ON THE DECOMPOSITION OF GROUPS

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1. Introduction. The problem in which we are interested is the following. Call an additively written group G finitely decomposable if $G = \sum G_i$ is the weak sum of finite groups G_i. Consider the following property.

Property P. Each subgroup of G having cardinality less than G is contained in a finitely decomposable direct summand of G.

Does Property P imply that G is finitely decomposable? We shall demonstrate that the answer is negative even in the commutative case. Our question is closely related to (**1**, Problem 5). In (**4**), an abelian group is called a Fuchs 5-group if every infinite subgroup of the group can be embedded in a direct summand of the same cardinality. The question of whether or not a Fuchs 5-group is in fact a direct sum of countable groups has been open for several years. Note that if a group has Property P, then it must be a Fuchs 5-group. Thus, in the present paper, we show that a Fuchs 5-group need not be a direct sum of countable groups. It is also shown that not every Q-group is a direct sum of cyclic groups; this settles (**4**, Question 5).

2. Description of G. Let Ω denote the first uncountable ordinal and for each ordinal α less than Ω let $P_\alpha = \{a_\alpha^0, a_\alpha^1, \ldots, a_\alpha^n, \ldots\}$ be a copy of the Prüfer group associated with a fixed prime p. Thus, the generators a_α^n of P_α are subjected only to the relations $pa_\alpha^0 = 0$ and $p^n a_\alpha^n = a_\alpha^0$ if n is a positive integer; recall that we are using the additive notation, therefore the superscript is only an index, not an exponent. Let $P = \sum_{\alpha<\Omega} P_\alpha$ be the (weak) sum of $\aleph_1$ copies of the Prüfer group. Working inside of P, we construct an ascending chain of subgroups. Define $G_0 = 0$ and $G_1 = \{a_0^0\} \subseteq P_0 \subseteq P$. If $\beta > 1$ is an ordinal not exceeding Ω, define G_β inductively in the following way. Write $\beta = \omega\lambda + n$, where $n < \omega$.

Case 1. $n > 1$. Define $G_\beta = G_{\beta-1} + \{a_{\beta-1}^{n-1}\}$.

Case 2. $n = 0$. Define $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$.

Case 3. $n = 1$. Since $\beta > 1$, $\lambda \geqq 1$ and $\beta - 1$ is a countable limit ordinal. Let $\sigma(1) < \sigma(2) < \ldots < \sigma(k) < \ldots$ be an increasing sequence of ordinals leading up to $\beta - 1$ such that $\sigma(k) = \omega\lambda_k + k$ for each positive integer k (there is no restriction on λ_k). For each positive integer j, define $\xi_{\beta-1}^j = \sum_{i<j} a_{\sigma(i)}^j - a_{\beta-1}^j$. We define $G_\beta = \{G_{\beta-1}, \xi_{\beta-1}^j\}$, $1 \leqq j < \omega$.

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Observe that $G_\beta \subseteq \sum_{\alpha<\beta} P_\alpha$ for each β; therefore, in Case 1 the sum is direct inside of $\sum_{\alpha<\beta} P_\alpha$. The group that we are interested in is $G_\Omega = \bigcup_{\alpha<\Omega} G_\alpha$; let $G = G_\Omega$. One may observe that $G \subseteq P = \sum_{\alpha<\Omega} P_\alpha$ and that $a_\alpha^0 \in G$ for each $\alpha < \Omega$.

We remark that the construction of G is similar to a construction in (**2**).

3. If $\alpha - 1$ exists, then G_α is a direct summand of G. Suppose that $\alpha < \Omega$ and that α is not a limit. Set $K_{\alpha,\alpha} = 0$. Now suppose that $\alpha < \beta \leqq \Omega$ and that for each γ such that $\alpha \leqq \gamma < \beta$ we have a decomposition

$$G_\gamma = G_\alpha + K_{\alpha,\gamma},$$

where the $K_{\alpha,\gamma}$'s satisfy the following conditions:

(a) $K_{\alpha,\delta} \subseteq K_{\alpha,\gamma}$ if $\alpha \leqq \delta \leqq \gamma$;

(b) $a_{\mu+1}^0 \in K_{\alpha,\gamma}$ if $\alpha \leqq \mu + 1 < \gamma$.

Write $\beta = \omega\lambda + n$, where $n < \omega$. If $n > 1$, we set $K_{\alpha,\beta} = K_{\alpha,\beta-1} + \{a_{\beta-1}^{n-1}\}$. Then $G_\beta = G_\alpha + K_{\alpha,\beta}$ and (a) and (b) hold for $\gamma \leqq \beta$. If $n = 0$, let $K_{\alpha,\beta} = \bigcup_{\gamma<\beta} K_{\alpha,\gamma}$. Again it is immediate that $G_\beta = G_\alpha + K_{\alpha,\beta}$ and that conditions (a) and (b) are valid for $\gamma \leqq \beta$.

Finally, we consider the case $n = 1$. Recall that in this case,

$$G_\beta = \{G_{\beta-1}, \xi_{\beta-1}{}^j\}, \qquad 1 \leqq j < \omega,$$

where $\xi_{\beta-1}{}^j = \sum_{i<j} a_{\sigma(i)}{}^j - a_{\beta-1}{}^j$ and $\{\sigma(i)\}$ is an increasing sequence of ordinals that leads up to $\beta - 1$ with $\sigma(i)$ congruent to i modulo ω for each i. Let k be the smallest positive integer such that $\sigma(k) \geqq \alpha$. Define

$$K_{\alpha,\beta} = \{K_{\alpha,\beta-1}, \xi_{\beta-1}{}^j, (a_{\sigma(i)}{}^i + \xi_{\beta-1}{}^i - p\xi_{\beta-1}{}^{i+1})\}, \quad 1 \leqq i < k,\ j \geqq k.$$

Observe that (a) and (b) remain valid for $\gamma \leqq \beta$; condition (b) is vacuously satisfied for $\gamma = \beta$ since $\beta - 1$ is a limit ordinal. We intend to show that $G_\beta = G_\alpha + K_{\alpha,\beta}$. Since $G_{\sigma(i)+1} = G_{\sigma(i)} + \{a_{\sigma(i)}{}^i\}$, we have: $a_{\sigma(i)}{}^i \in G_\alpha$ if $i < k$. Hence, $G_\beta = \{G_\alpha, K_{\alpha,\beta}\}$.

Suppose that $x \in G_\alpha \cap K_{\alpha,\beta}$. Write

$$x = y + \sum_{j\geqq k} t_j\xi_{\beta-1}{}^j + \sum_{i<k} s_i(a_{\sigma(i)}{}^i + \xi_{\beta-1}{}^i - p\xi_{\beta-1}{}^{i+1}),$$

where $y \in K_{\alpha,\beta-1}$. First, we shall show that

$$\sum_{i<k} s_i(a_{\sigma(i)}{}^i + \xi_{\beta-1}{}^i - p\xi_{\beta-1}{}^{i+1}) = 0.$$

Since $x \in G_\alpha$, note that

$$\sum_{j\geqq k} t_j a_{\beta-1}{}^j + \sum_{i<k} s_i(a_{\beta-1}{}^i - pa_{\beta-1}{}^{i+1}) = 0.$$

Because of the defining relations for the Prüfer group $P_{\beta-1}$, the equation $\sum_j r_j a_{\beta-1}{}^j = 0$ implies that r_j is divisible by p^j for each j. Therefore, it follows, by induction on i, that p^i divides s_i for each $i < k$. Let $s_i = p^i s_i'$. Now we have:

$$\sum_{j<k} s_j(a_{\sigma(j)}{}^j + \xi_{\beta-1}{}^j - p\xi_{\beta-1}{}^{j+1})$$
$$= \sum_{j<k} s_j'\left(p^j a_{\sigma(j)}{}^j + \sum_{i<j} p^j a_{\sigma(i)}{}^j - p^j a_{\beta-1}{}^j - \sum_{i\leqq j} p^{j+1} a_{\sigma(i)}{}^{j+1} + p^{j+1} a_{\beta-1}{}^{j+1}\right)$$
$$= \sum_{j<k} s_j'\left(\sum_{i\leqq j} a_{\sigma(i)}{}^0 - \sum_{i\leqq j} a_{\sigma(i)}{}^0 + a_{\beta-1}{}^0 - a_{\beta-1}{}^0\right) = 0.$$

Thus, $x = y + \sum_{j\geqq k} t_j \xi_{\beta-1}{}^j$. Since x and y are contained in $H_{\beta-1}$, it follows that $\sum_{j\geqq k} t_j a_{\beta-1}{}^j = 0$. Therefore, p^j divides t_j for each $j \geqq k$; write $t_j = p^j t_j'$. Also note that

$$x = y + \sum_{j\geqq k} t_j(\xi_{\beta-1}{}^j - a_{\beta-1}{}^j)$$
$$= y + \sum_{j\geqq k} t_j\left(\sum_{i<j} a_{\sigma(i)}{}^j\right)$$
$$= y + \sum_i \sum_{j\geqq \max(k,\, i+1)} t_j a_{\sigma(i)}{}^j.$$

Since $\sum_{j\geqq k} t_j a_{\beta-1}{}^j = 0$, the equation $\sum_{j\geqq k} t_j a_{\sigma(i)}{}^j = 0$ must also hold. Thus,

$$x = y + \sum_{i\geqq k}\left(\sum_{j\geqq i+1} t_j a_{\sigma(i)}{}^j\right) = y + \sum_{i\geqq k}\left(\sum_{j\geqq i+1} t_j' a_{\sigma(i)}{}^0\right).$$

By condition (b), $a_{\sigma(i)}{}^0 \in K_{\alpha,\beta-1}$ if $\alpha \leqq \sigma(i) < \beta - 1$. Thus, $a_{\sigma(i)}{}^0 \in K_{\alpha,\beta-1}$ if $i \geqq k$, and $x \in K_{\alpha,\beta-1}$ since $y \in K_{\alpha,\beta-1}$. We conclude that $x = 0$ since $G_\alpha \cap K_{\alpha,\beta-1} = 0$, and $G_\beta = G_\alpha + K_{\alpha,\beta}$. Taking $\beta = \Omega$, we have $G = G_\alpha + K_{\alpha,\Omega}$.

4. G has no elements of infinite height. An element $g \in G$ is said to have infinite height if there exists, for each positive integer m, an element $x_m \in G$ such that $p^m x_m = g$. We shall show that zero is the only element of G that has infinite height. Suppose that x has infinite height in G. Since $G_{\alpha+1}$ is a direct summand of G for each countable α, it follows that x has infinite height in G_β for some countable β. In fact, if β is chosen to be the smallest ordinal such that x has infinite height in G_β and if $\beta = \omega\lambda + n$, then n must necessarily be equal to one; obviously, in the other cases ($n = 0$ and $n > 1$) no elements of infinite height can be introduced in the construction of G_β.

Let $x = y + \sum t_j \xi_{\beta-1}{}^j$, where $y \in G_{\beta-1}$ and, as before,

$$\xi_{\beta-1}{}^j = \sum_{i<j} a_{\sigma(i)}{}^j - a_{\beta-1}{}^j.$$

Suppose that $p^m x_m = x$, where $x_m \in G_\beta$ for each positive integer m. Write $x_m = y_m + \sum t_{j,m} \xi_{\beta-1}{}^j$, where $y_m \in G_{\beta-1}$. Then

$$\sum p^m t_{j,m} a_{\beta-1}{}^j = \sum t_j a_{\beta-1}{}^j.$$

Since $t_j = 0$ for all but a finite number of j and since $p^m a_{\beta-1}{}^j = 0$ if $m > j$, we can choose m such that $t_j = 0$ if $j \geqq m$ and obtain

$$\sum_{j \geqq m} p^m t_{j,m} a_{\beta-1}{}^j - \sum_{j<m} t_j a_{\beta-1}{}^j = 0.$$

This implies that p^j divides t_j for each j and that p^{j-m} divides $t_{j,m}$ if $j \geqq m$. Write $t_j = p^j t_j'$ and $p^m t_{j,m} = t_{j,m}' p^j$. Then

$$\sum_{j \geqq m} t_{j,m}' a_{\beta-1}{}^0 = \sum_{j<m} t_j' a_{\beta-1}{}^0 \quad \text{and} \quad \sum_{j \geqq m} t_{j,m}' \equiv \sum_{j<m} t_j' = c \mod (p)$$

for all sufficiently large m. Since y is a fixed element of $G_{\beta-1}$, we can choose $m > 1$ such that $y \in G_{\sigma(m-1)}$. From

$$p^m y_m + p^m \sum t_{j,m} \xi_{\beta-1}{}^j = y + \sum_{j<m} t_j \xi_{\beta-1}{}^j,$$

we obtain

$$p^m y_m + \sum_{j \geqq m} t_{j,m}' \Big(\sum_{i<j} a_{\sigma(i)}{}^0 \Big) = y + \sum_{j<m} t_j' \Big(\sum_{i<j} a_{\sigma(i)}{}^0 \Big).$$

Thus,

$$p^m y_m + \sum_{i<m} \Big(\sum_{j \geqq m} t_{j,m}' \Big) a_{\sigma(i)}{}^0 + \sum_{i \geqq m} \Big(\sum_{j>i} t_{j,m}' \Big) a_{\sigma(i)}{}^0 = y + \sum_{j<m} t_j' \Big(\sum_{i<j} a_{\sigma(i)}{}^0 \Big) \in G_{\sigma(m-1)}.$$

Consider the decompositions

$$G_{\sigma(m-1)+1} = G_{\sigma(m-1)} + \{a_{\sigma(m-1)}{}^{m-1}\}$$

and

$$G_{\beta-1} = G_{\sigma(m-1)} + \{a_{\sigma(m-1)}{}^{m-1}\} + K_{\sigma(m-1)+1,\beta-1},$$

where $K_{\sigma(m-1)+1,\beta-1}$ is as defined in § 3. Recall condition (b), which implies that $a_{\sigma(i)}{}^0 \in K_{\sigma(m-1)+1,\beta-1}$ if $i \geqq m$. Therefore,

$$\sum_{i \geqq m} \Big(\sum_{j>i} t_{j,m}' \Big) a_{\sigma(i)}{}^0 \in K_{\sigma(m-1)+1,\beta-1}.$$

If $y_m = u + v + w$, where $u \in G_{\sigma(m-1)}$, $v \in \{a_{\sigma(m-1)}{}^{m-1}\}$, and $w \in K_{\sigma(m-1)+1,\beta-1}$, then it follows that

$$c a_{\sigma(m-1)}{}^0 = \Big(\sum_{j \geqq m} t_{j,m}' \Big) a_{\sigma(m-1)}{}^0 = p^m v = 0.$$

Hence $c = 0 \mod (p)$, and

$$p^m y_m + \sum_{i \geqq m} \Big(\sum_{j>i} t_{j,m}' \Big) a_{\sigma(i)}{}^0 = y + \sum_{j<m} t_j' \Big(\sum_{i<j} a_{\sigma(i)}{}^0 \Big).$$

Now we see that

$$p^m u = y + \sum_{j<m} t_j' \Big(\sum_{i<j} a_{\sigma(i)}{}^0 \Big) \in G_{\sigma(m-1)}.$$

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This implies that

$$y + \sum_{j<m} t_j'\left(\sum_{i<j} a_{\sigma(i)}{}^0\right) = 0$$

since it is an element of infinite height in $G_{\beta-1}$. Thus,

$$x = y + \sum_{j<m} t_j \xi_{\beta-1}{}^j = y + \sum_{j<m} t_j\left(\sum_{i<j} a_{\sigma(i)}{}^j\right) - \sum_{j<m} t_j a_{\beta-1}{}^j$$
$$= y + \sum_{j<m} t_j'\left(\sum_{i<j} a_{\sigma(i)}{}^0\right) - \sum_{j<m} t_j' a_{\beta-1}{}^0 = 0 - c a_{\beta-1}{}^0 = 0.$$

5. If β is a countable limit ordinal, then $G_{\beta+1}/G_\beta$ has elements of infinite height. If β is a countable limit ordinal, then $G_{\beta+1} = \{G_\beta, \xi_\beta{}^j\}$, $j \geqq 1$, where

$$\xi_\beta{}^j = \sum_{i<j} a_{\sigma(i)}{}^j - a_\beta{}^j$$

and $\sigma(1) < \sigma(2) < \ldots < \sigma(k) < \ldots$ is an increasing sequence of ordinals leading up to β with the property that $\sigma(k) = \omega\lambda_k + k$ for each k. Since $G_{\sigma(i)+1} = G_{\sigma(i)} + \{a_{\sigma(i)}{}^i\}$, $a_{\sigma(i)}{}^0 \in G_{\sigma(i)+1} \subseteq G_\beta$. Thus, we have:

$$p^j \xi_\beta{}^j + G_\beta = p^j\left(\sum_{i<j} a_{\sigma(i)}{}^j - a_\beta{}^j\right) + G_\beta$$
$$= \left(\sum_{i<j} a_{\sigma(i)}{}^0 - a_\beta{}^0\right) + G_\beta$$
$$= -a_\beta{}^0 + G_\beta.$$

We have shown that $-a_\beta{}^0 \neq 0$ has infinite height in $G_{\beta+1}/G_\beta$.

6. The existence theorem. The following theorem summarizes the results that we have established thus far.

THEOREM 1. *There exists a commutative group G without elements of infinite height such that G is the union (direct limit) of an ascending chain*

$$0 = G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_\alpha \subset \ldots, \qquad \alpha < \Omega,$$

of subgroups G_α of G satisfying the following conditions:

(1) *G_α is a countable direct sum of finite cyclic groups for each $\alpha < \Omega$;*
(2) *$G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ if β is a limit ordinal;*
(3) *$G_{\beta+1}/G_\beta$ has elements of infinite height if β is a limit ordinal;*
(4) *G_α is a direct summand of G if α is not a limit ordinal.*

We emphasize that the group G has no elements of infinite height and remark that it is easy to give such examples satisfying conditions (1)–(3), but to obtain (4), as well, is another matter.

7. G satisfies Property P but is not finitely decomposable. In this section, we show that the group G which we have constructed has Property P but that it is not a direct sum of finite groups. We can prove a little more.

THEOREM 2. *If G is a group which satisfies the hypotheses of Theorem 1, then G satisfies Property P but G cannot be decomposed into a direct sum of countable groups.*

Proof. Let H be a subgroup of G having cardinality less than G. Since $|G| \leqq \sum_{\alpha<\Omega}|G_\alpha| \leqq \aleph_0\aleph_1 = \aleph_1$, we observe that H is countable. Hence, there exists a countable ordinal α such that $H \subseteq G_\alpha \subseteq G_{\alpha+1}$; however, $G_{\alpha+1}$ is a finitely decomposable direct summand of G. We conclude that G satisfies Property P.

Using the back-and-forth method (see, for example, (**3**)), we shall show that G is not countably decomposable. Suppose that $G = \sum_{i\in I} A_i$, where A_i is (at most) countable for each $i \in I$; the index set I is arbitrary. Let $\alpha(0) = 1$ and let $I(1)$ be the unique minimal subset of I such that $G_{\alpha(0)} \subseteq \sum_{I(1)} A_i$. Since $G_{\alpha(0)}$ is countable, so is $I(1)$. Thus, $H_1 = \sum_{I(1)} A_i$ is countable and is, therefore, contained in G_α for some countable α. Let $\alpha(1)$ be the first ordinal greater than $\alpha(0)$ such that $H_1 \subseteq G_{\alpha(1)}$. Since $G_{\alpha(1)}$ is countable, there exists a unique minimal countable subset $I(2)$ of I such that $G_{\alpha(1)} \subseteq \sum_{I(2)} A_i$. Set $H_2 = \sum_{I(2)} A_i$, and let $\alpha(2)$ be the smallest ordinal greater than $\alpha(1)$ such that $H_2 \subseteq G_{\alpha(2)}$. Inductively, define $\alpha(n)$ and $I(n)$, for each positive integer n, in the way that we have indicated. Since $\{\alpha(n)\}$ is a strictly increasing sequence of countable ordinals, $\alpha(\omega) = \sup\{\alpha(n)\}_{n<\omega}$ is a countable limit ordinal. Let

$$I(\omega) = \bigcup_{n<\omega} I(n).$$

Since $G_{\alpha(\omega)} = \bigcup_{n<\omega} G_{\alpha(n)}$ and since

$$\sum_{I(n)} A_i \subseteq G_{\alpha(n)} \subseteq \sum_{I(n+1)} A_i,$$

we conclude that $G_{\alpha(\omega)} = \sum_{I(\omega)} A_i$. Thus, $G/G_{\alpha(\omega)} \cong \sum_{I-I(\omega)} A_i$, a direct summand of G. It follows that $G_{\alpha(\omega)+1}/G_{\alpha(\omega)}$ is without elements of infinite height since a subgroup of a group without elements of infinite height is again a group without elements of infinite height. This contradicts condition (3) of Theorem 1, and Theorem 2 is proved.

8. Related results. From Theorem 2 and the Pontryagin duality, we have the following result.

THEOREM 3. *There exists an inverse system*

$$H_1 \xleftarrow{\phi_1{}^2} H_2 \leftarrow \ldots \leftarrow H_\alpha \xleftarrow{\phi_\alpha{}^\beta} H_\beta \leftarrow \ldots, \qquad \beta < \Omega,$$

with epic bonding maps $\phi_\alpha{}^\beta$ such that the following conditions are satisfied:

(i) *H_α is a direct product (= strong direct sum) of finite cyclic groups for each $\alpha < \Omega$;*

(ii) *if* β *is a limit ordinal,* $\bigcap_{\alpha<\beta} \operatorname{Ker} \phi_\alpha{}^\beta = 0$*;*

(iii) *if* H *is the limit of the inverse system* $(H_\alpha;\ \phi_\alpha{}^\beta)$ *and if* α *is not a limit ordinal, then* H_α *is a direct factor of* H*. In fact, if* $\pi_\alpha\colon H \to H_\alpha$ *is the natural map, then*

$$\operatorname{Ker} \pi_\alpha \to H \to H_\alpha$$

splits if α *is not a limit. Hence,*

$$\operatorname{Ker} \phi_\alpha{}^\beta \to H_\beta \to H_\alpha$$

splits if α *is not a limit;*

(iv) *although* H *is compact,* H *is not, as a topological group, a direct product of finite cyclic groups.*

THEOREM 4. *If* $\aleph$ *is any uncountable cardinal, there exists a Fuchs* 5*-group of cardinality* $\aleph$ *that is not a direct sum of countable groups.*

Proof. Let G be a group satisfying the conditions of Theorem 2. In particular, we can take G to be the group described in § 2. Since G satisfies Property P, G is certainly a Fuchs 5-group. It is elementary to show that a direct sum of Fuchs 5-groups is a Fuchs 5-group. Thus, $\sum_{\aleph} G$ is a Fuchs 5-group having cardinality $\aleph$ since G has cardinality $\aleph_1$. Since G is not a direct sum of countable groups, we conclude that $\sum_{\aleph} G$ is not a direct sum of countable groups by a theorem of Kaplansky (**5**).

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