \in I \implies a\theta \in I_{j+1},$$

so $I_j\theta \subseteq I_{j+1}$. Thus $(I_j)_{j \geq n_0}$ is a θ -sequence of right ideals of M . By assumption, there exists $n \geq n_0$ with a finite set $Y \subseteq I_n$ such that $I_{n+r} = (Y\theta^r)M$ for all $r \geq 0$. Since M is weakly right noetherian, for each $j \in \{n_0, \dots, n-1\}$ there exists a finite set $X_j \subseteq I_j$ such that $I_j = X_jM$. Writing $Y = X_n$, we claim that I is generated by the finite set

$$X = \bigcup_{j=n_0}^n \{(j, x, 0) : x \in X_j\}.$$

Let $(j, a, k) \in I$. Then $a \in I_j$ and $j \geq n_0$. If $j < n$, then $a = xm$ for some $x \in X_j$ and $m \in M$, so we have

$$(j, a, k) = (j, x, 0)(0, m, k) \in XM.$$

If $j \geq n$, then $a = (x\theta^{j-n})m$ for some $x \in X_n$ and $m \in M$, and hence

$$(j, a, k) = (n, x, 0)(j-n, m, k) \in XM,$$

as required. □

Corollary 6.25. *Let M be a monoid and let $\theta : M \rightarrow U(M)$ be a homomorphism. Then $BR(M, \theta)$ is weakly right noetherian if and only if M is weakly right noetherian.*

Proof. The forward direction follows immediately from Theorem 6.24. For the converse, let $(I_j)_{j \in \mathbb{N}_0}$ be a θ -sequence of right ideals of M . For any $j \in \mathbb{N}_0$, we have $I_j\theta \in U(M)$ and $I_j\theta \subseteq I_{j+1}$, so $I_{j+1} \cap U(M) \neq \emptyset$. Since I_{j+1} is a right ideal of M , it follows that $I_{j+1} = M$. Thus $M = I_1 = I_2 = \dots$, and $I_{1+r} = (\{1_M\}\theta^r)M$ for all $r \geq 0$. Hence, $BR(M, \theta)$ is weakly right noetherian by Theorem 6.24. □

We deduce from Theorem 6.24 that the Bruck–Reilly extension of a weakly right noetherian monoid (indeed, even a finite monoid) need not be weakly right noetherian.

Lemma 6.26. *Let M be a monoid. Suppose there exists a right ideal I of M , an element $a \in M \setminus I$, and a monoid homomorphism $\theta : M \rightarrow M$ such that $(I \cup \{a\})\theta \subseteq I$. Then $BR(M, \theta)$ is not weakly right noetherian.*

Proof. Let J be the right ideal $aM \cup I$, and consider the infinite θ -sequence J, J, \dots . Since $J\theta \subseteq I \subsetneq J$, this sequence does not satisfy the condition in (2) of Theorem 6.24, so $BR(M, \theta)$ is not weakly right noetherian. \square

Remark 6.27. The condition of Lemma 6.26 is satisfied by the bicyclic monoid: let $a = (1, 0)$, let $I = (2, 0)B$, and let $\theta : B \rightarrow B$ be given by $(i, j)\theta = (2i, 2j)$.

This condition is also satisfied by any monoid M such that $M \setminus U(M)$ is an ideal containing an idempotent e and an element $a \notin eM$ (e.g. the 2-element null semigroup with an identity adjoined). Indeed, let $I = eM$ and let $\theta : M \rightarrow M$ be the endomorphism given by $U(M)\theta = \{1\}$ and $(M \setminus U(M))\theta = \{e\}$. Then $(I \cup \{a\})\theta = \{e\} \subseteq I$.

7. Regular semigroups

In this section, we study weakly right noetherian regular semigroups. We begin with a necessary and sufficient condition for a regular semigroup to be weakly right noetherian. We then focus our attention on certain classes of regular semigroups, including inverse semigroups, completely regular semigroups and regular semigroups with a principal series.

Theorem 7.1. *The following are equivalent for a regular semigroup S :*

- (1) S is weakly right noetherian;
- (2) for every subset $U \subseteq E(S)$, there exists a finite set $X \subseteq U$ with the following property: for each $u \in U$ there exists $x \in X$ such that $u = xu$.

Proof. (1) \Rightarrow (2). Let U be a subset of $E(S)$, and let I be the right ideal US^1 of S . Since S is weakly right noetherian, there exists a finite subset $X \subseteq U$ such that $I = XS^1$. For each $u \in U$, we have that $u = xs$ for some $x \in X$ and $s \in S^1$, and hence $u = x^2s = xu$.

(2) \Leftarrow (1). Let I be a right ideal of S . By assumption, there exists a finite set $X \subseteq I \cap E(S)$ satisfying the property in (2). We claim that $I = XS$. Indeed, let $a \in I$. Let b be an inverse of a , so $a = aba$. Then $ab \in I \cap E(S)$, so there exists $x \in X$ such that $ab = x(ab)$. Thus $a = aba = xa \in XS$, as required. \square

Let S be a regular semigroup, and let $T = \langle E(S) \rangle$ be the subsemigroup of S generated by its set of idempotents. Then T is regular by [7, Corollary 2]. Since $E(S) = E(T)$, we immediately deduce from Theorem 7.1:

Corollary 7.2. *A regular semigroup S is weakly right noetherian if and only if its subsemigroup $T = \langle E(S) \rangle$ is weakly right noetherian.*

Corollary 7.3. *An inverse semigroup S is weakly right noetherian if and only if its semilattice of idempotents $E(S)$ is weakly noetherian.*

Corollary 7.4. *An inverse semigroup is weakly right noetherian if and only if it is weakly noetherian.*

Remark 7.5. Corollary 7.4 does not hold for regular semigroups in general. Indeed, any infinite right zero semigroup is weakly right noetherian but not weakly left noetherian.

Let X be an infinite set, and let $X^{-1} = \{x^{-1} : x \in X\}$ be a set disjoint from X . The *polycyclic monoid* over X , denoted by P_X , is the monoid with zero defined by the presentation

$$\langle X, X^{-1} \mid xx^{-1} = 1, xy^{-1} = 0 (x, y \in X, x \neq y) \rangle.$$

This presentation yields the normal form $\{u^{-1}v : u, v \in X^*\} \cup \{0\}$ for P_X [19, Section 1]. The monoid P_X is an inverse monoid with a single non-zero \mathcal{D} -class (see [22, Section 1.3] or [2, p. 478]). In the case that $|X| = 1$, P_X is the bicyclic monoid with zero adjoined. It turns out that this is the only case where P_X is weakly noetherian.

Proposition 7.6. *The polycyclic monoid P_X is weakly noetherian if and only if $|X| = 1$.*

Proof. If $|X| = 1$, then P_X is weakly noetherian by Lemma 6.23 and Corollaries 5.5 and 7.4.

Suppose $|X| \geq 2$. Choose distinct elements $x, y \in X$. We claim that the infinite set $\{y^{-i}x^{-1}xy^i : i \in \mathbb{N}\} \subseteq E(P_X)$ is an antichain. Indeed, since $xy^{-1} = yx^{-1} = 0$, we deduce that for any $i \neq j$,

$$(y^{-i}x^{-1}xy^i)(y^{-j}x^{-1}xy^j) = y^{-i}x^{-1}(xy^{i-j}x^{-1})xy^j = 0.$$

It now follows from Proposition 6.6 that $E(P_X)$ is not weakly noetherian, and hence P_X is not weakly noetherian by Corollary 7.3. □

We now show that the principal factors of a regular semigroup inherit the property of being weakly right noetherian.

Lemma 7.7. *Let S be a regular semigroup. If S is weakly right noetherian, then so are all its principal factors.*

Proof. Let J be a \mathcal{J} -class of S , fix $x \in J$, and let $T = S^1xS^1$. Now, T is a union of \mathcal{J} -classes of S , of which J is the unique maximal one. Since any pair of elements of S that are inverses of each other must belong to the same \mathcal{J} -class, it follows that T is regular, and hence T is weakly right noetherian by Corollary 5.7. Then the principal factor of J is weakly right noetherian by Lemma 4.1, since it is a Rees quotient of T . □

Lemma 7.7 and Corollary 4.8 together yield:

Corollary 7.8. *Let S be a regular semigroup with a principal series. Then S is weakly right noetherian if and only if all its principal factors are weakly right noetherian.*

A semigroup S is said to be *completely semisimple* if all its principal factors are completely 0-simple or completely simple.

Corollary 7.9. *Let S be a completely semisimple semigroup with a principal series. Then S is weakly right noetherian if and only if it has finitely many \mathcal{R} -classes.*

Proof. If S is weakly right noetherian, then every principal factor of S has finitely many \mathcal{R} -classes by Lemma 7.8 and Corollary 6.21. It follows that every \mathcal{J} -class is a finite union of \mathcal{R} -classes. Since S has finitely many \mathcal{J} -classes, we conclude that it has finitely many \mathcal{R} -classes.

The converse follows from Corollary 3.3. □

Remark 7.10. Corollary 7.9 does not hold if we remove the condition that S has a principal series. Indeed, there exist infinite weakly noetherian semilattices (in which $\mathcal{J} = \mathcal{R}$ is the identity relation).

Also, Corollary 7.9 does not hold for general regular semigroups with a principal series. For example, the bicyclic monoid has a single \mathcal{J} -class and is weakly right noetherian, but it has infinitely many \mathcal{R} -classes.

A semigroup is said to be *completely regular* if it is a union of groups. Completely regular semigroups have the following characterization.

Theorem 7.11 (Howie [13, Theorem 4.1.3]). *Every completely regular semigroup is a semilattice of completely simple semigroups.*

From Lemmas 6.7 and 6.11, and Corollary 6.21, we deduce:

Proposition 7.12. *Let S be a completely regular semigroup, and let $S = \mathcal{S}(Y, S_\alpha)$ be its decomposition into a semilattice of completely simple semigroups. If S is weakly right noetherian, then Y is weakly noetherian and each S_α has finitely many \mathcal{R} -classes.*

We shall see that the converse of Proposition 7.12 does not hold.

In the remainder of this section, we focus our attention on *strong* semilattices of completely simple semigroups. For more information about the structure of such semigroups, see [23, Section IV.4].

A *Clifford semigroup* is an inverse completely regular semigroup. It follows from Theorem 7.11 that Clifford semigroups are precisely the semilattices of groups. In fact, Clifford semigroups are *strong* semilattices of groups [11, Theorem III.2.12]. If $S = \mathcal{S}(Y, G_\alpha)$ is a semilattice of groups, it is clear that Y is isomorphic to $E(S)$, so Corollary 7.3 yields:

Corollary 7.13. *Let S be a Clifford semigroup with decomposition $S = \mathcal{S}(Y, G_\alpha)$ into a semilattice of groups. Then S is weakly right noetherian if and only if Y is weakly noetherian.*

In what follows we shall provide necessary and sufficient conditions for a general strong semilattice of completely simple semigroups to be weakly right noetherian. We use the following folklore result, which we prove for completeness.

Lemma 7.14. *Let $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of completely simple semigroups. Then each of Green's relations is a congruence on S . Furthermore, we have*

$S/\mathcal{J} = S/\mathcal{D} = Y$; S/\mathcal{R} is a strong semilattice of left zero semigroups; S/\mathcal{L} is a strong semilattice of right zero semigroups; and S/\mathcal{H} is a strong semilattice of rectangular bands.

Proof. It is clear that $\mathcal{D} = \mathcal{J}$ and $S/\mathcal{J} = Y$. We prove that \mathcal{R} is a congruence. A dual argument proves that \mathcal{L} is a congruence, and hence $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ is congruence.

For each $\alpha \in Y$, let $S_\alpha = \mathcal{M}(G_\alpha; I_\alpha, J_\alpha; P_\alpha)$. Recall that \mathcal{R} is a left congruence on S , so we just need to show that it is a right congruence. For $\alpha \in Y$, we write $\mathcal{R}_\alpha = \mathcal{R}_{S_\alpha}$. Since S_α is regular, we have that $\mathcal{R}_\alpha = \mathcal{R} \cap (S_\alpha \times S_\alpha)$ [13, Proposition 2.4.2]. Note that \mathcal{R}_α is a congruence on S_α .

Let $(a, b) \in \mathcal{R}$ and $c \in S$. Since Y is \mathcal{R} -trivial, we must have that $(a, b) \in \mathcal{R}_\alpha$ for some $\alpha \in Y$. Now, the element c belongs to some S_β , $\beta \in Y$. Certainly $(a\phi_{\alpha,\alpha\beta}, b\phi_{\alpha,\alpha\beta}) \in \mathcal{R}_{\alpha\beta}$. Since $\mathcal{R}_{\alpha\beta}$ is a congruence, we have

$$(ac, bc) = ((a\phi_{\alpha,\alpha\beta})(c\phi_{\beta,\alpha\beta}), (b\phi_{\alpha,\alpha\beta})(c\phi_{\beta,\alpha\beta})) \in \mathcal{R}_{\alpha\beta} \subseteq \mathcal{R},$$

as required.

It can be easily shown that $S/\mathcal{R} \cong \mathcal{S}(Y, I_\alpha, \psi_{\alpha,\beta})$, where the I_α are considered as left zero semigroups and each $\psi_{\alpha,\beta} : I_\alpha \rightarrow I_\beta$ is defined as follows: $i_\alpha\psi_{\alpha,\beta} = k_\beta$ if for some (and hence all) $j_\alpha \in J_\alpha$, we have

$$(i_\alpha, p_{j_\alpha i_\alpha}^{-1}, j_\alpha)\phi_{\alpha,\beta} = (k_\beta, p_{l_\beta k_\beta}^{-1}, l_\beta)$$

for some $l_\beta \in J_\beta$. Dually, we have that S/\mathcal{L} is a strong semilattice of right zero semigroups. Finally, we have $S/\mathcal{H} \cong \mathcal{S}(Y, B_\alpha, \theta_{\alpha,\beta})$, where each $B_\alpha = I_\alpha \times J_\alpha$ is a rectangular band and each $\theta_{\alpha,\beta} : B_\alpha \rightarrow B_\beta$ is defined as follows: $(i_\alpha, j_\alpha)\theta_{\alpha,\beta} = (k_\beta, l_\beta)$ if

$$(i_\alpha, p_{j_\alpha i_\alpha}^{-1}, j_\alpha)\phi_{\alpha,\beta} = (k_\beta, p_{l_\beta k_\beta}^{-1}, l_\beta).$$

This completes the proof. □

Theorem 7.15. *Let S be a strong semilattice of completely simple semigroups, and let $S/\mathcal{R} = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$ be the decomposition of S/\mathcal{R} into a strong semilattice of left zero semigroups. Then S is weakly right noetherian if and only if the following conditions hold:*

- (1) Y is weakly noetherian;
- (2) each S_α is finite;
- (3) there exists a finite subsemilattice Y_0 of Y with the following property: for each $\beta \in Y$, there exists $\alpha \in Y_0$ such that $\alpha \geq \beta$ and $\phi_{\alpha,\beta}$ is surjective.

Proof. Green’s relation \mathcal{R} is a congruence on S by Lemma 7.14. Hence, by Lemma 4.3, S is weakly right noetherian if and only if S/\mathcal{R} is weakly right noetherian. Therefore, it suffices to consider the case that $S = S/\mathcal{R}$.

(\Rightarrow) (1) and (2) follow immediately from Proposition 7.12. For (3), we have that $S = XS$ for some finite subset $X \subseteq S$. Let Y_0 be the semigroup generated by $\{\alpha \in Y : X \cap S_\alpha \neq \emptyset\}$. Since X is finite and Y is locally finite (that is, every finitely generated subsemigroup is finite), we conclude that Y_0 is finite.

Now consider $\beta \in Y$, and let $S_\beta = \{b_1, \dots, b_n\}$. For each $i \in \{1, \dots, n\}$, there exists $x_i \in X$ such that $b_i \in x_i S$, which implies that $b_i = x_i b_i$. Let $x_i \in S_{\alpha_i}$. Since S_β is a left zero semigroup, we have $b_i = (x_i \phi_{\alpha_i, \beta}) b_i = x_i \phi_{\alpha_i, \beta}$. Now set $\alpha = \alpha_1 \cdots \alpha_n$. Then $\alpha \in Y_0$, and since $\alpha_i \geq \beta$ for all $i \in \{1, \dots, n\}$, we deduce that $\alpha \geq \beta$. For each $i \in \{1, \dots, n\}$, we have $b_i = (x_i \phi_{\alpha_i, \alpha}) \phi_{\alpha, \beta}$, so $\phi_{\alpha, \beta}$ is surjective.

(\Leftarrow) Let I be a right ideal of S . For each $\alpha \in Y_0$ and $a \in S_\alpha$, define a set

$$U_a = \{\beta \in Y : \beta \leq \alpha, a \phi_{\alpha, \beta} \in I\},$$

and let I_a be the right ideal $U_a Y$ of Y . Since Y is weakly noetherian, there exists a finite set $X_a \subseteq U_a$ such that $I_a = X_a Y$. We claim that I is generated by the finite set

$$X = \{a \phi_{\alpha, \beta} : a \in S_\alpha, \alpha \in Y_0, \beta \in X_a\}.$$

Indeed, let $c \in I \cap S_\gamma$. There exists $\alpha \in Y_0$ such that $\alpha \geq \gamma$ and $\phi_{\alpha, \gamma}$ is surjective. In particular, there exists $a \in S_\alpha$ such that $a \phi_{\alpha, \gamma} = c$. Then $\gamma \in I_a$, so $\gamma = \beta \gamma$ for some $\beta \in X_a$. It follows that

$$c = c^2 = (a \phi_{\alpha, \gamma}) c = ((a \phi_{\alpha, \beta}) \phi_{\beta, \gamma}) (c \phi_{\gamma, \gamma}) = (a \phi_{\alpha, \beta}) c \in X S,$$

completing the proof of this direction and of the theorem. □

Corollary 7.16. *Let $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha, \beta})$ be a strong semilattice of completely simple semigroups. If S is weakly right noetherian, then the set $\{|S_\alpha/\mathcal{R}| : \alpha \in Y\}$ is bounded.*

Proof. As in the proof of Lemma 7.14, we have that S/\mathcal{R} is a strong semilattice of semigroups $\mathcal{S}(Y, T_\alpha, \psi_{\alpha, \beta})$, where $T_\alpha = S_\alpha/\mathcal{R}$. By Lemma 7.12, each T_α is finite. Let Y_0 be as stated in Theorem 7.15. We claim that $\{|T_\alpha| : \alpha \in Y\}$ is bounded above by

$$N = \max\{|T_\alpha| \in \mathbb{N} : \alpha \in Y_0\}.$$

Indeed, for each $\beta \in Y$ there exists $\alpha \in Y_0$ such that $\alpha \geq \beta$ and $\psi_{\alpha, \beta} : T_\alpha \rightarrow T_\beta$ is surjective; thus $|T_\beta| \leq |T_\alpha| \leq N$. □

Given Theorem 7.15, we can show that the converse of Proposition 7.12 does not hold, even in the case that S is a band.

Example 7.17. Let $S_i = \{x_i, y_i\}$ ($i \in \mathbb{N}$) be disjoint copies of the 2-element left zero semigroup. Let $\phi_{i, i}$ be the identity map on S_i , and for $i < j$ let $\phi_{i, j} : S_i \rightarrow S_j$ be the homomorphism given by $x_i \phi_{i, j} = y_i \phi_{i, j} = x_j$. Then we have a strong semilattice of semigroups $S = \mathcal{S}(Y, S_i, \phi_{i, j})$, where Y is the infinite descending chain (\mathbb{N}, \geq) . Clearly, S does not satisfy condition (3) of Theorem 7.15, so it is not weakly right noetherian.

We end this section with an example demonstrating that Corollary 7.16 does not hold for completely regular semigroups in general.

Example 7.18. Let $S_i = \{x_{i,1}, \dots, x_{i,i}\}$ for each $i \in \mathbb{N}$, and let $S = \bigcup_{i \in \mathbb{N}} S_i$. For $i, j \in \mathbb{N}$, $k \in \{1, \dots, i\}$ and $l \in \{1, \dots, j\}$, define

$$x_{i,k}x_{j,l} = \begin{cases} x_{j,l} & \text{if } i < j, \\ x_{i,k} & \text{if } i \geq j. \end{cases}$$

It can be shown that this multiplication is associative by an exhaustive case analysis. It is easy to see that S is a semilattice of semigroups $\mathcal{S}(Y, S_i)$, where $Y = (\mathbb{N}, \geq)$ and each S_i is a left zero semigroup.

We now prove that S is weakly right noetherian. Let I be a right ideal of S , and let i be minimal such that $I \cap S_i \neq \emptyset$. We claim that $I = (I \cap S_i)S$. Indeed, if $x_{j,l} \in I$, then $j \geq i$. If $i = j$, then $x_{j,l} \in I \cap S_i$. Otherwise, we have $x_{j,l} = x_{i,1}x_{j,l} \in (I \cap S_i)S$, as required.

8. Commutative semigroups

In this section, we consider weakly noetherian commutative semigroups. We begin by presenting the basic structure theory of commutative semigroups in terms of archimedean semigroups.

An *archimedean semigroup* is a commutative semigroup S with the following property: for each $a, b \in S$, there exist $n \in \mathbb{N}$ and $s \in S$ such that $a^n = bs$. For instance, the free monogenic semigroup is archimedean.

Theorem 8.1 (Grillet [11, Theorem IV.2.2]). *Every commutative semigroup is a semilattice of archimedean semigroups.*

We now characterize archimedean semigroups with an idempotent. We need the following definition.

A semigroup S with zero 0 is said to be *nilpotent* if for every $s \in S$ there exists $n \in \mathbb{N}$ such that $s^n = 0$.

Lemma 8.2 (Grillet [11, Proposition IV.2.3]). *A semigroup S is archimedean with idempotent if and only if S is either an abelian group or an ideal extension of an abelian group by a commutative nilpotent semigroup.*

In general, archimedean semigroups can have a rather complex structure. We refer the reader to [11, Chapter IV] for more information.

In order for a commutative semigroup to be weakly noetherian, it is *not* necessary that all its archimedean components be weakly noetherian, as demonstrated by the following example.

Example 8.3. Let FC_2 denote the free commutative semigroup on two generators a and b . We have that FC_2 is weakly noetherian by Theorem 3.8. It is easy to see that FC_2 has three archimedean components: $A = \langle a \rangle$, $B = \langle b \rangle$ and $C = \{a^i b^j : i, j \geq 1\}$. The infinite set $\{ab^i, a^i b : i \geq 1\}$ consists of all the indecomposable elements of C , so C is not weakly noetherian by Lemma 3.10.

The next example shows that a commutative semigroup may not be weakly noetherian even if its structure semilattice and archimedean components are all weakly noetherian.

Example 8.4. Let $S_i = \langle a_i \rangle$ ($i \in \mathbb{N}$) be disjoint copies of the free monogenic semigroup \mathbb{N} , which is weakly noetherian by Theorem 3.8. For $i < j$, let $\phi_{i,j} : S_i \rightarrow S_j$ be the isomorphism given by $a_i \mapsto a_j$. Let S be the strong semilattice of archimedean semigroups $\mathcal{S}(Y, S_i, \phi_{i,j})$, where $Y = (\mathbb{N}, \geq)$. Then S contains an infinite set $\{a_i : i \in \mathbb{N}\}$ of indecomposable elements, and is hence not weakly noetherian by Lemma 3.10.

We now state the main result of this section.

Theorem 8.5. *Let S be a commutative semigroup with finitely many archimedean components. Then S is weakly noetherian if and only if S/\mathcal{H} is finitely generated.*

Remark 8.6. Note that if S is a commutative semigroup such that S/\mathcal{H} is finitely generated, then S has finitely many archimedean components. Indeed, let $S = \mathcal{S}(Y, S_\alpha)$ be a decomposition of S into a semilattice of archimedean semigroups. It can be easily shown that if two elements of S are \mathcal{H} -related, then they belong to the same archimedean component. Thus $S/\mathcal{H} = \mathcal{S}(Y, T_\alpha)$ for some semigroups T_α . Hence, Y is a homomorphic image of S/\mathcal{H} . Since S/\mathcal{H} is finitely generated, we conclude that Y is finite.

It follows that Theorem 8.5 does not hold if the condition that S has finitely many archimedean components is dropped, since there certainly exist weakly noetherian commutative semigroups with infinitely many archimedean components; e.g. infinite weakly noetherian semilattices.

In order to prove Theorem 8.5, we first state and prove a few lemmas.

Lemma 8.7. *Let T be a commutative nilpotent semigroup. If T is finitely generated as a right ideal, then it is finite.*

Proof. Since T is finitely generated as a right ideal, there exists a finite set $X \subseteq T$ such that $T = XT^1$. Let $U = \langle X \rangle$. For each $x \in X$, let

$$m(x) = \min\{n \in \mathbb{N} : x^n = 0\},$$

and let $N = \prod_{x \in X} m(x)$. It can be easily shown that $|U| \leq N$. We claim that $T = U$. Suppose for a contradiction that $T \neq U$, and let $a \in T \setminus U$. We have that $a = x_1 t_1$ for some $x_1 \in X$ and $t_1 \in T^1$. Since $a \notin U$, we have that $t_1 \in T \setminus U$. By a similar argument, there exist $x_2 \in X$ and $t_2 \in T \setminus U$ such that $t_1 = x_2 t_2$. Continuing in this way, for each $n \in \mathbb{N}$ there exist $x_1, \dots, x_n \in X$ and $t_n \in T \setminus U$ such that $a = (x_1 \cdots x_n) t_n$. However, we have that $x_1 \cdots x_N = 0$ and hence $a = 0$, which is a contradiction. \square

Lemma 8.8. *Let S be an archimedean semigroup with idempotent. Then S is weakly noetherian if and only if S is either a group or an ideal extension of a group by a finite nilpotent semigroup.*

Proof. Every group is weakly noetherian, so assume that S is not a group. By Lemma 8.2, there exists a group G that is an ideal of S such that $T = S/G$ is a nilpotent semigroup. If S is weakly noetherian, it follows from Lemmas 4.1 and 8.7 that T is finite. Conversely, if T is finite, then S is weakly noetherian by Corollary 5.5. \square

Lemma 8.9. *Let S be an archimedean semigroup without idempotent. Then S is weakly noetherian if and only if it is finitely generated.*

Proof. The reverse implication follows from Theorem 3.8, so we just need to prove the direct implication.

Let a be a fixed element of S . The Tamura order on S (with respect to a) is defined by

$$x \leq_a y \iff x = a^n y \text{ for some } n \geq 0.$$

By [11, Section IV.4], there exists a set M of maximal elements of S (under \leq_a), where $a \in M$, such that every element of S can be written in the form $p_n = a^n p$ with $n \geq 0$ and $p \in M$, and the set $I = S \setminus M$ is an ideal. The Rees quotient S/I is nilpotent, since S is an archimedean semigroup, and hence it is finite by Lemmas 4.1 and 8.7. Therefore, the set M is finite and $S = \langle M \rangle$ is finitely generated. □

Lemma 8.10. *Let S be a weakly noetherian commutative semigroup with no non-trivial subgroups, and let $S = \mathcal{S}(Y, S_\alpha)$ be a decomposition of S into a semilattice of archimedean semigroups. Let $\beta \in Y$ and let T be the subsemigroup $\bigcup_{\alpha \geq \beta} S_\alpha$ of S . Then there exists a finite set $X \subseteq S_\beta$ such that $S_\beta = \langle X \rangle (T \setminus S_\beta)^1$.*

Proof. Since the complement of T is an ideal of S , we have that T is weakly noetherian by Corollary 5.10. Note that S_β is an ideal of T .

Suppose first that S_β has an idempotent. Then S_β is a nilpotent semigroup with zero 0. Since T is weakly noetherian, we have that $S_\beta = XT^1$ for some finite set $X \subseteq S_\beta \setminus \{0\}$. Then $S_\beta = (XS_\beta^1)(T \setminus S_\beta)^1$. By the same argument as the one in Lemma 8.7, we have that XS_β^1 is the finite semigroup $\langle X \rangle$.

Now suppose that S_β has no idempotent. Let M be the set of maximal elements of S_β under the Tamura order with respect to an element $a \in S_\beta$, and let $I = S_\beta \setminus M$. Then I is an ideal of T . We have that T/I is weakly noetherian by Lemma 4.1, and S_β/I is a nilpotent semigroup with zero 0. By the same argument as above, there exists a finite set $X \subseteq M = S_\beta \setminus \{0\}$ such that $S_\beta/I = \langle X \rangle (T \setminus S_\beta)^1$. We may assume without loss of generality that $a \in X$. Since every element of S_β can be written as $a^n p$ for some $n \geq 0$ and $p \in M$, it follows that $S_\beta = \langle X \rangle (T \setminus S_\beta)^1$. □

Remark 8.11. Given Lemma 8.10, one might be tempted to think that in a weakly noetherian commutative semigroup with no non-trivial subgroups, every archimedean semigroup is contained in a finitely generated subsemigroup. However, this is not the case. Indeed, letting $Y = (\mathbb{N}, \geq)$ and recalling Construction 6.8, we have that $U = \mathcal{U}(Y, \text{id})$ is weakly noetherian by Proposition 6.9. Clearly, U is commutative. It can be easily shown that U is locally finite, so its infinite archimedean component N_Y is not contained in a finitely generated subsemigroup.

We are now in a position to prove the main result of this section.

Proof of Theorem 8.5. (\Rightarrow) Let $S = \mathcal{S}(Y, S_\alpha)$, where Y is finite, be the decomposition of S into a semilattice of archimedean semigroups. Let T denote the quotient S/\mathcal{H} . Then T is a semilattice of archimedean semigroups $\mathcal{S}(Y, T_\alpha)$ where each T_α is \mathcal{H} -trivial. We have that T is weakly noetherian by Lemma 4.1.

We prove that T is finitely generated by induction on $|Y|$. Suppose that $|Y| = 1$, so that T is an archimedean semigroup. If T has an idempotent, then it follows from Lemma 8.8 that T is finite. If T has no idempotent, then $T \cong S$ is finitely generated by Lemma 8.9.

Now suppose that $|Y| > 1$. Let 0 be the minimal element of Y . By Lemma 8.10, there exists a finite set $X_0 \subseteq T_0$ such that $T_0 = \langle X_0 \rangle (T \setminus T_0)^1$. Let $\alpha_1, \dots, \alpha_k$ be the elements of Y that only 0 is strictly less than. For each $i \in \{1, \dots, k\}$, define

$$T_i = \bigcup_{\alpha \geq \alpha_i} T_\alpha.$$

We have that T_i is a subsemigroup of T whose complement is an ideal, so it is weakly noetherian by Corollary 5.10. By the inductive hypothesis, T_i is generated by some finite set X_i . It follows that $T = \bigcup_{i=0}^k T_i$ is generated by the finite set $\bigcup_{i=0}^k X_i$, as required.

(\Leftarrow) If S/\mathcal{H} is finitely generated, then it is weakly noetherian by Theorem 3.8, and hence S is weakly noetherian by Lemma 4.3. □

From the proof of Theorem 8.5, we deduce a couple of corollaries. The first concerns *complete* semigroups; that is, commutative semigroups in which every archimedean component contains an idempotent.

Corollary 8.12. *Let S be a complete semigroup with finitely many idempotents. Then S is weakly noetherian if and only if S/\mathcal{H} is finite.*

Proof. We have that $S/\mathcal{H} = \mathcal{S}(Y, T_\alpha)$, where Y is finite and each T_α is a nilpotent semigroup. The direct implication is proved by a similar induction argument to the one in the proof of Theorem 8.5. Notice that $\langle X_0 \rangle$ is finite, and also $T \setminus T_0$ is finite since it is a finite union of semigroups that are finite by the inductive hypothesis. It follows that $T_0 = \langle X_0 \rangle (T \setminus T_0)^1$ is finite, and hence $T = T_0 \cup (T \setminus T_0)$ is finite. □

On the other extreme we have:

Corollary 8.13. *Let S be an idempotent-free commutative semigroup with finitely many archimedean components. Then S is weakly noetherian if and only if it is finitely generated.*

Proof. This proof is essentially the same as that of Theorem 8.5, except we prove by induction that S is finitely generated, rather than S/\mathcal{H} . □

We conclude this section by exhibiting an example of an idempotent-free commutative semigroup that is weakly noetherian but not finitely generated.

Example 8.14. For each $i \in \mathbb{N}$, let $S_i = \langle a_i \rangle$ be a copy of the free monogenic semigroup \mathbb{N} , and let S be the disjoint union of the semigroups S_i . Define a multiplication on S , extending those on each S_i , as follows: for each $i, j, m, n \in \mathbb{N}$ with $i < j$, let

$$a_i^m a_j^n = a_j^n = a_j^n a_i^m.$$

It is easy to see that, with this multiplication, S is an idempotent-free commutative semigroup. Moreover, we have that $S = \mathcal{S}(Y, S_i)$ where $Y = (\mathbb{N}, \geq)$. Since Y is infinite,

S is not finitely generated. It can be easily shown that every ideal of S has the form

$$a_i^m S^1 = \{a_i^n : n \geq m\} \cup \left(\bigcup_{j>i} S_j \right)$$

for some $i, m \in \mathbb{N}$. In particular, every ideal of S is principal, so S is weakly noetherian.

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