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Abstract

We study the possible weights of an irreducible two-dimensional mod p representation of $\mathrm{Gal}(\overline{F}/F)$ which is modular in the sense that it comes from an automorphic form on a definite quaternion algebra with centre F which is ramified at all places dividing p, where F is a totally real field. In most cases we determine the precise list of possible weights; in the remaining cases we determine the possible weights up to a short and explicit list of exceptions.

1. Introduction

Let F be a totally real field and let p be a prime number. In this paper we formulate, and largely prove, an analogue of the weight part of Serre's conjecture [Ser87] for automorphic forms on quaternion algebras over F which are ramified at all places dividing p.

In recent years, a great deal of attention has been given to the problem of generalising the weight part of Serre's conjecture beyond the case of $\mathrm{GL}(2, \mathbb{Q})$, beginning with the seminal paper [BDJ10] which considered the situation for Hilbert modular forms. Let G_F denote the absolute Galois group of F; then to any irreducible modular representation

$$\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

there is associated a set of weights $W(\overline{\rho})$, the set of weights in which $\overline{\rho}$ is modular (see § 2 for the definitions of weights and of what it means for $\overline{\rho}$ to be modular of a certain weight). Under the assumption that p is unramified in F, the paper [BDJ10] associated to $\overline{\rho}$ a set of weights $W^?(\overline{\rho})$, and conjectured that $W^?(\overline{\rho}) = W(\overline{\rho})$. Schein [Sch08] subsequently proposed a generalisation that, in the tame case (where the restrictions of $\overline{\rho}$ to inertia subgroups at places dividing p are semisimple), removes the restriction that p be unramified in F. When p is either unramified or totally ramified in F many cases of these conjectures have been proved, in [Gee06, GS] respectively, but the general case has so far been out of reach.

As far as we know, there is no corresponding conjecture in the literature for automorphic forms on quaternion algebras that are ramified at p (although the results in the case $F = \mathbb{Q}$ are easily deduced from the discussion in [Kha01, § 4]). We specify a conjectural set of weights $W^?(\overline{\rho})$, depending only on the restrictions of $\overline{\rho}$ to decomposition subgroups at places dividing p. In the case that these restrictions are semisimple, the conjecture is completely explicit (and depends only on the restrictions to inertia subgroups). In the general case the set $W^?(\overline{\rho})$ is defined in terms of the existence of certain potentially Barsotti–Tate lifts of specific type, and so depends on some rather delicate questions involving extensions of crystalline characters.

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In fact, we always make a definition in terms of the existence of certain potentially Barsotti–Tate lifts, and in the semisimple case we make this description explicit by means of calculations with Breuil modules and strongly divisible modules.

We assume that p is odd, and that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. We make a mild additional assumption if $p=5$. All of these restrictions are imposed by our use of the modularity lifting theorems of [Kis09b] (or rather, by their use in [Gee]). Under these assumptions, we are able to prove that if for each place $v|p$, $\overline{\rho}|_{G_{F_v}}$ is not of the form $\binom{\psi_1\ *}{0\ \psi_2}$ with ψ_1/ψ_2 equal to the mod p cyclotomic character, then $W^?(\overline{\rho})=W(\overline{\rho})$. In the exceptional cases we establish that $W(\overline{\rho})\subset W^?(\overline{\rho})$, with equality up to a short list of possible exceptions (for example, in the case that there is only one place of F above p there is only one exception). Our techniques are analogous to those of [Gee06, GS]. As in those papers, the strategy is to construct modular lifts of $\overline{\rho}$ which are potentially semistable of specific type, using the techniques of Khare–Wintenberger, as explained in [Gee].

The significant advantage in the present situation over those considered in [Gee06, GS] is that the property of being modular of a specific weight corresponds exactly to the property of having a lift of some specific type (in the case considered in [Gee06] this correspondence was considerably weaker). Accordingly, we have no regularity assumption on the weights, we do not have to use Buzzard's 'weight cycling' technique, and especially we do not need to make any restriction on the splitting behaviour of p in F.

In the case that the restrictions of $\overline{\rho}$ to decomposition groups at places dividing p are all semisimple, we establish an explicit description of $W^?(\overline{\rho})$ by a computation in two stages. In one step we make use of Breuil modules with descent data, in the same style as analogous computations in the literature; our calculations are more complicated than those made in the past, however, as we have no restrictions on the ramification or inertial degrees of our local fields.

For the second step, we have to exhibit potentially Barsotti–Tate lifts of the appropriate types. Writing down such lifts is rather non-trivial. We accomplish this by means of an explicit construction of corresponding strongly divisible modules; again, these calculations are more complicated than those in the literature, because we make no restrictions on the ramification or inertial degrees of our local fields.

We also note that while we work throughout with definite quaternion algebras, it should not be difficult to extend our results to indefinite algebras; one needs only to establish the analogue of Lemma 2.1 (see for example [DT94, proof of Lemma 6] for the case $F=\mathbb{Q}$).

We now detail the outline of the paper. In § 2 we give our initial definitions and notation. In particular, we introduce spaces of algebraic modular forms on definite quaternion algebras, and we explain what it means for $\overline{\rho}$ to be modular of a specific weight.

In § 3 we explain which tame lifts we will need to consider, and the relationship between the existence of modular lifts of specified types and the property of being modular of a certain weight. This amounts to recalling certain concrete instances of the local Langlands and Jacquet–Langlands correspondences for GL_2 and local-global compatibility. All of this material is completely standard.

Section 4 begins the work of establishing an explicit description of $W^?(\overline{\rho})$, by finding necessary conditions for the existence of potentially Barsotti–Tate lifts of particular type, via calculations with Breuil modules. The sufficiency of these conditions is established in § 6, by writing down explicit strongly divisible modules. Both sections make use of a lemma relating the type of the lifts to the descent data on the Breuil modules and strongly divisible modules, which is established in § 5.

These local calculations are summarised in § 7. Finally, in § 8 we prove our main theorem.

2. Notation and assumptions

Let p be an odd prime. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We will consider all finite extensions of $\mathbb{Q}$ (respectively $\mathbb{Q}_p$) to be contained in $\overline{\mathbb{Q}}$ (respectively $\overline{\mathbb{Q}}_p$). If K is such an extension, we let G_K denote its absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. Let F be a totally real field. Let $\overline{\rho}: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Assume from now on that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible. If $p=5$ and the projective image of $\overline{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, assume further that $[F(\zeta_5):F]=4$. We normalise the isomorphisms of local class field theory so that a uniformiser corresponds to a geometric Frobenius element.

We wish to discuss the Serre weights of $\overline{\rho}$ for quaternion algebras ramified at all places dividing p. We choose to work with totally definite quaternion algebras. We recall the basic definitions and results that we need.

Let D be a quaternion algebra with centre F which is ramified at all infinite places of F and at a set Σ of finite places which contains all primes dividing p. Fix a maximal order $\mathcal{O}_D$ of D and for each finite place $v \notin \Sigma$ fix an isomorphism $\mathcal{O}_{D,v} := (\mathcal{O}_D)_v \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$. For any finite place v let π_v denote a uniformiser of F_v.

Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup, with each $U_v \subset \mathcal{O}_{D,v}^\times$. Furthermore, assume that $U_v = \mathcal{O}_{D,v}^\times$ for all $v \in \Sigma$.

Take A a topological $\mathbb{Z}_p$-algebra. For each place $v|p$ fix a continuous representation $\sigma_v : U_v \to \mathrm{Aut}(W_v)$ with W_v a finite A-module. Let σ denote the representation $\bigotimes_{v|p} \sigma_v$ of $U_p := \prod_{v|p} U_v$, acting on $W_\sigma := \bigotimes_{v|p} W_v$. We regard σ as a representation of U in the obvious way (that is, we let U_v act trivially if $v \nmid p$). Fix also a character $\psi : F^\times \backslash (\mathbb{A}_F^f)^\times \to A^\times$ such that for any finite place v of F, $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times}$ is multiplication by ψ^{-1}. Then we can think of W_σ as a $U(\mathbb{A}_F^f)^\times$-module by letting $(\mathbb{A}_F^f)^\times$ act via ψ^{-1}.

Let $S_{\sigma,\psi}(U, A)$ denote the set of continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \to W_\sigma$$

such that for all $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have

$$f(gu) = \sigma(u)^{-1} f(g) \quad \text{for all } u \in U,$$
$$f(gz) = \psi(z) f(g) \quad \text{for all } z \in (\mathbb{A}_F^f)^\times.$$

We can write $(D \otimes_F \mathbb{A}_F^f)^\times = \coprod_{i \in I} D^\times t_i U(\mathbb{A}_F^f)^\times$ for some finite index set I and some $t_i \in (D \otimes_F \mathbb{A}_F^f)^\times$. Then we have

$$S_{\sigma,\psi}(U, A) \xrightarrow{\sim} \bigoplus_{i \in I} W_\sigma^{(U(\mathbb{A}_F^f)^\times \cap t_i^{-1} D^\times t_i)/F^\times},$$

the isomorphism being given by the direct sum of the maps $f \mapsto f(t_i)$. From now on we make the following assumption.

$$\text{For all } t \in (D \otimes_F \mathbb{A}_F^f)^\times \text{ the group } (U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t)/F^\times = 1.$$

One can always replace U by a subgroup (satisfying the above assumptions, and without changing U_p) for which this holds (cf. [Kis09a, § 3.1.1]). Under this assumption $S_{\sigma,\psi}(U, A)$ is a finite A-module, and the functor $W_\sigma \mapsto S_{\sigma,\psi}(U, A)$ is exact in W_σ.

We now define some Hecke algebras. Let S be a set of finite places containing Σ and the primes v of F such that $U_v \neq \mathcal{O}_{D,v}^\times$. Let $\mathbb{T}_{S,A}^{\mathrm{univ}} = A[T_v, S_v]_{v \notin S}$ be the commutative polynomial ring in the formal variables T_v, S_v. Consider the left action of $(D \otimes_F \mathbb{A}_F^f)^\times$ on W_σ-valued functions on $(D \otimes_F \mathbb{A}_F^f)^\times$ given by $(gf)(z) = f(zg)$. Then we make $S_{\sigma,\psi}(U, A)$ a $\mathbb{T}_{S,A}^{\text{univ}}$-module by letting S_v act via the double coset $U\left(\begin{smallmatrix}\pi_v & 0\\ 0 & \pi_v\end{smallmatrix}\right)U$ and T_v via $U\left(\begin{smallmatrix}\pi_v & 0\\ 0 & 1\end{smallmatrix}\right)U$. These are independent of the choices of π_v. We will write $\mathbb{T}_{\sigma,\psi}(U, A)$ or $\mathbb{T}_{\sigma,\psi}(U)$ for the image of $\mathbb{T}_{S,A}^{\text{univ}}$ in End $S_{\sigma,\psi}(U, A)$.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_{S,A}^{\text{univ}}$. We say that $\mathfrak{m}$ is *in the support of* (σ, ψ) if $S_{\sigma,\psi}(U, A)_\mathfrak{m} \neq 0$. Now let $\mathcal{O}$ be the ring of integers in $\overline{\mathbb{Q}}_p$, with residue field $\mathbb{F} = \overline{\mathbb{F}}_p$, and suppose that $A = \mathcal{O}$ in the above discussion, and that σ has open kernel and is free as an $\mathcal{O}$-module. Consider a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{S,\mathcal{O}}^{\text{univ}}$ with residue field $\mathbb{F}$ which is in the support of (σ, ψ). Then there is a semisimple Galois representation $\overline{\rho}_\mathfrak{m} : G_F \to \mathrm{GL}_2(\mathbb{F})$ associated to $\mathfrak{m}$ which is characterised up to conjugacy by the property that if $v \notin S$ then $\overline{\rho}_\mathfrak{m}|_{G_{F_v}}$ is unramified, and if Frob_v is an arithmetic Frobenius at v then the characteristic polynomial of $\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_v)$ is the image of $X^2 - T_v X + S_v \mathbf{N}v$ in $\mathbb{F}[X]$.

We have the following basic lemma.

Lemma 2.1. *Let $\psi : F^\times \backslash (\mathbb{A}_F^f)^\times \to \mathcal{O}^\times$ be a continuous character, and write $\overline{\psi}$ for the composite of ψ with the projection $\mathcal{O}^\times \to \mathbb{F}^\times$. Fix a representation σ' of U_p on a finite free $\mathcal{O}$-module $W_{\sigma'}$, and an irreducible representation σ of U_p on a finite free $\mathbb{F}$-module W_σ. Suppose that we have $\sigma'|_{U_v \cap \mathcal{O}_{F_v}^\times} = \psi^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$ and $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times} = \overline{\psi}^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$ for all finite places v.*

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_{S,\mathcal{O}}^{\text{univ}}$.

Suppose that W_σ occurs as a U_p-module subquotient of $W_{\sigma'} \otimes \mathbb{F}$. If $\mathfrak{m}$ is in the support of $(\sigma, \overline{\psi})$, then $\mathfrak{m}$ is in the support of (σ', ψ).

Conversely, if $\mathfrak{m}$ is in the support of (σ', ψ), then $\mathfrak{m}$ is in the support of $(\sigma, \overline{\psi})$ for some irreducible U_p-module subquotient W_σ of $W_{\sigma'} \otimes \mathbb{F}$.

Proof. The first part is proved just as in [Kis09b, Lemma 3.1.4], and the second part follows, for example from [AS86, Proposition 1.2.3], or from a basic commutative algebra argument. □

We are now in a position to define what it means for a representation to be modular of some weight. Let $v|p$ be a place of F, so that $U_v = \mathcal{O}_{D,v}^\times$. Let σ_v be an irreducible $\mathbb{F}$-representation of U_v. Note that if Π_v is a uniformiser of $\mathcal{O}_{D,v}$, then $k_{2,v} := \mathcal{O}_{D,v}/\Pi_v$ is a finite field, a quadratic extension of the residue field k_v of F_v. The kernel of the reduction map $U_v \to k_{2,v}^\times$ is a pro-p group, so σ_v factors through this kernel, and is a representation of the finite abelian group $k_{2,v}^\times$. It is therefore one-dimensional. Let $\sigma = \bigotimes_{v|p} \sigma_v$, which we will regard as an $\mathcal{O}$-module via the natural map $\mathcal{O} \to \mathbb{F}$.

Definition 2.2. We say that $\overline{\rho}$ is *modular of weight* σ if for some D, S, U, ψ, and $\mathfrak{m} \subset \mathbb{T}_{S,\mathcal{O}}^{\text{univ}}$ as above, we have $S_{\sigma,\psi}(U, \mathcal{O})_\mathfrak{m} \neq 0$ and $\overline{\rho}_\mathfrak{m} \cong \overline{\rho}$.

(Here $\overline{\rho}_\mathfrak{m}$ is characterised as above, and exists by Lemma 2.1 and the remarks above.) Write $W(\overline{\rho})$ for the set of weights σ for which $\overline{\rho}$ is modular of weight σ. Assume from now on that $\overline{\rho}$ is modular of some weight, and fix D, S, U, ψ, $\mathfrak{m}$ as in the definition.

We also have the following useful lemma, which was first observed by Serre in the case $F = \mathbb{Q}$ (see remark (11) in Serre's letter to Tate in [Ser96]). For each place $v|p$ we let q_v denote the order of the residue field k_v of F_v. If σ_v is an irreducible $\mathbb{F}$-representation of $\mathcal{O}_{D,v}^\times$, then by the remarks above it is a character of $k_{2,v}^\times$. Thus $\sigma_v^{q_v}$ is another irreducible $\mathbb{F}$-representation, and $\sigma_v^{q_v^2} = \sigma_v$. Serre observed that the set $W(\overline{\rho})$ is preserved by this operation. This is essentially a consequence of the structure of $\mathcal{O}_{D,v}$. Let $K_{2,v} = W(k_{2,v})[1/p]$, a subfield of D_v. Note that there is a choice of uniformiser Π_v of D_v with the property that conjugation by Π_v preserves $K_{2,v}$, and acts on it via a non-trivial involution. In particular, the induced action on $k_{2,v}$ is via the q_vth power map.

Lemma 2.3. *Let v be a place of F dividing p, and let σ be a weight as above. Let $\sigma' = \sigma_v^{q_v} \otimes_{w|p, w \neq v} \sigma_w$. Then $\overline{\rho}$ is modular of weight σ if and only if it is modular of weight σ'.*

Proof. It suffices to exhibit a bijection

$$\theta : S_{\sigma,\psi}(U, \mathbb{F}) \to S_{\sigma',\psi}(U, \mathbb{F})$$

which commutes with the action of $\mathbb{T}^{\text{univ}}_{S,\mathcal{O}}$. Let $\Pi \in (D \otimes_F \mathbb{A}^f)^\times$ be trivial away from v, and equal to Π_v at v, where Π_v is as in the previous paragraph. Then we define θ by

$$(\theta f)(x) := f(x\Pi).$$

It is straightforward to check that this map has the required properties; the key point is that if $u \in U$, then

$$\begin{aligned}(\theta f)(xu) &= f(xu\Pi)\\ &= f(x\Pi(\Pi^{-1}u\Pi))\\ &= \sigma(\Pi^{-1}u\Pi)^{-1} f(x\Pi)\\ &= \sigma'(u)^{-1} f(x\Pi)\\ &= \sigma'(u)^{-1}(\theta f)(x).\end{aligned}$$

□

3. Weights are controlled by lifts of tame type

Continue to let v be a place of F that divides p. We distinguish two types of irreducible $\mathbb{F}$-representations σ_v of U_v. Recall that any such representation is one-dimensional, and factors through $k_{2,v}^\times$, with $k_{2,v}$ a quadratic extension of k_v.

Definition 3.1. We say that σ_v is of *type I* if it does not factor through the norm $k_{2,v}^\times \to k_v^\times$. Otherwise, we say that it is of *type II*.

We now recall some facts about the local Langlands and local Jacquet–Langlands correspondences. Let K be a finite extension of $\mathbb{Q}_p$, let L be an unramified quadratic extension of K, and let D be a non-split quaternion algebra over K. Consider L as a subfield of D. Let k be the residue field of K, of cardinality q. If π is an irreducible admissible ($\mathbb{C}$- or $\overline{\mathbb{Q}}_p$-valued) representation of $D^\times$, we let $\text{JL}(\pi)$ be the corresponding representation of $\text{GL}_2(K)$. If π is an irreducible admissible representation of $\text{GL}_2(K)$, we let $\text{LL}(\pi)$ denote the corresponding representation of the Weil group W_K of K. Let $N_D : D^\times \to K^\times$ be the reduced norm. As usual we identify characters of $L^\times$ or $K^\times$ with characters of the corresponding Weil groups via local class field theory. If χ is a character of $L^\times$ which does not factor through the norm to $K^\times$, we denote the corresponding supercuspidal representation of $\text{GL}_2(K)$ by W_χ.

- If χ is a character of $K^\times$, then $\text{JL}(\chi \circ N_D) = (\chi \circ \det) \otimes \text{St}$, where St is the Steinberg representation.
- Suppose that χ is a character of $L^\times$ of conductor 1. Then $\text{LL}(\pi)|_{I_K} \cong \chi|_{I_K} \oplus \chi|_{I_K}^q$ if and only if $\pi = W_{\chi'}$ for some unramified twist χ' of χ. (See [BM02, § A.3.2].)
- If χ is a character of $L^\times$ of conductor 1, then $\text{JL}^{-1}(W_\chi)$ is two-dimensional. Furthermore, $\text{JL}^{-1}(W_\chi)|_{\mathcal{O}_L^\times} \cong \chi|_{\mathcal{O}_L^\times} \oplus \chi|^q_{\mathcal{O}_L^\times}$. (See [Pra90, § 7].)

We now recall some definitions relating to potentially semistable lifts of particular type. We use the conventions of [Sav05].

Definition 3.2. Let τ_v be an inertial type. We say that a lift $\rho : G_{F_v} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ of $\overline{\rho}|_{G_{F_v}}$ is *parallel potentially Barsotti–Tate* (respectively *parallel potentially semistable*) of type τ_v if ρ is potentially Barsotti–Tate (respectively potentially semistable with all Hodge–Tate weights equal to 0 or 1), has determinant a finite order character of order prime to p times the cyclotomic character, and the corresponding Weil–Deligne representation, when restricted to I_{F_v}, is isomorphic to τ_v.

Note that for a two-dimensional de Rham representation with all Hodge–Tate weights equal to 0 or 1, the condition that all pairs of labeled Hodge–Tate weights are $\{0, 1\}$ is equivalent to the condition that the determinant is the product of the cyclotomic character, a finite order character, and an unramified character; the condition of being parallel is slightly stronger than this.

If σ_v is an irreducible $\mathbb{F}$-representation of U_v, we will consider the inertial type of I_{F_v} given by $\widetilde{\sigma}_v \oplus \widetilde{\sigma}_v^{q_v}$, where a tilde denotes a Teichmüller lift (considered as a representation of I_{F_v} via local class field theory).

Lemma 3.3. *The representation $\overline{\rho}$ is modular of weight $\sigma = \bigotimes_{v|p} \sigma_v$ if and only if $\overline{\rho}$ lifts to a modular Galois representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ which for all places $v|p$ is parallel potentially Barsotti–Tate of type $\widetilde{\sigma}_v \oplus \widetilde{\sigma}_v^{q_v}$ at v if σ_v is of type I, and is parallel potentially semistable of type $\widetilde{\sigma}_v \oplus \widetilde{\sigma}_v^{q_v}$ at v, but not potentially crystalline, if σ_v is of type II.*

Proof. We first tackle the only if direction. If σ_v is of type I then we choose an arbitrary extension of $\widetilde{\sigma}_v$ to a character of $F_{v,2}^{\times}$, where $F_{v,2}$ is the unramified quadratic extension of F_v, and if σ_v is of type II then we choose an arbitrary extension of $\widetilde{\sigma}_v$ to a character of $F_v^{\times}$. We continue to denote these extensions by $\widetilde{\sigma}_v$. We apply Lemma 2.1, with $\mathfrak{m}$ and $\overline{\psi}$ chosen such that $\overline{\rho}_{\mathfrak{m}} \cong \overline{\rho}$ and $S_{\sigma,\overline{\psi}}(U, \mathbb{F})_{\mathfrak{m}} \neq 0$ (such a maximal ideal exists by the assumption that $\overline{\rho}$ is modular of weight σ), and

$$\sigma' = \bigotimes_{v|p} \sigma'_v$$

where

$$\sigma'_v = (\mathrm{JL}^{-1}(W_{\widetilde{\sigma}_v}))|_{\mathcal{O}_{D,v}^{\times}}$$

if σ_v is of type I, and

$$\sigma'_v = \widetilde{\sigma}_v$$

if σ_v is of type II. We take the ψ of Lemma 2.1 to be the Teichmüller lift of $\overline{\psi}$. The correspondence between the algebraic modular forms considered in § 2 and automorphic representations of $D^{\times}$ is explained in [Kis09b, § 3.1.14] (there is a running assumption in that paper that D is split at all places dividing p, but it is not needed in this discussion, and if one sets the representation $W_{\tau^{\mathrm{alg}}}$ of *loc. cit.* to be the trivial representation the discussion goes through immediately in our case), and we see that after choosing an isomorphism $\overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ there is an automorphic representation π of $D^{\times}$ whose weight is the trivial representation, whose Hecke polynomials at unramified places lift the characteristic polynomials of the corresponding Frobenius elements for $\overline{\rho}$, and such that for each place $v|p$, π_v is either $\mathrm{JL}^{-1}(W_{\chi})$ for χ an unramified twist of $\widetilde{\sigma}_v$ if σ_v is of type I, or an unramified twist of $\widetilde{\sigma}_v$ if σ_v is of type II. (To see this in the case that σ_v has type I, note that if $\pi_v|_{\mathcal{O}_{D,v}^\times}$ contains $(\mathrm{JL}^{-1}(W_{\widetilde{\sigma}_v}))|_{\mathcal{O}_{D,v}^\times}$, then the conductor of π_v is at most the conductor of $(\mathrm{JL}^{-1}(W_{\widetilde{\sigma}_v}))|_{\mathcal{O}_{D,v}^\times}$. By the results recalled in [Pra90, § 7], we see that π_v must be of the form W_χ for χ a character of $F_{2,v}^\times$ of conductor 1. Since (for example by the character formulae in [Pra90, § 7]) $W_{\widetilde{\sigma}_v} \cong W_{\widetilde{\sigma}_v^{q_v}}$, the result follows from the third bullet point above.)

Applying the Jacquet–Langlands correspondence [JL70, Theorem 16.1] we see that there is an automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_F)$ with the same infinitesimal character as the trivial representation, whose Hecke polynomials at unramified places lift the characteristic polynomials of the corresponding Frobenius elements for $\overline{\rho}$, and such that for each place $v|p$, π_v is either W_χ for χ an unramified twist of $\widetilde{\sigma}_v$ if σ_v is of type I, or an unramified twist of $(\widetilde{\sigma}_v \circ \det) \otimes \mathrm{St}$ if σ_v is of type II. The compatibility of the local and global Langlands correspondences at places dividing p (see [Kis08]), and the results on the form of the local Langlands correspondence recalled above, show that the Galois representation corresponding to π' gives a representation of the required form (note that the Galois representation has determinant $\psi\epsilon$, so is indeed parallel).

For the converse, we may reverse the above argument, and we see that Lemma 2.1 guarantees that $\overline{\rho}$ is modular of a weight $\mu = \bigotimes_{v|p} \mu_v$, where for each $v|p$ if σ_v is of type II then $\mu_v = \sigma_v$, and if σ_v is of type I then $\mu_v = \sigma_v$ or $\sigma_v^{q_v}$. The result then follows from Lemma 2.3. □

This motivates the following definition of $W^?(\overline{\rho})$.

Definition 3.4. For each place $v|p$, let $W^?(\overline{\rho}|_{G_{F_v}})$ denote the set of σ_v of type I such that $\overline{\rho}|_{G_{F_v}}$ has a parallel potentially Barsotti–Tate lift of type $\widetilde{\sigma}_v \oplus \widetilde{\sigma}_v^{q_v}$, together with the set of σ_v of type II such that $\overline{\rho}|_{G_{F_v}}$ has a parallel potentially semistable lift of type $\widetilde{\sigma}_v \oplus \widetilde{\sigma}_v^{q_v}$ which is not potentially crystalline. Let $W^?(\overline{\rho})$ be the set of weights $\sigma = \bigotimes_{v|p} \sigma_v$ with $\sigma_v \in W^?(\overline{\rho}|_{G_{F_v}})$ for all $v|p$.

Note that by Lemma 3.3 we have $W(\overline{\rho}) \subset W^?(\overline{\rho})$. We will prove under a mild hypothesis that $W(\overline{\rho}) = W^?(\overline{\rho})$ in § 8. In the intervening sections we will give an explicit description of $W^?(\overline{\rho})$ in the case that $\overline{\rho}|_{G_{F_v}}$ is semisimple for each $v|p$. It is already possible to see that weights of type II are rather rare.

Lemma 3.5. *If $\overline{\rho}$ is modular of weight $\sigma = \bigotimes_{v|p} \sigma_v$, and σ_v is of type II, then*

$$\overline{\rho}|_{I_{F_v}} \cong \sigma_v \begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$$

where σ_v is regarded as a character of I_{F_v} via local class field theory, and ϵ is the cyclotomic character.

Proof. This follows from Lemma 3.3, and the well-known fact that two-dimensional semistable non-crystalline p-adic representations with all pairs of labeled Hodge–Tate weights equal to $\{0, 1\}$ are unramified twists of an extension of the trivial character by the cyclotomic character. □

4. Necessary conditions

4.1 Breuil modules with descent data

Let k be a finite extension of $\mathbb{F}_p$, define $K_0 = W(k)[1/p]$, and let K be a finite totally ramified extension of K_0 of degree e'. Suppose that L is a subfield of K containing $\mathbb{Q}_p$ such that K/L is Galois and tamely ramified. Assume further that there is a uniformiser π of $\mathcal{O}_K$ such that $\pi^{e(K/L)} \in L$, where $e(K/L)$ is the ramification degree of K/L, and fix such a π. Since K/L

is tamely ramified, the category of Breuil modules with coefficients and descent data is easy to describe (see [Sav08]). Let k_E be a finite extension of $\mathbb{F}_p$. The objects of the category $\mathrm{BrMod}_{\mathrm{dd},L}$ are quadruples $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}, \phi_1, \{\widehat{g}\})$ consisting of the following.

– A finitely generated $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$-module $\mathcal{M}$, free over $k[u]/u^{e'p}$.

– A $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$-submodule $\mathrm{Fil}^1\mathcal{M}$ of $\mathcal{M}$ containing $u^{e'}\mathcal{M}$.

– A k_E-linear and ϕ-semilinear map $\phi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ with image generating $\mathcal{M}$ as a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$-module. (Here $\phi : k[u]/u^{e'p} \to k[u]/u^{e'p}$ is the pth power map.)

– For the $g \in \mathrm{Gal}(K/L)$, additive bijections $\widehat{g} : \mathcal{M} \to \mathcal{M}$ that preserve $\mathrm{Fil}^1\mathcal{M}$, commute with the ϕ_1- and k_E-actions, and satisfy $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(K/L)$. Furthermore $\widehat{1}$ is the identity, and if $a \in k \otimes_{\mathbb{F}_p} k_E$, $m \in \mathcal{M}$ then $\widehat{g}(au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i \widehat{g}(m)$.

The category $\mathrm{BrMod}_{\mathrm{dd},L}$ is equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ together with a k_E-action and descent data on the generic fibre from K to L (this equivalence depends on π).

We choose in this paper to adopt the conventions of [BM02, Sav05], rather than those of [BCDT01]; thus, rather than working with the usual contravariant equivalence of categories, we work with a covariant version of it, so that our formulae for generic fibres will differ by duality and a twist from those following the conventions of [BCDT01]. To be precise, we obtain the associated G_L-representation (which we will refer to as the generic fibre) of an object of $\mathrm{BrMod}_{\mathrm{dd},L}$ via the functor $T^L_{\mathrm{st},2}$.

Let E be a finite extension of $\mathbb{Q}_p$ with integers $\mathcal{O}_E$, maximal ideal $\mathfrak{m}_E$, and residue field k_E. Recall from [Sav05, § 2] that the functor $D^K_{\mathrm{st},2}$ is an equivalence of categories between the category of E-representations of G_L which are semistable when restricted to G_K and have Hodge–Tate weights in $\{0, 1\}$, and the category of weakly admissible filtered (ϕ, N)-modules D with descent data and E-coefficients such that $\mathrm{Fil}^0(K \otimes_{K_0} D) = K \otimes_{K_0} D$ and $\mathrm{Fil}^2(K \otimes_{K_0} D) = 0$.

Suppose that ρ is a representation in the source of $D^K_{\mathrm{st},2}$. Write $S = S_{K,\mathcal{O}_E}$ (notation and terminology in this paragraph are as in [Sav05, § 4]). Then $T^L_{\mathrm{st},2}$ is an essentially surjective functor from strongly divisible modules $\mathcal{M}$ (with $\mathcal{O}_E$-coefficients and descent data) in $S[1/p] \otimes_{K_0 \otimes E} D^K_{\mathrm{st},2}(\rho)$ to Galois-stable $\mathcal{O}_E$-lattices in ρ. This functor is compatible with reduction modulo $\mathfrak{m}_E$, so that applying $T^L_{\mathrm{st},2}$ to the object $(k \otimes_{\mathbb{F}_p} k_E)[u]/(u^{e'p}) \otimes_{S/\mathfrak{m}_E S} (\mathcal{M}/\mathfrak{m}_E\mathcal{M})$ of $\mathrm{BrMod}_{\mathrm{dd},L}$ yields a reduction modulo p of ρ (see [Sav05, Corollary 4.12, Proposition 4.13]).

To simplify notation, for the remainder of the paper we write simply $\mathcal{M}/\mathfrak{m}_E\mathcal{M}$ for the above reduction modulo $\mathfrak{m}_E$ of $\mathcal{M}$ in $\mathrm{BrMod}_{\mathrm{dd},L}$ (we will never mean the literal $S/\mathfrak{m}_E S$-module). Let ℓ be the residue field of L, and let ur_λ denote the unramified character of G_L sending an arithmetic Frobenius element to λ. Define $\mathrm{N}_{\ell/\mathbb{F}_p,k_E} : (\ell \otimes_{\mathbb{F}_p} k_E)^\times \to (\mathbb{F}_p \otimes_{\mathbb{F}_p} k_E)^\times \cong k_E^\times$ to be the norm map $x \mapsto \prod_{\beta \in \mathrm{Gal}(\ell/\mathbb{F}_p)} \beta(x)$, with each β acting trivially on k_E.

The following lemma is a more precise version of [GS, Lemma 5.2].

Lemma 4.1. *Let $\overline{\chi} : \mathrm{Gal}(K/L) \to k_E^\times$ be a character, and for $c \in (\ell \otimes_{\mathbb{F}_p} k_E)^\times$ let $\mathcal{M}(\overline{\chi}, c)$ denote the Breuil module with k_E-coefficients and descent data from K to L that is free of rank one with generator v and*

$$\mathrm{Fil}^1\mathcal{M}(\overline{\chi}, c) = \mathcal{M}(\overline{\chi}, c), \quad \phi_1(v) = cv, \quad \widehat{g}(v) = (1 \otimes \overline{\chi}(g))v$$

for $g \in \mathrm{Gal}(K/L)$. Then $T^L_{\mathrm{st},2}(\mathcal{M}(\overline{\chi}, c)) = \mathrm{ur}_\lambda \cdot \overline{\chi}$, where $\lambda = \mathrm{N}_{\ell/\mathbb{F}_p,k_E}(c)^{-1}$.

Proof. This statement is exactly the same as [GS, Lemma 5.2], except that here we determine the unramified character multiplying $\overline{\chi}$. We return ourselves to the proof of that statement, and in particular we re-adopt the notation from that proof, so that χ is the Teichmüller lift of $\overline{\chi}$, the element $\widetilde{c} \in (W(\ell) \otimes_{\mathbb{Z}_p} \mathcal{O}_E)^\times$ is a lift of c, and $D := D(\chi, \widetilde{c}\,)$ is a filtered ϕ-module of rank one over $K_0 \otimes_{\mathbb{Q}_p} E$ with descent data χ and generator $\mathbf{v}$ such that $\phi(\mathbf{v}) = p\widetilde{c}\mathbf{v}$; moreover the filtration on $D_K = K \otimes_{K_0} D$ vanishes in degree two. It is possible to choose $\widetilde{c}$ to be an element of finite multiplicative order, and we do so.

The representation $V^L_{\mathrm{st},2}(D)$ giving rise to D is equal to $(B_{\mathrm{st}} \otimes_{K_0} D)^{\phi=p}_{N=0} \cap \mathrm{Fil}^1(B_{\mathrm{dR}} \otimes_K D_K)$ (see the definition after [Sav05, Corollary 2.10]), and so is generated by some $\alpha\mathbf{v}$ with $\alpha \in (\mathrm{Fil}^0 B_{\mathrm{cris}}) \otimes_{\mathbb{Q}_p} E$; then $p\alpha\mathbf{v} = \phi(\alpha\mathbf{v}) = \phi(\alpha)\widetilde{c}p\mathbf{v}$, so that $\phi(\alpha)\widetilde{c} = \alpha$. If $f = [\ell : \mathbb{F}_p]$ it follows that $\phi^{(f)}(\alpha) = \mathrm{N}_{L_0/\mathbb{Q}_p,E}(\widetilde{c}\,)^{-1}\alpha$, where $L_0 = W(\ell)[1/p]$ and $\mathrm{N}_{L_0/\mathbb{Q}_p,E}$ is defined via the obvious analogy with $\mathrm{N}_{\ell/\mathbb{F}_p,k_E}$. Set $\widetilde{\lambda} = \mathrm{N}_{L_0/\mathbb{Q}_p,E}(\widetilde{c}\,)^{-1}$.

Since $\widetilde{c}$ has finite order we have $\phi^{(m)}(\alpha) = \alpha$ for some $m > 0$, and therefore α is an element of $(\mathrm{Fil}^0 B_{\mathrm{cris}})^{\phi^{(m)}=1} \otimes_{\mathbb{Q}_p} E = \mathbb{Q}_{p^m} \otimes_{\mathbb{Q}_p} E$. In particular, the action of crystalline Frobenius coincides with the action of an arithmetic Frobenius on α. As a result, if $g \in G_L$ is a lift of an nth power of arithmetic Frobenius with $n \in \mathbb{Z}$ then $g(\alpha\mathbf{v}) = \phi^{(nf)}(\alpha)\chi(g)\mathbf{v} = (\mathrm{ur}_{\widetilde{\lambda}} \cdot \chi)(g)\alpha\mathbf{v}$. Since $\widetilde{\lambda}$ lifts λ, the result follows by continuity. □

For the remainder of this paper we make the hypothesis that k_E is sufficiently large as to contain an embedding of k. Let σ_0 be a fixed choice of embedding $k \hookrightarrow k_E$ and recursively define $\sigma_{i+1}^p = \sigma_i$. If M is any $(k \otimes_{\mathbb{F}_p} k_E)$-module, we recall from [Sav08] that M decomposes as a direct sum $M = \bigoplus_{i=0}^{d-1} M_i$, where $d = [k : \mathbb{F}_p]$ and M_i is the k_E-submodule on which multiplication by $x \otimes 1$ for $x \in k$ is the same as multiplication by $1 \otimes \sigma_i(x)$. In fact there is a collection of idempotents $e_i \in k \otimes_{\mathbb{F}_p} k_E$ so that $M_i = e_i M$ and $\phi(e_i) = e_{i+1}$.

Suppose now that $\mathcal{M}$ is an object of $\mathrm{BrMod}_{\mathrm{dd},L}$. Note that ϕ_1 maps $(\mathrm{Fil}^1\mathcal{M})_i$ into $\mathcal{M}_{i+1}$. For $g \in G_L$ let $\overline{\eta}(g)$ be the image of $g(\pi)/\pi$ in (the $e(K/L)$th roots of unity of) k. The rank one objects of $\mathrm{BrMod}_{\mathrm{dd},L}$ are classified as follows.

PROPOSITION 4.2 [Sav08, Theorem 3.5]. *With our fixed choice of uniformiser π, every rank one object of $\mathrm{BrMod}_{\mathrm{dd},L}$ with descent data relative to L has the form:*

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}) \cdot v$;
- $(\mathrm{Fil}^1\mathcal{M})_i = u^{r_i}\mathcal{M}_i$;
- $\phi_1(\sum_{i=0}^{d-1} u^{r_i} e_i v) = cv$ *for some* $c \in (\ell \otimes_{\mathbb{F}_p} k_E)^\times$; *and*
- $\widehat{g}(v) = \sum_{i=0}^{d-1} (\overline{\eta}(g)^{k_i} \otimes 1)e_i v$ *for all* $g \in \mathrm{Gal}(K/L)$.

Here $0 \leqslant r_i \leqslant e'$ *and* $0 \leqslant k_i < e(K/L)$ *are sequences of integers satisfying* $k_i \equiv p(k_{i-1} + r_{i-1}) \pmod{e(K/L)}$; *furthermore the sequences* r_i, k_i *are periodic with period dividing* $f = [\ell : \mathbb{F}_p]$.

COROLLARY 4.3. *In the above proposition, suppose that* $e(K/L)$ *is divisible by* $p^f - 1$. *Define* $s_0 = p(p^{f-1}r_0 + \cdots + r_{f-1})/(p^f - 1)$ *and* $\lambda = \mathrm{N}_{\ell/\mathbb{F}_p,k_E}(c)^{-1}$. *Then* $T^L_{\mathrm{st},2}(\mathcal{M}) = (\sigma_0 \circ \overline{\eta}^{k_0+s_0}) \cdot \mathrm{ur}_\lambda$.

Remark 4.4. According to [Sav08, Remark 3.6], the congruences

$$k_i \equiv p(k_{i-1} + r_{i-1}) \qquad (\mathrm{mod}\ e(K/L))$$

imply that

$$p^{f-1}r_0 + \cdots + r_{f-1} \equiv 0 \qquad (\mathrm{mod}\ p^f - 1),$$

and k_0 is a solution to $-p(p^{f-1}r_0 + \cdots + r_{f-1}) \equiv (p^f - 1)k_0 \pmod{e(K/L)}$. It follows that s_0 is an integer; moreover $(p^f - 1)(k_0 + s_0) \equiv 0 \pmod{e(K/L)}$, so that the image of $\overline{\eta}^{k_0+s_0}$ lies in $\ell^\times$ and $\overline{\eta}^{k_0+s_0}$ is actually a character.

Proof of Corollary 4.3. We proceed as in [Sav08, Example 3.7].

Define $s_i = p(r_i p^{f-1} + \cdots + r_{i+f-1})/(p^f - 1)$ with subscripts taken modulo f, and observe that $(k_i + s_i) \equiv p^i(k_0 + s_0) \pmod{e(K/L)}$. Let $\overline{\chi} = \sigma_0 \circ \overline{\eta}^{k_0+s_0}$. We check that there is a morphism $\mathcal{M}(\overline{\chi}, c) \to \mathcal{M}$ with $\mathcal{M}(\overline{\chi}, c)$ as in Lemma 4.1 (except that here we will use w to denote its generator, since v is now our generator of $\mathcal{M}$).

The morphism will send w to $\sum_i u^{s_i} e_i v$. One checks easily that this is a morphism of Breuil modules. Indeed: the filtration is preserved since $s_i \geqslant r_i$; the morphism commutes with ϕ_1 because $s_{i+1} = p(s_i - r_i)$; and to check that the morphism commutes with descent data, use the fact that $\widehat{g}(w) = (1 \otimes (\sigma_0 \circ \overline{\eta}^{k_0+s_0}(g)))w = \sum_i (\overline{\eta}^{k_i+s_i}(g) \otimes 1)e_i w$.

Now the claim follows immediately from Lemma 4.1 and an application of [Sav04, Proposition 8.3]. (This last step uses our running hypothesis that $p > 2$.) □

4.2 Necessary conditions: notation and preliminaries

Let $\mathfrak{p}$ be a prime of F lying above p, and $\pi_{\mathfrak{p}} \in \mathfrak{p}$ our chosen uniformiser. Suppose that the residue field of $F_{\mathfrak{p}}$ has order $q = p^f$.

In the remainder of § 4, we consider the following situation. Let L be the unramified quadratic extension of $F_{\mathfrak{p}}$, and K the splitting field of $u^{p^{2f}-1} - \pi_{\mathfrak{p}}$ over L. Let ϖ be a choice of $\pi_{\mathfrak{p}}^{1/(p^{2f}-1)}$ in K. Let k denote the residue field of K, and if $g \in \mathrm{Gal}(K/F_{\mathfrak{p}})$ then as before we let $\overline{\eta}(g)$ be the image of $g(\varpi)/\varpi$ in k. Suppose that $F_{\mathfrak{p}}$ has absolute ramification index e, and write $e' = e(p^{2f} - 1)$. (We alert the reader that in what follows, the fields $F_{\mathfrak{p}}$ and L will both take turns being used in the role of the field L of the previous subsection.)

Suppose that k embeds into k_E. By Proposition 4.2, any rank one Breuil module $\mathcal{M}$ with k_E-coefficients and descent data from K to L may be written in the form:

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e(p^{2f}-1)p}) \cdot v$;
- $(\mathrm{Fil}^1 \mathcal{M})_i = u^{r_i} \mathcal{M}_i$;
- $\phi_1(u^{r_i} e_i v) = (1 \otimes \gamma_i) e_{i+1} v$ for some $\gamma_i \in k_E^\times$;
- $\widehat{g}(\sum_{i=0}^{2f-1} e_i v) = \sum_{i=0}^{2f-1} (\overline{\eta}(g)^{k_i} \otimes 1) e_i v$ for all $g \in \mathrm{Gal}(K/L)$.

Here the k_i, r_i are any integers with $k_i \in [0, p^{2f} - 1)$ and $r_i \in [0, e(p^{2f} - 1)]$ satisfying $k_{i+1} \equiv p(k_i + r_i) \pmod{p^{2f} - 1}$. For $g \in \mathrm{Gal}(K/L)$ we write $\widehat{g}(e_i v) = (1 \otimes \chi_i(g)) e_i v$ where $\chi_i = \sigma_i \circ \overline{\eta}^{k_i}$. Note that χ_i, defined on $\mathrm{Gal}(K/L)$, is a homomorphism.

Let $\chi : I_{F_{\mathfrak{p}}} \to \mathcal{O}_E^\times$ be an inertial character with $\chi = \chi^{q^2}$ but $\chi \neq \chi^q$, and let $\overline{\chi}$ denote its reduction modulo the maximal ideal of $\mathcal{O}_E$. In what follows we will be concerned with Breuil modules $\mathcal{M}$ as above that have the extra property $\chi_i \in \{\overline{\chi}, \overline{\chi}^q\}$ for all i. In the remainder of this subsection we introduce some notation that is special to this situation (and that will be used repeatedly throughout the rest of the paper), and we derive a variant of Corollary 4.3.

Let $\eta_i = (\sigma_i \circ \overline{\eta})|_{I_{F_{\mathfrak{p}}}}$ for $0 \leqslant i < 2f$ be a system of fundamental characters of niveau $2f$ of $I_{F_{\mathfrak{p}}}$; note that $\eta_i^p = \eta_{i-1}$. Then $\omega_i = \eta_i \eta_{i+f}$ for $0 \leqslant i < f$ is a system of fundamental characters of niveau f. Write

$$\overline{\chi} = \prod_{i=0}^{2f-1} \eta_i^{c_i} \tag{4.5}$$

with $0 \leqslant c_i \leqslant p-1$; since $\overline{\chi}$ is non-trivial, this is unambiguous. We let J be the set of $i \in \{0, \ldots, 2f-1\}$ such that $\chi_i = \overline{\chi}$. References to elements of J should always be taken modulo $2f$, so that e.g. if $i = 2f-1$ then $i+1$ refers to 0.

The congruence $k_{i+1} \equiv p(k_i + r_i) \pmod{p^{2f}-1}$ is equivalent to the relation $\chi_{i+1} = \chi_i \eta_i^{r_i}$. If $i \in J$ and $i+1 \in J$, or if $i \notin J$ and $i+1 \notin J$, this gives $r_i \equiv 0 \pmod{p^{2f}-1}$. In either case, write $r_i = (p^{2f}-1)x_i$ for some $0 \leqslant x_i \leqslant e$. If $i \in J$ and $i+1 \notin J$, we see that

$$r_i = (p^{2f}-1)x_i + (p^f-1)(p^{f-1}(c_{i+f+1} - c_{i+1}) + p^{f-2}(c_{i+f+2} - c_{i+2}) + \cdots + (c_i - c_{i+f}))$$

for some x_i, and if $i \notin J$ and $i+1 \in J$, then

$$r_i = (p^{2f}-1)x_i + (p^f-1)(p^{f-1}(c_{i+1} - c_{i+f+1}) + p^{f-2}(c_{i+2} - c_{i+f+2}) + \cdots + (c_{i+f} - c_i)).$$

Since the expression $(p^f-1)(p^{f-1}(c_{i+f+1} - c_{i+1}) + p^{f-2}(c_{i+f+2} - c_{i+2}) + \cdots + (c_i - c_{i+f}))$ is non-zero and is strictly between $1 - p^{2f}$ and $p^{2f}-1$, we allow either $0 \leqslant x_i \leqslant e-1$ or $1 \leqslant x_i \leqslant e$, depending on whether the sign of this expression is positive or negative. If $i \in J$ and $i+1 \notin J$, then the allowable range is $0 \leqslant x_i \leqslant e-1$ precisely when there is a $j \geqslant 1$ with $c_{i+k} = c_{i+k+f}$ for all $1 \leqslant k < j$ and $c_{i+j+f} > c_{i+j}$, and the situation is reversed in the case $i \notin J$ and $i+1 \in J$. We summarise these conditions in the following definition.

DEFINITION 4.6. Fix J and $\overline{\chi}$ as above. We say that $x_i \in \{0, 1, \ldots, e\}$ is *allowable* in each of the following situations, and *not allowable* otherwise.

- $i, i+1 \in J$ or $i, i+1 \notin J$.
- $i \in J$, $i+1 \notin J$: we require $x_i \neq e$ if there is $j \geqslant 1$ with $c_{i+k} = c_{i+k+f}$ for all $1 \leqslant k < j$ and $c_{i+j} < c_{i+j+f}$; we require $x_i \neq 0$ otherwise.
- $i \notin J$, $i+1 \in J$: we require $x_i \neq 0$ if there is $j \geqslant 1$ with $c_{i+k} = c_{i+k+f}$ for all $1 \leqslant k < j$ and $c_{i+j} < c_{i+j+f}$; we require $x_i \neq e$ otherwise.

Here subscripts should be taken modulo $2f$. We also say x_i is not allowable if $x_i \notin \{0, 1, \ldots, e\}$. We say that the list $x_0, \ldots, x_{2f-1}$ is allowable if each x_i is allowable.

Thus a rank one Breuil module $\mathcal{M}$ with the property that $\chi_i \in \{\overline{\chi}, \overline{\chi}^q\}$ for all i gives rise to a set J and an allowable collection $x_0, \ldots, x_{2f-1}$. Conversely, it is straightforward to check that this construction can be reversed: any J and any allowable list $x_0, \ldots, x_{2f-1}$, together with any choice of γ_i's, determines a Breuil module $\mathcal{M}$ with the desired property.

Let ψ denote the restriction to inertia of $T_{\mathrm{st},2}^L(\mathcal{M})$, and note from Corollary 4.3 that ψ depends only on the χ_i's and r_i's, or equivalently only on J and the x_i's.

LEMMA 4.7. *Let $\mathcal{M}$ be a rank one Breuil module with k_E-coefficients and descent data from K to L with $\chi_i \in \{\overline{\chi}, \overline{\chi}^q\}$ for all i. Then*

$$\psi = \prod_{i \in J} \eta_i^{c_i} \prod_{i \notin J} \eta_i^{c_{i+f}} \prod_{i=0}^{2f-1} \eta_i^{x_i}. \tag{4.8}$$

Proof. Recall that

$$s_0 = \frac{p}{(p^{2f}-1)}(r_0 p^{2f-1} + r_1 p^{2f-2} + \cdots + r_{2f-1}),$$

so that $\psi = \eta_0^{k_0+s_0} = \chi_0 \eta_0^{s_0}$. Write s_0 as $p^{2f}x_0 + p^{2f-1}x_1 + \cdots + px_{2f-1}$ plus a linear expression in the c_i's.

We compute the coefficient of c_0 in this linear expression. For each transition $i \in J$, $i+1 \notin J$ with $i \in [0, f)$, the coefficient of c_0 in $p^{2f-i}r_i$ is $p^{2f}(p^f - 1)$; on the other hand for each transition $i \notin J$, $i+1 \in J$ with $i \in [0, f)$ the coefficient of c_0 in $p^{2f-i}r_i$ is $-p^{2f}(p^f - 1)$.

For $i \in [f, 2f-1)$ the respective coefficients are $-p^f(p^f-1)$ for transitions $i \in J$, $i+1 \notin J$, and $p^f(p^f-1)$ for the reverse. As a consequence:

- if $0, f \in J$ or $0, f \notin J$ then the net number of transitions out of J from $i=0$ to $i=f$ is zero, and similarly from f to $2f$. In either case the coefficient of c_0 in s_0 is zero;
- if $0 \in J$ and $f \notin J$, then the net number of transitions out of J from $i=0$ to $i=f$ is 1, and from $i=f$ to $i=2f$ is -1. In this case the coefficient of c_0 in s_0 is $(p^{2f}(p^f-1) + p^f(p^f-1))/(p^{2f}-1) = p^f$;
- similarly if $0 \notin J$ and $f \in J$, the coefficient of c_0 in s_0 is $-p^f$.

From (4.5) and the definition of J, the contribution of c_0 to χ_0 is $\eta_0^{c_0}$ if $0 \in J$ and $\eta_0^{p^f c_0}$ if $0 \notin J$. Thus the total contribution of c_0 to $\psi = \chi_0\eta_0^{s_0}$ is:

- $\eta_0^{c_0}$ if $0 \in J$, $f \in J$;
- $\eta_0^{c_0}\eta_0^{p^f c_0} = \eta_0^{c_0}\eta_f^{c_0}$ if $0 \in J$, $f \notin J$;
- $\eta_0^{-p^f c_0}\eta_0^{p^f c_0} = 1$ if $0 \notin J$, $f \in J$;
- $\eta_0^{p^f c_0} = \eta_f^{c_0}$ if $0 \notin J$, $f \notin J$.

In each case we obtain precisely the contribution of c_0 to the first two products on the right-hand side of (4.8). The lemma follows by cyclic symmetry, together with the fact that η_0 raised to the power $p^{2f}x_0 + \cdots + px_{2f-1}$ is the third product on the right-hand side of (4.8). □

4.3 Necessary conditions: the reducible case

Suppose that we have $\overline{\rho} : G_{F_{\mathfrak{p}}} \to \mathrm{GL}_2(k_E)$ with k_E a finite field into which k may be embedded, and assume that $\overline{\rho}$ is the reduction modulo $\mathfrak{m}_E$ of a parallel potentially Barsotti–Tate representation ρ of type $\chi \oplus \chi^q$. Let $\mathcal{H}$ be the $\mathfrak{m}_E$-torsion of the Barsotti–Tate group over $\mathcal{O}_K$ corresponding to ρ; then $\mathcal{H}$ is a finite flat group scheme over $\mathcal{O}_K$ with descent data to $F_{\mathfrak{p}}$, and $\overline{\rho}$ is the generic fibre of $\mathcal{H}$.

In this subsection we suppose that $\overline{\rho} \cong \left(\begin{smallmatrix}\psi_1 & * \\ 0 & \psi_2\end{smallmatrix}\right)$ is reducible, and we wish to restrict the possibilities for ψ_1 and ψ_2. Note that by a standard scheme-theoretic closure argument, ψ_1 corresponds to a finite flat subgroup scheme $\mathcal{G}$ of $\mathcal{H}$. Let $\mathcal{M}$ be the rank-one Breuil module with k_E-coefficients and descent data from K to $F_{\mathfrak{p}}$ corresponding to $\mathcal{G}$, and let χ_i for $i = 0, \ldots, 2f-1$ be defined as in the previous subsection. It follows from Corollary 5.2, which does not depend on anything in this paper before § 5, that the descent data for $\mathcal{H}$ is of the form $\overline{\chi} \oplus \overline{\chi}^q$, so that we have $\chi_i \in \{\overline{\chi}, \overline{\chi}^q\}$ for all i. Therefore we may define J and $x_0, \ldots, x_{2f-1}$ as in the previous subsection, and the analysis of the previous subsection applies to $\mathcal{M}$.

Since the descent data on $\mathcal{M}$ is from K to $F_{\mathfrak{p}}$ and not simply from K to L, we in fact have from Proposition 4.2 that $r_{i+f} = r_i$ and $k_{i+f} = k_i$ for all i, or equivalently $\chi_i = \chi_{i+f}^q$ and $x_i = x_{i+f}$ for all i. In particular for all i we have exactly one of $i, i+f$ in J, and $x_{i+f} = x_i$ is allowable if and only if x_i is. Letting π denote the natural projection from $\mathbb{Z}/2f\mathbb{Z}$ to $\mathbb{Z}/f\mathbb{Z}$, we deduce from Lemma 4.7 that $\psi_1|_{I_{F_\mathfrak{p}}}$ has the form

$$\psi_1|_{I_{F_\mathfrak{p}}} = \prod_{i\in J} \omega_{\pi(i)}^{c_i} \prod_{i=0}^{f-1} \omega_i^{x_i} \tag{4.9}$$

where J contains exactly one of $i, i+f$ for all i, and $x_0, \ldots, x_{f-1}$ are allowable for $\overline{\chi}$ and J.

PROPOSITION 4.10. *If $e \geqslant p-1$, then (for fixed $\overline{\chi}$) any inertial character of niveau f occurs as the right-hand side of (4.9) for some choice of J with exactly one of $i, i+f \in J$ for all i, and some allowable values $x_0, \ldots, x_{f-1}$.*

Proof. The proposition is immediate if $e \geqslant p$, because for *any* J the allowable range for each x_i contains p consecutive integers; so we suppose that $e = p-1$, where the matter is more delicate. Observe that the claim is invariant under twisting $\overline{\chi}$ by a character ω of niveau f: replacing $\overline{\chi}$ with $\omega\overline{\chi}$ replaces each χ_i with $\omega\chi_i$, leaving the possibilities for the integers r_i and s_0 arising from the relevant Breuil modules unchanged. The claim is similarly invariant under replacing $\overline{\chi}$ with $\overline{\chi}^p$. As a consequence of these observations we may suppose without loss of generality that $c_0, \ldots, c_{f-1} = 0$ while $c_{2f-1} \neq 0$.

Consider first the set $J = \{0, \ldots, f-1\}$. The allowable range for x_{f-1} is $[1, p-1]$ (since there is some $1 \leqslant j \leqslant f$ with $c_{f-1+j} > 0$ while each $c_{j-1} = 0$), and $x_0, \ldots, x_{f-2}$ can range over $[0, p-1]$. Writing the right-hand side of (4.9) as ω_{f-1} raised to the power $p^{f-1}x_0 + \cdots + px_{f-2} + x_{f-1}$, we see that the exponent of ω_{f-1} obtains every integer value in $[0, p^f-1]$ except those divisible by p.

Now consider the sets $J = \{2f-i, \ldots, 2f-1, 0, \ldots, f-i-1\}$ with $1 \leqslant i \leqslant f-1$. The allowable range for x_{f-i-1} is $[0, p-2]$ since $c_{2f-1} > 0$ while each $c_{f-i}, \ldots, c_{f-1}$ equals 0; for each other x_i the allowable range is $[0, p-1]$. For this choice of J the right-hand side of (4.9) becomes ω_{f-1} raised to the power

$$(p^{i-1}c_{2f-i} + \cdots + c_{2f-1}) + (p^{f-1}x_0 + \cdots + px_{f-2} + x_{f-1}). \tag{4.11}$$

The right-hand term varies over all integers in the range $[0, p^f-1]$ except those whose p^i-coefficient in base p is $p-1$. In particular the base p sum in (4.11) does not have a carry from the p^i-place to the p^{i+1}-place. Since $c_{2f-1} \neq 0$, it follows that the values taken by (4.11) (with allowable $x_0, \ldots, x_{f-1}$) include all integers in $[0, p^f-1]$ that are exactly divisible by p^i.

All together, we find that for suitable choices of J the right-hand side of (4.9) when written as a power of ω_{f-1} can take every exponent in the range $[1, p^f-1]$. This is a complete set of powers of ω_{f-1}. □

4.4 Necessary conditions: the irreducible case

We retain the notation and hypotheses of the previous subsection, but now we consider the case of an irreducible $\overline{\rho}$. In this case, $\overline{\rho}|_{G_L} \equiv \left(\begin{smallmatrix}\psi_1 & 0\\ 0 & \psi_2\end{smallmatrix}\right)$ with $\psi_2 = \psi_1^q$. Again, we examine the possibilities for $\psi_1|_{I_{F_\mathfrak{p}}}$. Let $\mathcal{H}$ be the finite flat group scheme with generic fibre descent data from K to L corresponding to $\overline{\rho}|_{G_L}$, and let $\mathcal{G}$ be the finite flat subgroup scheme corresponding to ψ_1. Note that the descent data on $\mathcal{H}$ must extend to $\mathrm{Gal}(K/F_\mathfrak{p})$ while the descent data on $\mathcal{G}$ must not.

Let $\mathcal{M}$ be the Breuil module with k_E-coefficients and descent data from K to L corresponding to $\mathcal{G}$. It follows once again from Corollary 5.2 that the descent data for $\mathcal{H}$ is of the form $\overline{\chi} \oplus \overline{\chi}^q$, so that we have $\chi_i \in \{\overline{\chi}, \overline{\chi}^q\}$ for all i. Thus the analysis of § 4.2 applies to $\mathcal{M}$, and we may employ the notation of that subsection; in particular we take $J = \{i : \chi_i = \overline{\chi}\}$.

Remark 4.12. In an earlier version of this paper we claimed to show that $\mathcal{H}$ must decompose as a product $\mathcal{G} \times \mathcal{G}'$ where $\mathcal{G}'$ is the finite flat subgroup scheme with descent data from K to L

corresponding to ψ_2, in which case it would follow that $i \in J$ if and only if $i+f \in J$, and also that $x_i + x_{i+f} = e$ for all i. Our proof of this claim was in error and we do not know whether or not the claim is true. Nevertheless we have the following proposition.

Proposition 4.13. *There exists $J \subset \{0, \ldots, 2f-1\}$ with $i \in J$ if and only if $i+f \in J$, and an allowable list $x_0, \ldots, x_{2f-1}$ with $x_i + x_{i+f} = e$ for all i, such that*

$$\psi_1|_{I_L} = \prod_{i \in J} \eta_i^{c_i} \prod_{i \notin J} \eta_i^{c_{i+f}} \prod_i \eta_i^{x_i}. \tag{4.14}$$

Per Remark 4.12, the J in the proposition may not necessarily be the J coming from $\mathcal{M}$. The proof of Proposition 4.13 occupies the remainder of this section.

Lemma 4.15. *If $\overline{\rho}$ has a parallel potentially Barsotti–Tate lift of type $\chi \oplus \chi^q$, then $\det \overline{\rho}|_{I_{F_\mathfrak{p}}} = \epsilon \cdot \overline{\chi}^{q+1}$, where ϵ is the mod p cyclotomic character.*

Proof. This follows at once from Definition 3.2 and the results of [CDT99, § B.2]. □

By Lemma 4.15, we must have $(\psi_1|_{I_L})^{q+1} = \epsilon \cdot \overline{\chi}^{q+1}$. A straightforward computation shows that for any J with $i \in J$ if and only if $i+f \in J$, and any $x_0, \ldots, x_{2f-1}$ with $x_i + x_{i+f} = e$ for all i, the character ψ on the right-hand side of (4.14) has $\psi^{q+1} = \epsilon \cdot \overline{\chi}^{q+1}$ as well; this uses the fact that $\epsilon = \omega_0^e \cdots \omega_{f-1}^e$.

Lemma 4.16. *If $e \geqslant p-1$, then as J varies over subsets of $\{0, \ldots, 2f-1\}$ with $i \in J$ if and only if $i+f \in J$, and $x_0, \ldots, x_{2f-1}$ varies over allowable lists for J with $x_i + x_{i+f} = e$ for all i, the right-hand side of (4.14) varies over all inertial characters ψ with $\psi^{q+1} = \epsilon \cdot \overline{\chi}^{q+1}$. In particular Proposition 4.13 is true if $e \geqslant p-1$.*

Proof. First take $J = \{0, \ldots, 2f-1\}$, so that any $x_0, \ldots, x_{2f-1} \in [0, e]$ are allowable. We let $x_0, \ldots, x_{f-1}$ vary over $[0, e]$ and take $x_{i+f} = e - x_i$, and we consider the characters ψ that occur.

If $X = p^{f-1} x_0 + \cdots + x_{f-1}$ then $\psi = \overline{\chi} \cdot \eta_{2f-1}^{e(p^f-1)/(p-1)} \cdot \eta_{f-1}^{(1-p^f)X}$ and depends only on X (mod p^f+1). If $e \geqslant p$ then as $x_0, \ldots, x_{f-1}$ range over the interval $[0, e]$, the integer X ranges over an interval that includes $[0, p^f]$, and ψ ranges over all p^f+1 inertial characters ψ with $\psi^{q+1} = \epsilon \cdot \overline{\chi}^{q+1}$.

If instead $e = p-1$, then X only ranges over the interval $[0, p^f-1]$, and we obtain all possibilities for ψ satisfying the condition on ψ^{q+1} except $\psi = \overline{\chi}$. However, performing the same analysis with $J = \emptyset$ gives us all possibilities for ψ except $\psi = \overline{\chi}^q$; in particular since $\overline{\chi} \neq \overline{\chi}^q$ we obtain $\psi = \overline{\chi}$ as a possibility with $J = \emptyset$. □

Before continuing with the proof of Proposition 4.13, we make the following observation. Suppose that $i \in J$ if and only if $i+f \in J$, and $x_0, \ldots, x_{2f-1}$ is an allowable list such that the product ψ on the right-hand side of (4.14) satisfies $\psi^{q+1} = \epsilon \overline{\chi}^{q+1}$. If $e < p-1$, then the condition $x_i + x_{i+f} = e$ must be satisfied automatically. Indeed, the condition $\psi^{q+1} = \epsilon \overline{\chi}^{q+1}$ comes down to $\prod_{i=0}^{f-1} \omega_i^{x_i + x_{i+f}} = \prod_{i=0}^{f-1} \omega_i^e$. Since $x_i + x_{i+f} \in [0, 2e]$ and $e < p-1$, the only possibility is $x_i + x_{i+f} = e$ for all i.

Proof of Proposition 4.13. Thanks to Lemma 4.16 we may assume $e < p-1$. Let J be any subset of $\{0, \ldots, 2f-1\}$, let $x_0, \ldots, x_{2f-1}$ be allowable for J, and write

$$\psi = \prod_{i \in J} \eta_i^{c_i} \prod_{i \notin J} \eta_i^{c_{i+f}} \prod_i \eta_i^{x_i}.$$

We wish to prove that if $\psi^{q+1} = \epsilon \cdot \overline{\chi}^{q+1}$ (so that, for instance, J and $x_0, \dots, x_{2f-1}$ might be the data associated to $\mathcal{M}$) then there exists some J' with $i \in J'$ if and only if $i + f \in J'$, and allowable $x'_0, \dots, x'_{2f-1}$ such that if we write

$$\psi' = \prod_{i \in J'} \eta_i^{c_i} \prod_{i \notin J'} \eta_i^{c_{i+f}} \prod_i \eta_i^{x'_i}$$

then in fact we have $\psi = \psi'$. (Then the desideratum $x'_i + x'_{i+f} = e$ also holds, by the observation immediately before we began the remainder of the proof.)

Assuming that J does not already satisfy $i \in J$ if and only if $i + f \in J$, it suffices to produce J' and allowable $x'_0, \dots, x'_{2f-1}$ such that $\psi' = \psi$ and J' has more pairs $(i, i+f)$ with $i \in J'$ if and only if $i + f \in J'$ than J does. (Then repetition of this step will complete the argument.) This is what we now carry out.

Let S be the set of indices i such that c_i appears as an exponent twice in the product for ψ (equivalently, such that $i \in J$ and $i + f \notin J$), and similarly let T be the set of indices i such that c_i occurs zero times (equivalently, $i \notin J$ and $i + f \in J$). Note that $S = f + T$ (with the obvious meaning for this notation). Then the condition on ψ^{q+1} is

$$\prod_{i=0}^{f-1} \omega_i^{x_i + x_{i+f}} \prod_{i \in T} \omega_i^{-c_i} \prod_{i \in S} \omega_i^{c_i} = \prod_{i=0}^{f-1} \omega_i^e$$

which we re-write as

$$\prod_{i=0}^{f-1} \omega_i^{x_i + x_{i+f} - e \pm [c_i - c_{i+f}]} = 1$$

where the brackets around $c_i - c_{i+f}$ denote that the term may not occur (in this case, it occurs with sign + if $i \in S$, with sign − if $i \in T$, and not at all if i is in neither S nor T).

Each exponent in this product lies in the interval $[-e - p + 1, e + p - 1] \subset [-(2p-3), (2p-3)]$ since $e < p - 1$. Now, if $\prod_{i=0}^{f-1} \omega_i^{y_i} = 1$ then the vector $(y_0, \dots, y_{f-1})$ must be an integral linear combination

$$a_0(p, 0, \dots, 0, -1) + a_1(-1, p, 0, \dots, 0) + \dots + a_{f-1}(0, \dots, 0, -1, p).$$

It is easy to check that if each y_i lies in $[-(2p-3), (2p-3)]$ then in fact each a_i must be 0 or ± 1.

Writing the vector $(x_i + x_{i+f} - e \pm [c_i - c_{i+f}])_i$ as such a linear combination, we have

$$x_i + x_{i+f} - e \pm [c_i - c_{i+f}] = pa_i - a_{i+1}$$

for all $i \in \{0, \dots, 2f-1\}$ where we conventionally set $a_{i+f} := a_i$; take all subscripts modulo $2f$; and the sign is + if $i \in S$ and − if $i \in T$, and 0 otherwise.

Choose any *maximal* interval $[j', j] \subset \mathbb{Z}$ such that $i \notin J$ and $i + f \in J$ for $i \in [j', j]$. (As usual, we abuse notation and take all indices modulo $2f$.) By definition this interval cannot contain both i and $i + f$ for any i, so it contains at most f integers. Now if $i \in [j', j]$ we have $i \in T$ and $i + f \in S$, so that in fact

$$x_i + x_{i+f} - e - (c_i - c_{i+f}) = pa_i - a_{i+1} \tag{4.17}$$

for $i \in [j', j]$.

First consider the case $j = j' + (f-1)$. Define $J' = \{0, \dots, 2f-1\}$, and set:

- $x'_i = x_i - (c_i - c_{i+f}) - pa_i + a_{i+1}$ if $i \in [j', j'+f-1)$;
- $x'_{j'+f-1} = x_{j'+f-1} - (c_{j'+f-1} - c_{j'-1}) - pa_{j'-1}$;
- $x'_{j'-1} = x_{j'-1} + a_{j'}$; and
- $x'_i = x_i$ for all other indices i.

One checks easily that $\psi' = \psi$, and by construction $x'_i + x'_{i+f} = e$ for all i. We next verify that $x'_{j'-1}$ remains in the interval $[0, e]$. Note that by our choice of interval $[j', j]$ we have $j' - 1 \in J$ while $j' \notin J$. If we had $a_{j'} = -1$ then (4.17) for $i = j'$ implies $c_{j'} > c_{j'+f}$, and according to the definition of allowability we must have $x_{j'-1} > 0$; hence $x'_{j'-1}$ remains non-negative. Similarly if $a_{j'} = 1$ we still have $x'_{j'-1} \leqslant e$. This completes the verification.

Since $x'_i = x_i$ for $i \notin [j'-1, j'+f-1]$, we in fact have $x'_i \in [0, e]$ for all $i \in [j'-f, j'-1]$. However, $x'_i + x'_{i+f} = e$ for all i, and one of the two summands always lies in $[0, e]$; therefore so does the other. Since $J' = \{0, \dots, 2f-1\}$ the list $x'_0, \dots, x'_{2f-1}$ is allowable and we are done.

Henceforth suppose that $j - j' < f - 1$. Consider the following two 'moves'.

(i) Set $J' = J \cup \{j', \dots, j\}$, and define:
   - $x'_i = x_i - (c_i - c_{i+f}) - pa_i + a_{i+1}$ if $i \in [j', j)$;
   - $x'_j = x_j - (c_j - c_{j+f}) - pa_j$;
   - $x'_{j'-1} = x_{j'-1} + a_{j'}$;
   - $x'_i = x_i$ for all remaining indices.

(ii) Set $J' = J \backslash \{j'+f, \dots, j+f\}$ and define:
   - $x'_{i+f} = x_{i+f} - (c_i - c_{i+f}) - pa_i + a_{i+1}$ if $i \in [j', j)$;
   - $x'_{j+f} = x_{j+f} - (c_j - c_{j+f}) - pa_j$;
   - $x'_{j'+f-1} = x_{j'+f-1} + a_{j'}$;
   - $x'_i = x_i$ for all remaining indices.

In either case we have $\psi' = \psi$. For $i \in [j', j)$ we have $x'_i + x'_{i+f} = e$, from which it follows that $x'_i, x'_{i+f} \in [0, e]$ (since at least one is in that interval); moreover $i, i+1$ are either both in J' or both not in J' for $i \in [j', j)$ or $[j'+f, j+f)$. Thus x'_i, x'_{i+f} are allowable for $i \in [j', j)$.

To decide whether the list $x'_0, \dots, x'_{2f-1}$ is allowable, the only issue that remains is the allowability of $x'_{j'-1}$ and x'_j after move (i), or of $x'_{j'+f-1}$ and x'_{j+f} after move (ii). Note that $x'_{j'-1}$ and x'_j are not the same object since $j - j' < f - 1$, and similarly for $x'_{j'+f-1}$ and x'_{j+f}. We will argue that at least one of these two pairs must be allowable.

First consider move (i) and the allowability of x'_j. We have $x'_j + x'_{j+f} = e - a_{j+1}$, so in particular $x'_j \in [-1, e+1]$. Note that if $a_{j+1} \neq 0$ then the last term on the left-hand side of

$$x_{j+1} + x_{j+f+1} - e \pm [c_{j+1} - c_{j+f+1}] = pa_{j+1} - a_{j+2}$$

must be non-zero, so $j+1$ is in S or T. By maximality of $[j', j]$ we have $j+1 \notin T$, so $j+1 \in S$ and the sign $\pm$ must be $+$. In particular either $a_{j+1} = 1$ and $c_{j+1} > c_{j+f+1}$, or $a_{j+1} = -1$ and $c_{j+1} < c_{j+f+1}$.

There are several conceivable ways that x'_j might be non-allowable.

- If $x'_j = -1$, then $x'_{j+f} = x_{j+f} = e$ and $a_{j+1} = 1$. We have seen that $a_{j+1} = 1$ implies $j+f+1 \notin J$ and $c_{j+1} > c_{j+f+1}$. However, since $j+f \in J$, under these conditions $x_{j+f} = e$ would not have been allowable to begin with. Thus $x'_j = -1$ cannot occur.

– If $x'_j = e+1$, then $x'_{j+f} = x_{j+f} = 0$ and $a_{j+1} = -1$. We have seen that $a_{j+1} = -1$ implies $j+f+1 \notin J$ and $c_{j+1} < c_{j+f+1}$. However, since $j+f \in J$, under these conditions $x_{j+f} = 0$ would not have been allowable to begin with. Thus $x'_j = e+1$ cannot occur.

– If $x'_j = 0$ and is not allowable, then since $j \in J'$ we must have $j+1 \notin J'$. By maximality of $[j', j]$ we have $j+f+1 \notin J'$ and $j+1 \notin S \cup T$. In particular $a_{j+1} = 0$ and $x'_{j+f} = x_{j+f} = e$. The allowability of $x_{j+f} = e$ when $j+f \in J$, $j+f+1 \notin J$ implies the allowability of $x'_j = 0$ when $j \in J'$, $j+1 \notin J'$, a contradiction.

– If $x'_j = e$ and is not allowable, then since $j \in J'$ we must have $j+1 \notin J'$. By maximality of $[j', j]$ we have $j+f+1 \notin J'$ and $j+1 \notin S \cup T$. In particular $a_{j+1} = 0$ by the remarks above, and $x'_{j+f} = x_{j+f} = 0$. The allowability of $x_{j+f} = 0$ when $j+f \in J$, $j+f+1 \notin J$ implies the allowability of $x'_j = e$ when $j \in J'$, $j+1 \notin J'$, a contradiction.

We deduce that *in all cases, x'_j is allowable after move (i)*. By an identical argument, *in all cases x'_{j+f} is allowable after move (ii).*

Now consider move (i) and the allowability of $x'_{j'-1} = x_{j'-1} + a_{j'}$. Note that if $a_{j'} \neq 0$ then the last term on the left-hand side of

$$x_{j'} + x_{j'+f} - e - (c_{j'} - c_{j'+f}) = pa_{j'} - a_{j'+1}$$

must be positive if $a_{j'} = 1$ and negative if $a_{j'} = -1$. That is, if $a_{j'} = 1$ then $c_{j'} < c_{j'+f}$ and if $a_{j'} = -1$ then $c_{j'} > c_{j'+f}$. There are again several conceivable ways that $x'_{j'-1}$ might be non-allowable.

– If $x'_{j'-1} = -1$, then $a_{j'} = -1$ and $x_{j'-1} = 0$. We obtain $c_{j'} > c_{j'+f}$. Since $j' \notin J$, if we had $j'-1 \in J$ it would contradict the allowability of $x_{j'-1} = 0$. Hence in this case we must have had $j'-1 \notin J$ to begin with.

– If $x'_{j'-1} = 0$ or e, then since $j' \in J'$, in order to be non-allowable we must have $j'-1 \notin J'$, and so $j'-1 \notin J$.

– If $x'_{j'-1} = e+1$, then $a_{j'} = 1$ and $x_{j'-1} = e$. We obtain $c_{j'} < c_{j'+f}$. Since $j' \notin J$, if we had $j'-1 \in J$ it would contradict the allowability of $x_{j'-1} = e$. Hence yet again we must have had $j'-1 \notin J$.

We deduce that *in all cases, $x'_{j'-1}$ is allowable after move (i) provided that $j'-1 \in J$*. By an identical argument, *in all cases, $x'_{j'+f-1}$ is allowable after move (ii) provided that $j'+f-1 \notin J$.*

By maximality of $[j', j]$, we must have either $j'-1 \in J$ or $j'+f-1 \notin J$. Therefore at least one of moves (i) and (ii) results in an allowable collection J' and $x'_0, \dots, x'_{2f-1}$ with $\psi' = \psi$. After such a move, the set T for J' is strictly smaller than it was for J. The result follows. □

5. Descent data on strongly divisible modules and Galois types

For this section only, let $F/\mathbb{Q}_p$ be a finite extension. Suppose that K/F is a tamely ramified Galois extension with ramification index $e(K/F)$. Suppose moreover that there exists a uniformiser $\pi \in \mathcal{O}_K$ with $\pi^{e(K/F)} \in L$, where L is the maximal unramified extension of F contained in K; then $K = L(\pi)$ and L contains all of the $e(K/F)$th roots of unity. Let k denote the residue field of K (also equal to the residue field of L), and K_0 the maximal unramified extension of $\mathbb{Q}_p$ contained in K. Let $E/\mathbb{Q}_p$ be a finite extension.

Suppose that ρ is a potentially Barsotti–Tate representation $G_F \to \mathrm{GL}_n(E)$ that becomes Barsotti–Tate over K. We assume as usual that K_0 embeds into the coefficients E.

Write $D := D^K_{\mathrm{st},2}(\rho)$ and let $\mathcal{N}$ be a strongly divisible module with descent data over $S := S_{K,\mathcal{O}_E}$ contained in $S[1/p] \otimes_{K_0 \otimes_{\mathbb{Q}_p} E} D$.

Let τ be the inertial Galois type of ρ. Note that τ factors through $I_L/I_K \cong \mathrm{Gal}(K/L)$. Since $\mathrm{Gal}(K/L)$ is abelian, τ decomposes as a direct sum of n characters $\chi_i : I_L \to \mathcal{O}_E^\times$, and we use the isomorphism $I_L/I_K \cong \mathrm{Gal}(K/L)$ to identify each χ_i as a character of $\mathrm{Gal}(K/L)$.

Proposition 5.1. *We have $\tau = \chi_1 \oplus \cdots \oplus \chi_n$ if and only if there is an S-basis $v_1, \ldots, v_n$ of $\mathcal{N}$ such that the descent data acts on $\mathcal{N}$ via $\widehat{g} \cdot v_i = (1 \otimes \chi_i(g))v_i$ for all $g \in \mathrm{Gal}(K/L)$.*

Proof. For each embedding $\sigma : K_0 \to E$, let e_σ denote the corresponding idempotent in $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, so that $S \cong \bigoplus_\sigma e_\sigma S$ with each $e_\sigma S$ a local domain. Since $\widehat{g}$ fixes each e_σ, we see that $\widehat{g}$ acts separately on each $e_\sigma \mathcal{N}$.

Suppose we know that $\mathcal{N}$ has an S-basis $v'_1, \ldots, v'_n$ on which $\widehat{g} \cdot v'_i = \psi_i(g) v'_i$ for some characters $\psi_i : \mathrm{Gal}(K/L) \to (W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E)^\times$. The argument in the first three paragraphs of [GS, proof of Proposition 6.6] proves that D has a $K_0 \otimes_{\mathbb{Q}_p} E$-basis on which $\widehat{g}$ acts via the maps ψ_i. This is enough for the 'if' direction; for the 'only if' direction, recall that by definition of τ, we know that D also has a $K_0 \otimes_{\mathbb{Q}_p} E$-basis on which $\widehat{g}$ acts via the maps $1 \otimes \chi_i$. Since $K_0 \otimes_{\mathbb{Q}_p} E$ is not a domain it may not quite be the case that $\psi_i = 1 \otimes \chi_i$, but at least for each σ the multiset $\{e_\sigma \psi_i\}$ is equal to the multiset $\{\chi_i\}$: that is, we may relabel $e_\sigma v'_1, \ldots, e_\sigma v'_n$ as $v_{1,\sigma}, \ldots, v_{n,\sigma}$ in such a way that $\widehat{g} \cdot v_{i.\sigma} = \chi_i(g) v_{i,\sigma}$. If we define $v_i = \sum_\sigma v_{i,\sigma}$ then $v_1, \ldots, v_n$ is the desired basis. Thus we are reduced to the statement at the beginning of the paragraph.

In particular it is enough to show for each σ that the free $e_\sigma S$-module $e_\sigma \mathcal{N}$ of rank n has a $e_\sigma S$-basis on which $\widehat{g}$ acts by characters $\mathrm{Gal}(K/L) \to \mathcal{O}_E^\times$. Let $e_\sigma S_0$ be the subring of $e_\sigma S$ consisting of power series in $u^{e(K/F)}$. Observe that $e_\sigma S$ is *free* of rank $e(K/F)$ as an $e_\sigma S_0$-module, with basis $1, \ldots, u^{e(K/F)-1}$; this is because $e(K/F)$ divides the absolute ramification index of K, so that if p^α exactly divides u^{m-1} in S and a larger power of p divides u^m, then m is divisible by $e(K/F)$.

We now regard $e_\sigma \mathcal{N}$ as a free $e_\sigma S_0$-module of rank $e(K/F)n$. Note that $\mathrm{Gal}(K/L)$ acts trivially on $e_\sigma S_0$, so that $e_\sigma \mathcal{N}$ is actually a $\mathrm{Gal}(K/L)$-representation over $e_\sigma S_0$. Since $\mathrm{Gal}(K/L)$ is abelian and $p \nmid \#\mathrm{Gal}(K/L)$, and since $\mathcal{O}_E^\times$ contains the $e(K/F)$th roots of unity, the module $e_\sigma \mathcal{N}$ actually has a simultaneous $e_\sigma S_0$-basis of eigenvectors $y_1, \ldots, y_{e(K/F)n}$ for the action of $\mathrm{Gal}(K/L)$. Relabel the elements y_i so that $y_1, \ldots, y_n$ are a basis for the k_E-vector space $e_\sigma \mathcal{N}/(\mathfrak{m}_{e_\sigma S}) e_\sigma \mathcal{N}$; here $\mathfrak{m}_{e_\sigma S}$ is the maximal ideal of $e_\sigma S$ and k_E is the residue field of E. By Nakayama's lemma [Mat89, Theorem 2.3], the elements $y_1, \ldots, y_n$ are the desired $e_\sigma S$-basis of $e_\sigma \mathcal{N}$. □

The following corollary is immediate.

Corollary 5.2. *Let $\overline{\mathcal{N}}$ denote the Breuil module with descent data corresponding to the π_E-torsion in the Barsotti–Tate group corresponding to $\rho|_{G_K}$. If $\tau = \chi_1 \oplus \cdots \oplus \chi_n$, then $\overline{\mathcal{N}}$ has a $(k \otimes k_E)[u]/(u^{e'p})$-basis $\overline{v}_1, \ldots, \overline{v}_n$ such that $\widehat{g} \cdot \overline{v}_i = (1 \otimes \overline{\chi}_i(g))\overline{v}_i$ for all $g \in \mathrm{Gal}(K/L)$. (Here e' is the absolute ramification index of K.)*

Let $e_i \in k \otimes k_E$ be one of our usual idempotents. Since the descent data fixes e_i, we see in particular that the descent data acts via $\overline{\chi}_1 \oplus \cdots \oplus \overline{\chi}_n$ on each piece $e_i \overline{\mathcal{N}}$ of $\overline{\mathcal{N}}$.

6. Local lifts

6.1 Lifts of certain rank-two Breuil modules

We continue to use the following notation from §§ 4.2–4.4. Let L be the unramified quadratic extension of $F_\mathfrak{p}$, and K the splitting field of $u^{p^{2f}-1} - \pi_\mathfrak{p}$ over L. Let ϖ be a choice of $\pi_\mathfrak{p}^{1/(p^{2f}-1)}$ in K. Let k_0 and k denote the residue fields of $F_\mathfrak{p}$ and K respectively. If $g \in \mathrm{Gal}(K/F_\mathfrak{p})$ then we define $\eta(g) = g(\varpi)/\varpi \in W(k)$, so that $\overline{\eta}(g)$ is the image of $\eta(g)$ in k. Suppose that $F_\mathfrak{p}$ has absolute ramification index e, and write $e' = (p^{2f}-1)e$.

Let $E_0(u)$ be an Eisenstein polynomial for $\pi_\mathfrak{p}$, so that $E(u) = E_0(u^{p^{2f}-1})$ is an Eisenstein polynomial for ϖ. Write $E(u) = u^{e'} + pF(u)$; then $F(u)$ is a polynomial in $u^{p^{2f}-1}$ over $W(k_0)$ whose constant term is a unit.

Let E, a finite extension of $\mathbb{Q}_p$, denote the coefficient field for our representations, with integer ring $\mathcal{O}_E$ and maximal ideal $\mathfrak{m}_E$. Enlarging E if necessary, we assume that a Galois closure of K embeds into E. In particular E is ramified and $W(k)$ embeds into E. Let k_E denote the residue field of E. Write $S = S_{K,\mathcal{O}_E}$ (notation as in [Sav05, § 4]). Recall that $\phi : S \to S$ is the $W(k)$-semilinear, $\mathcal{O}_E$-linear map sending u to u^p. The group $\mathrm{Gal}(K/F_\mathfrak{p})$ acts $W(k)$-semilinearly on S via $g \cdot u = (\eta(g) \otimes 1)u$. Set $c = (1/p)\phi(E(u)) \in S^\times$. Let $\varphi \in \mathrm{Gal}(K/F_\mathfrak{p})$ denote the element fixing ϖ and acting non-trivially on L, so that $\varphi^{-1} g \varphi = g^q$ for $g \in \mathrm{Gal}(K/L)$.

We now define a rank-two Breuil module $\overline{\mathcal{N}}$ over $(k \otimes k_E)[u]/(u^{e'p})$ with descent data from K to $F_\mathfrak{p}$ with generators $\bar{v}$ and $\bar{w}$, as follows. Choose $J \subset \{0, \dots, 2f-1\}$, and set $\chi_i = \overline{\chi}$ if $i \in J$ and $\chi_i = \overline{\chi}^q$ otherwise. For each i, choose $r_i \in [0, e']$ such that $\chi_{i+1} = \chi_i \eta_i^{r_i}$. (This is equivalent to choosing an allowable x_i for J and $\overline{\chi}$.) Set $r_i' = e' - r_i$ and $\chi_i' = \chi_i^q$, and note that $\chi_{i+1}' = \chi_i' \eta_i^{r_i'}$ since each r_i is divisible by $q-1$. We define $\overline{\mathcal{N}}$ as follows.

- The submodule $\mathrm{Fil}^1 \overline{\mathcal{N}}$ is generated by $\sum_{i=0}^{2f-1} u^{r_i} e_i \bar{v}$ and $\sum_{i=0}^{2f-1} u^{r_i'} e_i \bar{w}$.
- We have $\phi_1(u^{r_i} e_i \bar{v}) = (1 \otimes \gamma_i) e_{i+1} \bar{v}$ and $\phi_1(u^{r_i'} e_i \bar{w}) = (1 \otimes \gamma_i') e_{i+1} \bar{w}$.
- We have $\widehat{g}(e_i \bar{v}) = (1 \otimes \chi_i(g))(e_i \bar{v})$ and $\widehat{g}(e_i \bar{w}) = (1 \otimes \chi_i(g)^q)(e_i \bar{w})$ for $g \in \mathrm{Gal}(K/L)$.

Here $\gamma_i, \gamma_i' \in k_E^\times$. Finally, we assume that one of the following two sets of additional conditions holds:

$$\begin{cases} i \in J \text{ if and only if } i + f \notin J, \\ \chi_{i+f} = \chi_i^q, \\ r_i = r_{i+f} \text{ and } r_i' = r_{i+f}', \\ \gamma_i = \gamma_{i+f} \text{ and } \gamma_i' = \gamma_{i+f}', \\ \widehat{\varphi}(\bar{v}) = \bar{v} \text{ and } \widehat{\varphi}(\bar{w}) = \bar{w}; \end{cases} \tag{RED}$$

or

$$\begin{cases} i \in J \text{ if and only if } i + f \in J, \\ \chi_{i+f} = \chi_i, \\ r_i = r_{i+f}', \\ \gamma_i = \gamma_{i+f}', \\ \widehat{\varphi}(\bar{v}) = \bar{w} \text{ and } \widehat{\varphi}(\bar{w}) = \bar{v}. \end{cases} \tag{IRR}$$

The second line in each set of conditions is equivalent to the first. Note that the relation $\widehat{\varphi}^{-1} \circ \widehat{g} \circ \widehat{\varphi} = \widehat{g}^{\,q}$ holds in either case, and so the module $\overline{\mathcal{N}}$ so-defined is indeed a Breuil module with descent data from K to $F_\mathfrak{p}$. Since $r_{i+f}' = e' - r_{i+f}$, in case (IRR) the condition $r_i = r_{i+f}'$ is equivalent to $x_i + x_{i+f} = e$.

Theorem 6.1. *For each Breuil module $\overline{\mathcal{N}}$ as above, the generic fibre $\overline{\rho}$ of $\overline{\mathcal{N}}$ lifts to a parallel potentially Barsotti–Tate representation ρ with inertial type $\chi \oplus \chi^q$.*

Proof. We will show that $\overline{\mathcal{N}}$ lifts to a strongly divisible module $\mathcal{N}$ with $\mathcal{O}_E$-coefficients and tame descent data from K to $F_{\mathfrak{p}}$ such that $\rho' := T_{\mathrm{st},2}^{F_{\mathfrak{p}}}(\mathcal{N})[1/p]$ is a potentially Barsotti–Tate representation with inertial type $\chi \oplus \chi^q$ and with all its pairs of labeled Hodge–Tate weights equal to $\{0, 1\}$. The representation ρ' need not be parallel, but since $\det(\rho')$ is the product of the cyclotomic character, a finite order character of order prime to p, and an unramified character, we may take ρ to be the twist of ρ' by a suitable unramified character with trivial reduction modulo p.

Let $\mathcal{N}$ be the free S-module generated by v and w; as one would imagine, v, w will lift $\bar{v}, \bar{w}$ respectively. Let $e_i \in W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ also denote the idempotent lifting $e_i \in k \otimes_{\mathbb{F}_p} k_E$, and let $\widetilde{\sigma}_i : K_0 \hookrightarrow E$ be the embedding that lifts σ_i. We define $\mathrm{Fil}^1\mathcal{N}$ to be the sum of $(\mathrm{Fil}^1 S)\mathcal{N}$ and the submodules $\mathcal{N}_i'$ of $e_i\mathcal{N}$ defined for each $0 \leqslant i < 2f$ as follows.

If $\chi_i = \chi_{i+1}$, let $g_i h_i$ be a monic factorisation of $\widetilde{\sigma}_i(E_0(u))$ in $\mathcal{O}_E[u]$ such that $\deg(g_i) = r_i/(p^{2f} - 1)$ and $\deg(h_i) = r_i'/(p^{2f} - 1)$. This is where we use our hypothesis that E contains all conjugates of $\pi_{\mathfrak{p}}$ over $\mathbb{Q}_p$. Take $G_i = g_i(u^{p^{2f}-1})$ and $H_i = h_i(u^{p^{2f}-1})$, so that $G_i H_i = \widetilde{\sigma}_i(E(u))$, and let $\mathcal{N}_i'$ be the S-module generated by $(1 \otimes G_i(u))e_i v$ and $(1 \otimes H_i(u))e_i w$.

If $\chi_i \neq \chi_{i+1}$, let $y_i z_i$ be a factorisation of $-p\widetilde{\sigma}_i(F(u))$ in $\mathcal{O}_E[u]$ such that $y_i \in \mathfrak{m}_E$ and $z_i \in \mathfrak{m}_E[u]$ or vice-versa. This is where we use our hypothesis that E is ramified. Take $\mathcal{N}_i'$ to be the S-module generated by $e_i(u^{r_i}v + (1 \otimes y_i)w)$ and $e_i((1 \otimes z_i)v + u^{r_i'}w)$.

We impose the following extra conditions. If $\overline{\mathcal{N}}$ satisfies (RED), then we insist that $G_{i+f} = G_i$, $H_{i+f} = H_i$, $y_{i+f} = y_i$, and $z_{i+f} = z_i$; on the other hand if $\overline{\mathcal{N}}$ satisfies (IRR), then we require $G_i = H_{i+f}$ and $y_i = z_{i+f}$. Note that this is possible because $E(u)$ is defined over $W(k_0)$, so that $\widetilde{\sigma}_{i+f}(E(u)) = \widetilde{\sigma}_i(E(u))$.

We define descent data from K to $F_{\mathfrak{p}}$ on $\mathcal{N}$ as follows, semilinearly with respect to the action of $\mathrm{Gal}(K/F_{\mathfrak{p}})$ on S. Let $\widetilde{\chi}_i$ be the Teichmüller lift of χ_i.

- If $g \in \mathrm{Gal}(K/L)$, set $\widehat{g}(e_i v) = (1 \otimes \widetilde{\chi}_i(g))e_i v$ and $\widehat{g}(e_i w) = (1 \otimes \widetilde{\chi}_i(g)^q)e_i w$.
- If $\overline{\mathcal{N}}$ satisfies (RED) then set $\widehat{\varphi}(v) = v$ and $\widehat{\varphi}(w) = w$.
- If $\overline{\mathcal{N}}$ satisfies (IRR) then set $\widehat{\varphi}(v) = w$ and $\widehat{\varphi}(w) = v$.

In either of the last two cases, using the fact that φ acts trivially on $\mathcal{O}_E$ and takes e_i to e_{i+f}, one checks that $\widehat{\varphi}^{-1}\widehat{g}\widehat{\varphi} = \widehat{g}^{\,q}$, so that this descent data extends to $\mathrm{Gal}(K/F_{\mathfrak{p}})$ in a well-defined way. One checks with little difficulty from the definition of $\mathrm{Fil}^1\mathcal{N}$ and the conditions on $\overline{\mathcal{N}}$ that this descent data preserves $\mathrm{Fil}^1\mathcal{N}$. (Note in particular that $\mathrm{Gal}(K/F_{\mathfrak{p}})$ acts trivially on G_i, H_i, y_i, z_i since they are all polynomials in $u^{p^{2f}-1}$.)

Finally we wish to define a map $\phi : \mathcal{N} \to \mathcal{N}$, semilinear with respect to ϕ on S and such that $\phi_1 = (1/p)\phi|_{\mathrm{Fil}^1\mathcal{N}}$ is well defined and satisfies

$$\phi_1((1 \otimes G_i(u))e_i v) = \widetilde{\gamma}_i e_{i+1} v, \tag{6.2}$$
$$\phi_1((1 \otimes H_i(u))e_i w) = \widetilde{\gamma}_i' e_{i+1} w \tag{6.3}$$

if $\chi_i = \chi_{i+1}$ and

$$\phi_1(e_i(u^{r_i}v + (1 \otimes y_i)w)) = \widetilde{\gamma}_i e_{i+1} v, \tag{6.4}$$
$$\phi_1(e_i((1 \otimes z_i)v + u^{r_i'}w)) = \widetilde{\gamma}_i' e_{i+1} w \tag{6.5}$$

if $\chi_i \neq \chi_{i+1}$. Here $\widetilde{\gamma}_i, \widetilde{\gamma}'_i$ are lifts of γ_i, γ'_i to $\mathcal{O}_E^\times$ that satisfy $\widetilde{\gamma}_{i+f} = \widetilde{\gamma}_i$ and $\widetilde{\gamma}'_{i+f} = \widetilde{\gamma}'_i$ in case (RED) and $\widetilde{\gamma}'_{i+f} = \widetilde{\gamma}_i$ in case (IRR).

If $\chi_i = \chi_{i+1}$ we may satisfy (6.2) and (6.3) by setting

$$\phi(e_i v) = c^{-1}\phi(1 \otimes H_i(u))\widetilde{\gamma}_i e_{i+1} v,$$
$$\phi(e_i w) = c^{-1}\phi(1 \otimes G_i(u))\widetilde{\gamma}'_i e_{i+1} w.$$

If $\chi_i \neq \chi_{i+1}$, then since

$$(E(u) \otimes 1)e_i v = u^{r'_i}(e_i(u^{r_i}v + (1 \otimes y_i)w)) - y_i(e_i((1 \otimes z_i)v + u^{r'_i}w))$$

and similarly for $(E(u) \otimes 1)e_i w$, we should set

$$\phi(e_i v) = c^{-1}e_{i+1}(u^{pr'_i}\widetilde{\gamma}_i v - \phi(y_i)\widetilde{\gamma}'_i w),$$
$$\phi(e_i w) = c^{-1}e_{i+1}(u^{pr_i}\widetilde{\gamma}'_i w - \phi(z_i)\widetilde{\gamma}_i v).$$

Extending this map additively ϕ-semilinearly to all of $\mathcal{N}$, one checks that equations (6.2)–(6.5) hold, so that $\phi(\mathrm{Fil}^1\mathcal{N})$ is contained in $p\mathcal{N}$ and generates it over S. One checks furthermore that ϕ commutes with the descent data on $\mathcal{N}$ that was constructed in preceding paragraphs.

It is now evident that $(\mathcal{N}, \mathrm{Fil}^1\mathcal{N}, \phi)$ with the given descent data is a lift of $\overline{\mathcal{N}}$. It remains to check that $\mathcal{N}$ satisfies the rest of the axioms of a strongly divisible module with coefficients and descent data (namely, conditions (2), (5)–(8), and (12) of [Sav05, Definition 4.1]) and to prove our claims about the representation ρ'.

To check that $\mathrm{Fil}^1\mathcal{N} \cap I\mathcal{N} = I\mathrm{Fil}^1\mathcal{N}$ for an ideal $I \subset \mathcal{O}_E$, observe that it suffices to check separately for each i that $e_i\mathrm{Fil}^1\mathcal{N} \cap e_iI\mathcal{N} = e_iI\mathrm{Fil}^1\mathcal{N}$. If $\chi_i \neq \chi_{i+1}$ then this is follows by exactly the same argument as in the proof of [GS, Theorem 6.5] (the algebraic claim being made is literally identical). If $\chi_i = \chi_{i+1}$ then the argument is even easier. Each coset in $e_i(\mathrm{Fil}^1\mathcal{N}/(\mathrm{Fil}^1 S)\mathcal{N})$ has a representative of the form $e_i(aG_iv + bH_iw)$ with $a, b \in \mathcal{O}_E[u]$ such that $\deg(a) < \deg(H_i)$ and $\deg(b) < \deg(G_i)$: terms of higher degree can be absorbed into $(\mathrm{Fil}^1 S)\mathcal{N}$ by using the relation $E(u) \otimes 1 = 1 \otimes G_iH_i$ in e_iS. If $aG_ie_iv + bH_ie_iw + s_1v + s_2w$ lies in $e_iI\mathcal{N}$ (with $s_1, s_2 \in e_i\mathrm{Fil}^1 S$) then $aG_ie_i + s_1$ must lie in e_iIS; the same must be true of aG_ie_i and s_1 individually since they have no terms in common of the same degree in their unique expansions of the form $\sum_{j \geqslant 0} q_j(u)E(u)^j/j!$ with $\deg(q_j) < \deg(E(u))$. Then since G_i is monic the coefficients of a must lie in I. Similarly $s_2 \in e_iIS$ and the coefficients of b lie in I. It follows that $aG_ie_iv + bH_ie_iw + s_1v + s_2w$ actually lies in $e_iI\mathrm{Fil}^1\mathcal{N}$.

As for the axioms (5)–(8) and (12) of [Sav05, Definition 4.1] concerning the monodromy operator N, again we appeal to arguments in the proof of [GS, Theorem 6.5]: ignoring the action of $\mathcal{O}_E$ and the descent data and regarding $(\mathcal{N}, \mathrm{Fil}^1\mathcal{N}, \phi)$ simply as a strongly divisible $\mathbb{Z}_p$-module over K, it follows from [Bre00, Proposition 5.1.3(1)] that there exists a *unique* $W(k) \otimes \mathbb{Z}_p$-endomorphism $N : \mathcal{N} \to \mathcal{N}$ satisfying axioms (5)–(8) of [Sav05, Definition 4.1], except that we have axiom (5) only with respect to $s \in S_{K,\mathbb{Z}_p}$ until we know that N commutes with the action of $\mathcal{O}_E$. This commutativity, as well as the commutativity between N and the descent data (axiom (12)), follows by uniqueness of the operator N. This completes the proof that $\mathcal{N}$ is a strongly divisible module.

As before, set $\rho' = T_{\mathrm{st},2}^{F_\mathfrak{p}}(\mathcal{N})[1/p]$, the potentially Barsotti–Tate Galois representation associated to $\mathcal{N}$, and let $D := D_{\mathrm{st},2}^K(\rho')$. The claim that the Galois type of ρ' is $\chi \oplus \chi^q$ follows directly by Proposition 5.1 applied with respect to the S-basis $v' = \sum_{i \in J} e_iv + \sum_{i \notin J} e_iw$ and $w' = \sum_{i \notin J} e_iv + \sum_{i \in J} e_iw$ of $\mathcal{N}$.

The last thing to verify is that all pairs of labeled Hodge–Tate weights of ρ' are $\{0,1\}$. Recall that if we regard $K \otimes_{\mathbb{Q}_p} E$ as an $S[1/p]$-algebra via the map $u \mapsto \varpi \otimes 1$, then by [Bre97, Proposition 6.2.2.1] there is an isomorphism

$$f_\varpi : (K \otimes_{\mathbb{Q}_p} E) \otimes_{S[1/p]} \mathcal{N}[1/p] \cong D_K := K \otimes_{W(k)[1/p]} D$$

that identifies the filtrations on both sides. It follows that D_K is the free $(K \otimes_{\mathbb{Q}_p} E)$-module generated by $f_\varpi(v)$ and $f_\varpi(w)$, and we need to show that $\mathrm{Fil}^1 D_K$ is a free submodule of rank one in D_K. It suffices to check for each i that $e_i \mathrm{Fil}^1 D_K$ is a free $e_i(K \otimes_{\mathbb{Q}_p} E)$-submodule of rank one in $e_i D_K$. If $\chi_i \neq \chi_{i+1}$ then this follows as in the last paragraph of the proof of [GS, Proposition 6.6]: the images of $e_i(u^{r_i} v + y_i w)$ and $e_i(z_i v + u^{r'_i} w)$ under f_ϖ are scalar multiples of one another, and each generates a free submodule of rank one in $e_i \mathrm{Fil}^1 D_K$.

For the case $\chi_i = \chi_{i+1}$, we note that each of our idempotents $e_i \in K \otimes_{\mathbb{Q}_p} E$ decomposes as a sum of idempotents e_τ, where τ ranges over the $e(p^{2f}-1)$ embeddings $K \hookrightarrow E$ extending $\widetilde{\sigma}_i$. Since $\varpi \otimes 1 = 1 \otimes \tau(\varpi)$ in $e_\tau(K \otimes_{\mathbb{Q}_p} E)$, we deduce that the e_τ-component of $f_\varpi(G_i(u) e_i v)$ is non-zero precisely for those τ such that the root $\tau(\varpi)$ of $\widetilde{\sigma}_i(E(u))$ is not a root of $G_i(u)$, and similarly the e_τ-component of $f_\varpi(H_i(u) e_i w)$ is non-zero for those τ such that $\tau(\varpi)$ is not a root of $H_i(u)$. It follows that $e_i \mathrm{Fil}^1 D_K$ is free of rank one, generated by the image of $e_i(G_i(u) v + H_i(u) w)$ under f_ϖ. □

7. An explicit description of the set of weights

We maintain the notation of the previous three sections, so that F is totally real and $\mathfrak{p}|p$ is a place of F. The results of §§ 4 and 6 may be combined to give a complete description of when a semisimple two-dimensional mod p representation of $G_{F_\mathfrak{p}}$ admits a parallel potentially Barsotti–Tate lift of type $\chi \oplus \chi^q$ with $\chi \neq \chi^q$. In turn, this furnishes an explicit description of the conjectural set of weights for a global representation whose restriction to each decomposition group above p is semisimple.

Theorem 7.1. *Write $\overline{\chi} = \prod_{i=0}^{2f-1} \eta_i^{c_i}$, with $0 \leqslant c_i \leqslant p-1$. Assume that the local representation $\overline{\rho}$ is semisimple. Then $\overline{\rho}$ has a parallel potentially Barsotti–Tate lift of type $\chi \oplus \chi^q$ if and only if one of the following three possibilities holds.*

(i) *We have $e \geqslant p-1$ and $\det \overline{\rho}|_{I_{F_\mathfrak{p}}} = \epsilon \cdot \overline{\chi}^{q+1}$.*

(ii) *The representation $\overline{\rho} \cong \left(\begin{smallmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{smallmatrix}\right)$ is decomposable, with $(\psi_1\psi_2)|_{I_{F_\mathfrak{p}}} = \epsilon \cdot \overline{\chi}^{q+1}$, and*

$$\psi_1|_{I_{F_\mathfrak{p}}} = \prod_{i \in J} \omega_{\pi(i)}^{c_i} \prod_{i=0}^{f-1} \omega_i^{x_i}$$

where $J \subset \{0, \ldots, 2f-1\}$ is a subset with $i+f \in J$ if and only if $i \notin J$, where the x_i are allowable for J and $\overline{\chi}$ (as in Definition 4.6), and where π is the natural projection from $\mathbb{Z}/2f\mathbb{Z}$ to $\mathbb{Z}/f\mathbb{Z}$.

(iii) *The representation $\overline{\rho}$ is irreducible, and $\overline{\rho}|_{I_{F_\mathfrak{p}}} \cong \left(\begin{smallmatrix} \psi & 0 \\ 0 & \psi^q \end{smallmatrix}\right)$ with*

$$\psi = \prod_{i \in J} \eta_i^{c_i} \prod_{i \notin J} \eta_i^{c_{i+f}} \prod_i \eta_i^{x_i}$$

where $J \subset \{0, \ldots, 2f-1\}$ is a subset with $i+f \in J$ if and only if $i \in J$, and where the x_i are allowable for J and $\overline{\chi}$ (as in Definition 4.6) and satisfy $x_i + x_{i+f} = e$.

Note that by Lemma 4.16 and the observation following its proof, the condition in case (iii) that the x_i are non-negative integers satisfying $x_i + x_{i+f} = e$ may be replaced by the condition that $0 \leqslant x_i \leqslant e$ for all i and $\psi^{q+1}|_{I_{F_\mathfrak{p}}} = \epsilon \cdot \overline{\chi}^{q+1}$.

Proof. The necessity of these conditions follows from Lemma 4.15 and the discussions of §§ 4.3 and 4.4, particularly (4.9) and Proposition 4.13. For their sufficiency, consider first the case that $\overline{\rho}$ is irreducible. Then by the discussion of § 4.4, along with the conditions given in case (iii), there is a Breuil module $\overline{\mathcal{N}}$ as in § 6 so that the generic fibre of $\overline{\mathcal{N}}$ restricted to $I_{F_\mathfrak{p}}$ is $\overline{\rho}|_{I_{F_\mathfrak{p}}}$. Since we are in the irreducible case, the generic fibre of $\overline{\mathcal{N}}$ is an unramified twist of $\overline{\rho}$, and the representation coming from Theorem 6.1 applied to $\overline{\mathcal{N}}$ is an unramified twist of the desired lift of $\overline{\rho}$. This completes case (iii), and case (i) with $\overline{\rho}$ irreducible follows from case (iii) combined with Lemma 4.16.

In the case that $\overline{\rho}$ is reducible, note that case (i) will follow immediately from case (ii) and Proposition 4.10. For case (ii), observe that by the discussion in § 4.3, the generic fibre of the rank one Breuil module $\mathcal{M}$ of § 4.3 agrees with ψ_1 up to an unramified twist; but in fact by Corollary 4.3 the parameters γ_i may be chosen so that these characters agree on the whole group $G_{F_\mathfrak{p}}$. Choosing the parameters γ_i' similarly to suit ψ_2, the Breuil module $\mathcal{M}$ from § 4.3 may be extended to a Breuil module $\overline{\mathcal{N}}$ as in § 6, satisfying the conditions (RED), whose generic fibre is $\overline{\rho}$. The result again follows from Theorem 6.1. □

We now return to the situation where $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a global representation.

Theorem 7.2. *Suppose that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous, and that $\overline{\rho}|_{G_{F_v}}$ is semisimple for each $v|p$. Let $\sigma = \bigotimes_{v|p} \sigma_v$ be a weight. Then $\sigma \in W^?(\overline{\rho})$ if and only if for each $v|p$ we have one of the following.*

(i) *The weight σ_v is of type I, and the conditions of Theorem 7.1 apply with $\mathfrak{p} = v$ and $\chi = \widetilde{\sigma}_v$ (regarded as a character of I_{F_v} by local class field theory).*

(ii) *The weight σ_v is of type II, and*

$$\overline{\rho}|_{I_{F_v}} \cong \sigma_v \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$$

where σ_v is regarded as a character of I_{F_v} via local class field theory.

In particular, if for each $v|p$ the ramification index of F_v is at least $p-1$, and σ_v is of type I for each $v|p$, then $\sigma \in W^?(\overline{\rho})$ if and only if for each $v|p$ we have $\det \overline{\rho}|_{I_{F_v}} = \epsilon \cdot \sigma_v^{q+1}$.

Proof. This all follows immediately from Definition 3.4 and Theorem 7.1, except for the case that σ_v is of type II. In this case, the necessity of the given condition follows from Lemma 3.5, and the sufficiency is straightforward: twisting reduces to the case $\sigma_v = 1$, when the result follows from the existence of a non-crystalline extension of the trivial character by the cyclotomic character. □

8. Proof of the weight conjecture

Recall that we are assuming that F is a totally real field. We now prove in many cases that $W(\overline{\rho}) = W^?(\overline{\rho})$, by combining the results of earlier sections with the lifting machinery of Khare–Wintenberger, as interpreted by Kisin. In particular, we use the following result.

Definition 8.1. Let $v|p$. We say that a representation $\rho : G_{F_v} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is *ordinary* if $\rho|_{I_{F_v}}$ is an extension of a finite order character by a finite order character times the cyclotomic character.

Proposition 8.2. *Suppose that $p>2$ and that $\overline{\rho}:G_F\to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular. Assume that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. If $p=5$ and the projective image of $\overline{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, assume further that $[F(\zeta_p):F]=4$. Suppose that for each place $v|p$, θ_v is an inertial type for I_{F_v} such that $\overline{\rho}|_{G_{F_v}}$ has a non-ordinary parallel potentially Barsotti–Tate lift of type θ_v. Then $\overline{\rho}$ has a modular lift which is parallel potentially Barsotti–Tate of type θ_v for all $v|p$.*

Proof. This is a special case of [Gee, Corollary 3.1.7]. □

Recall that we defined the set of weights $W^?(\overline{\rho})$ conjecturally associated to $\overline{\rho}$ in § 3, and that in the case that the restrictions of $\overline{\rho}$ to decomposition groups above p are semisimple, Theorem 7.2 gives an explicit description of $W^?(\overline{\rho})$.

Theorem 8.3. *Suppose that $p>2$ and that $\overline{\rho}:G_F\to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular. Assume that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. If $p=5$ and the projective image of $\overline{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, assume further that $[F(\zeta_p):F]=4$. Then $W(\overline{\rho})\subset W^?(\overline{\rho})$. If $\sigma\in W^?(\overline{\rho})$, and $\sigma=\bigotimes_{v|p}\sigma_v$ with each σ_v of type I, then $\sigma\in W(\overline{\rho})$. In particular, if there are no places $v|p$ for which $\overline{\rho}|_{G_{F_v}}$ is a twist of an extension of the trivial character by the cyclotomic character, then $W(\overline{\rho})=W^?(\overline{\rho})$.*

Proof. The inclusion $W(\overline{\rho})\subset W^?(\overline{\rho})$ already follows from Lemma 3.3. The rest of the result follows from Lemma 3.3, Proposition 8.2 and Lemma 3.5, because if σ_v is of type I, any lift of type $\widetilde{\sigma}_v\oplus\widetilde{\sigma}_v^{q_v}$ only becomes crystalline over a non-abelian extension, and is thus certainly non-ordinary. □

Remark 8.4. It should be possible to improve this result to prove the equality $W^?(\overline{\rho})=W(\overline{\rho})$ under the assumption that $\overline{\rho}$ has a modular lift of parallel weight two which is ordinary at any place v such that there is an element of $W^?(\overline{\rho}|_{G_{F_v}})$ of type II. This would involve strengthening Proposition 8.2 to include potentially semistable lifts, and the use of $R=T$ theorems for Hida families. The required results are not in the literature in the appropriate level of generality, however.

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Appendix A. Corrigendum to [Sav05]

The second author wishes to take this opportunity to correct an error in [Sav05], as a consequence of which there is one more family of strongly divisible modules that must be studied by the methods of [Sav05]. Once this is done, the remaining claims of [Sav05] are unaffected. We adopt the notation of [Sav05] without further comment, and all numbered references are to that paper.

The mistake is in the statement and proof of Theorem 6.12(4). In the situation of that item, if $m=1+(p+1)j$, i.e., if $i=1$, then the two characters ω_2^{m+p} and ω_2^{pm+1} are both characters of niveau 1, and are equal; hence in this case the proof of Theorem 6.12(4) does *not* show that $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)|_{I_p}$ decomposes as a sum of two conjugate characters. In fact, for each choice c of square root of $\overline{w}$, the map $\mathcal{M}_2'\to\mathcal{M}_E(F_2/\mathbb{Q}_{p^2},e_2,c,m-1)$ extends to a map $\mathcal{M}'\to\mathcal{M}_E(F_2/\mathbb{Q}_p,e_2,c,j)$; by Proposition 5.4(1), we conclude when $i=1$ that

$$T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)\otimes_{\mathbf{k}_E}\overline{\mathbb{F}}_p\cong\lambda_{c^{-1}}\omega^{1+j}\oplus\lambda_{-c^{-1}}\omega^{1+j}.$$

This means that when $i=1$ and $\mathrm{val}(b)>0$ we still need to construct a strongly divisible lattice in $\mathcal{D}_{m,[1:b]}$ whose reduction modulo p has trivial endomorphisms; or, conversely, we need to study deformations of type $\widetilde{\omega}_2^m \oplus \widetilde{\omega}_2^{pm}$ (with $i=1$) of non-split residual representations of the form

$$\begin{pmatrix} \lambda_{c^{-1}}\omega^{1+j} & * \\ 0 & \lambda_{-c^{-1}}\omega^{1+j} \end{pmatrix}.$$

We rectify this omission now. Our statements are numbered to mesh with the original article if one drops the **A.** prefix.

Lemma A.6.7. (2) *If $i=1$, $\mathrm{val}_p(b)>0$, and w is a square in E, then there is $X \in S^{\times}_{F_2,\mathcal{O}_E}$ satisfying*

$$X(1\otimes wb) = 1\otimes w - \left(1+\frac{u^{pe_2}}{p}\right)X\phi(X).$$

Proof. The constant term of X may be taken to be $1\otimes x_0$ where x_0 is either root of $x_0^2+wbx_0-w$ in $\mathcal{O}_E^{\times}$. The recursion for the coefficient x_n of u^n is $x_n(x_0+wb) = \text{lower terms}$, and so the recursion can be solved to obtain $X\in S^{\times}_{F_2,\mathcal{O}_E}$. □

Moreover, since $\mathrm{val}_p(b)>0$, by putting the variable B for b we obtain an element X_B of $S_{F_2,\mathcal{O}_E[[B]]}$ which specialises to X under the map $\mathcal{O}_E[[B]]\to\mathcal{O}_E$ sending B to b. Note that the image of X in $(\mathbb{F}_{p^2}\otimes \mathbf{k}_E)[u]/u^{e_2p}$ is $1\otimes c$ with c a square root of $\overline{w}$. Assume henceforth that the coefficient field E contains a square root of w. Now Proposition 6.10 is modified as follows.

Proposition A.6.10. *In the case $i=1$ and $\mathrm{val}_p(b)>0$, we instead define*

$$\begin{aligned}\mathcal{M}_{m,[1:b]} &= S_{F_2,\mathcal{O}_E}\cdot g_1 + S_{F_2,\mathcal{O}_E}\cdot g_2\\ g_1 &= \mathbf{e}_1 + \frac{X}{pw}u^{p(p-1)}\mathbf{e}_2\\ g_2 &= \mathbf{e}_2,\end{aligned}$$

and this is a strongly divisible $\mathcal{O}_E$-module with descent data inside $\mathcal{D}_{m,[1:b]}$.

Proof. Put $\mathcal{M}=\mathcal{M}_{m,[1:b]}$. Observe that $h := u^{p-1}g_1 + (X/w+(1\otimes b))g_2$ lies in $\mathrm{Fil}^1\mathcal{M}$. Since $X/w+(1\otimes b)$ is a unit in $S_{F_2,\mathcal{O}_E}$ and g_1 does not lie in $\mathrm{Fil}^1\mathcal{M}$, we deduce that $\mathrm{Fil}^1\mathcal{M} = S_{F_2,\mathcal{O}_E}\cdot h + (\mathrm{Fil}^1 S_{F_2,\mathcal{O}_E})\mathcal{M}$. From this it is easy to check that $I\mathcal{M}\cap\mathrm{Fil}^1\mathcal{M} = I\mathrm{Fil}^1\mathcal{M}$. Finally, we compute that

$$\begin{aligned}\phi(g_1) &= \phi(X)u^{p^2(p-1)}g_1 + \left(1 - X\phi(X)\frac{u^{pe_2}}{pw}\right)g_2,\\ \phi(g_2) &= pwg_1 - Xu^{p(p-1)}g_2\end{aligned}$$

both lie in $\mathcal{M}$; using the defining relation for X we find $\phi_1(h) = (1\otimes w)X^{-1}g_1\in\mathcal{M}$ and conclude that $\mathcal{M}$ is a strongly divisible module. □

Now amend Theorem 6.12(4) so that it applies only to the case $i>1$, and add the following.

Theorem A.6.12. (5) *If $i=1$ and $\mathrm{val}_p(b)>0$, then $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)$ is independent of b and*

$$T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E) \cong \begin{pmatrix} \lambda_{-c^{-1}}\omega^{1+j} & * \\ 0 & \lambda_{c^{-1}}\omega^{1+j} \end{pmatrix}$$

with $*\neq 0$.

Proof. Write $\mathcal{M}' = T_0(\mathcal{M}/\mathfrak{m}_E)$. Then $\mathrm{Fil}^1\mathcal{M}'$ is generated by $u^{p-1}g_1 + c^{-1}g_2$ and $u^{e_2}g_1$, with $\phi_1(u^{p-1}g_1 + c^{-1}g_2) = cg_1$ and $\phi_1(u^{e_2}g_1) = u^{p^2(p-1)}cg_1 + g_2$. Note that $\phi_1(u^{p(p-1)}g_2) = -cg_2$. There is evidently a non-trivial map $\mathcal{M}' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, c, j)$ sending g_2 to 0 and g_1 to $u^{p^2}\mathbf{e}$. On the other hand if $f : \mathcal{M}' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, d, n)$ is a non-trivial map sending g_1 to $\alpha\mathbf{e}$ and g_2 to $\beta\mathbf{e}$, then α, β must both be polynomials in u^p since g_1, g_2 are in the image of ϕ_1. Now if $\beta \neq 0$ then the relation $f \circ \phi_1 = \phi_1 \circ f$ on $u^{p(p-1)}g_2$ implies that β is a unit times u^p; but then $f(u^{p-1}g_1 + c^{-1}g_2) \in \langle u^{e_2}\mathbf{e}\rangle$ implies that α has a linear term, a contradiction. Therefore $\beta = 0$, and then it is easy to check that $c = d$ and $j = n$. It follows that $* \neq 0$. □

(We also note the following typographical errors in the published version of the proof of Theorem 6.12(4): in the first sentence, the expression $\phi_1(u^{e_2})$ should be $\phi_1(u^{e_2}g_2)$; in the last sentence, the characters λ_c should both be $\lambda_{c^{-1}}$.)

The proof of Corollary 6.15(2) should then invoke Theorem 6.12(5) in lieu of Theorem 6.12(4) in the case of representations ρ to which Theorem 6.12(5) applies, noting that the two choices for x_0 lead to different reductions of ρ.

We now turn to deformation spaces of strongly divisible modules. The proof of the following proposition is identical to the proof that the corresponding module $\mathcal{M}_{m,[1:b]}$ of Proposition 6.10 is a strongly divisible module. As noted in Remark 6.20, we omit the description of N in the strongly divisible module below.

PROPOSITION A.6.21. *There exists a strongly divisible module with descent data and $\mathcal{O}_E[[B]]$-coefficients as follows.*

(vi) *If $i = 1$ and assuming that w is a square in E,*

$$\begin{aligned}
\mathcal{M}_X &= (S_{F_2,\mathcal{O}_E[[B]]}) \cdot g_1 \oplus (S_{F_2,\mathcal{O}_E[[B]]}) \cdot g_2,\\
\mathrm{Fil}^1\mathcal{M}_X &= S_{F_2,\mathcal{O}_E[[B]]} \cdot (u^{p-1}g_1 + (w^{-1}X_B + (1 \otimes B))g_2) + (\mathrm{Fil}^1 S_{F_2,\mathcal{O}_E[[B]]})\mathcal{M}_X,\\
\phi(g_1) &= \phi(X_B)u^{p^2(p-1)}g_1 + \left(1 - X_B\phi(X_B)\frac{u^{pe_2}}{pw}\right)g_2,\\
\phi(g_2) &= pwg_1 - X_B u^{p(p-1)}g_2,\\
\widehat{g}(g_1) &= (\widetilde{\omega}_2^m \otimes 1)g_1, \quad \widehat{g}(g_2) = (\widetilde{\omega}_2^{pm} \otimes 1)g_2.
\end{aligned}$$

Finally, one must amend the proof of Theorem 6.24 to include a proof that the canonical injection

$$R(2, \tau(\mathcal{M}_X), \overline{\rho}(\mathcal{M}_X))_{\mathcal{O}_E} \to \mathcal{O}_E[[B]]$$

is a surjection; this proceeds exactly along the strategy outlined in the proof of Theorem 6.24. Indeed, let $\mathcal{M}''$ denote the minimal Breuil module with descent data from F_2 to $\mathbb{Q}_p$ associated to the character $\lambda_{-c^{-1}}\omega^{1+j}$, with generator h such that $\phi_1(h) = -c^{-1}h$. Then a map $f : \mathcal{M}'' \to T_0(\mathcal{M}_X/(\mathfrak{m}_E, B^2))$ must send h to an element of the form $\alpha u^{e_2}g_1 + \beta(u^{p-1}g_1 + (w^{-1}X_B + B)g_2)$ (where, abusing notation, we identify elements of $S_{F_2,\mathcal{O}_E[[B]]}$ with their images in $(\mathbb{F}_{p^2} \otimes \mathbf{k}_E[B]/(B^2))[u]/u^{e_2p}$). Write $\alpha = \alpha_0 + B\alpha_1$ and $\beta = \beta_0 + B\beta_1$ to separate out the terms involving B. The relation $f(\phi_1(h)) = \phi_1(f(h))$ shows first that $\alpha_0 = au^p$, $\beta_0 = -au^{p^2}$ for some $a \in \mathbf{k}_E$, by considering the relation modulo B; then, after some algebra, the full relation eventually implies $a = 0$. Thus the image of f lies in $B \cdot T_0(\mathcal{M}_X/(\mathfrak{m}_E, B^2))$, as desired.

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