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ALMOST DIAGONAL SYSTEMS IN ASYMPTOTIC INTEGRATION

by H. GINGOLD

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1. Introduction

Consider the ordinary linear matrix differential system

$$\psi^{-1}(x)X' = A(x)X, X' = \frac{dX}{dx}, \tag{1.1}$$

$\psi(x)$ is a scalar mapping, X and $A(x)$ are n by n matrices. Both belong to $C^l([a,\infty))$ for some integer l. The stability and asymptotic behaviour of its solutions have been subject to much investigation. See Bellman **[2]**, Levinson **[24]**, Hartman and Wintner **[20]**, Devinatz **[9]**, Fedoryuk **[11]**, Harris and Lutz **[16, 17, 18]** and Cassell **[30]**. The special interest in eigenvalue problems and in the deficiency index problem stimulated a continued interest in asymptotic integration. See e.g. Naimark **[36]**, Eastham and Grundniewicz **[10]** and **[8, 9]**. Harris and Lutz **[16, 17, 18]** succeeded in explaining how to derive many known theorems in asymptotic integration by repeatedly using certain "$(I+Q)$" linear transformations.

Officially, the purpose of this note is to contribute a theorem in asymptotic integration which is best possible in a certain sense. Technically and practically our main goal is to present a method producing a certain linear transformation utilizing projections. The method will directly provide a fine bound on the derivative of this linear transformation as well as on the linear transformation, itself. This, when combined with a new simple but delicate lemma on the perturbation of eigenvalues, will produce the desired new result in asymptotic integration.

Before continuing the explanation of the nature of our results and relating them to previous works, we will adopt the following definitions and assumptions.

Let Y be an n by n matrix function. Given the differential system

$$\psi^{-1}(x)Y' = [D(x) + \psi^{-1}(x)V(x) + \psi^{-1}(x)R(x)]Y, \tag{1.2}$$

We adopt the following assumption throughout this work.

Assumption 1. *$D(x)$ is an n by n diagonal matrix and $V(x)$ is an n by n matrix whose diagonal elements are all* 0.

$$D(x) = \operatorname{diag}\{d_1(x)\dots d_n(x)\}, V(x) = (v_{jk}(x)), j,k = 1\dots n, v_{jj}(x) \equiv 0. \tag{1.3}$$

The mapping $\psi(x)$ *does not vanish on* $[a,\infty)$. *The mappings* $\psi(x)$ *and all the entries of* $D(x)$ *and* $V(x)$ *belong to* $C^1([a,\infty))$. *The entries of* $R(x)$ *belong to* $L^1[a,\infty)$.

Assume that

$$\lim_{x\to\infty}\psi^{-1}(x)V(x)=0 \tag{1.4}$$

Moreover, if $V(x)\notin L^1[a,\infty)$, assume

$$\lim_{x\to\infty}(d_j(x)-d_k(x))\neq 0,\ j,k=1\dots n,\ j\neq k. \tag{1.5}$$

We recommend the following nomenclature.

Definition 2. *We say that the differential system* (1.2) *is almost diagonal if a fundamental solution of* (1.2) *is given by*

$$Y=(I+P(x))\exp\int^x\psi(s)D(s)ds \tag{1.6}$$

where

$$P(x)=o(1),\ x\to\infty,\ P(x)\in C^1([\alpha,\infty))\quad \textit{for some}\quad \alpha\geqq a. \tag{1.7}$$

We also need the following assumption on $D(x)$.

Assumption 3. *For each pair of indices* $j,k,\ j\neq k,\ j,k=1\dots n$ *either*

$$\lim_{x\to\infty}\operatorname{Re}\int_a^x\psi(s)(d_j(s)-d_k(s))ds=\infty \ \textit{and}\ \int_{x_1}^{x_2}\psi(s)(d_j(s)-d_k(s))ds\geqq -K \ \textit{if}\ x_2\geqq x_1\geqq a \tag{1.8}$$

or

$$\operatorname{Re}\int_{x_1}^{x_2}\psi(s)(d_j(s)-d_k(s))\,ds\leqq K \quad \textit{for}\quad x_2\geqq x_1\geqq a. \tag{1.9}$$

When $\psi\equiv 1$ Assumption 3 coincides with conditions imposed by Levinson [**24**]. Let $\psi\equiv 1$. Then, under Assumptions 1, 3 and the additional assumption that $V(x)\in L^1[a,\infty)$ it follows from Levinson [**24**] that (1.2) is an almost diagonal system. By using [**24**] as a model, many works such as [**8, 16, 17, 18**] reduce the system (1.1) to one like (1.2) in which $V(x)\in L^1[a,\infty)$. This is a *final stage* of the asymptotic integration. The reduction is achieved via a sequence of linear transformations

$$X=T_1(x)\dots T_k(x)Y, \tag{1.10}$$

where k is an integer, $T_1(x)\dots T_k(x)$ are certain "computable" n by n matrices and $d_1(x)\dots d_n(x)$ are certain "computable" mappings which may coincide with certain eigenvalues. The larger the value of k, the more labour needs to be invested in the

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calculations. A particularly tedious part of the work is the calculation of the eigenvalues of certain n by n matrices and the verification of Assumption 3. It is worth noticing that in works **[16, 17, 18]** Assumption 3 is imposed on a system derived from (1.1). It is not always clear what conditions are to be imposed on the elements of $A(x)$ to guarantee that Assumption 3 will hold. In the case of a *sequence* of linear transformations being applied to (1.1), Assumption 3 is not guaranteed to hold for the reduced system if the eigenvalues of $A(x)$ satisfy Assumption 3.

We have an exception in the case where for all j, k,

$$|\operatorname{Re}(\lambda_j(x)-\lambda_k(x))| \geqq \mu > 0, j, k = 1 \ldots n, j \neq k, \tag{1.11}$$

μ is a constant independent of x. Then, if (1.11) holds with $\lambda_j(x)$ the eigenvalues of $A(x)$, under fairly general conditions, (1.11) will also hold with $d_j(x)$ from the reduced system (1.2) replacing $\lambda_j(x)$. This in turn will guarantee the validity of Assumption 3. Therefore, it is important to recognize, as soon as possible, an almost diagonal system. Failing to do so will result in the *additional labour of computing* the eigenvalues of $(D(x)+\psi^{-1}(x)V(x))$ and finding out if Assumption 3 holds for a newly derived differential system.

The above has provided a major source of motivation for us to obtain a finer estimate on the perturbation of certain eigenvalues and a finer estimate on the derivative of a certain linear transformation. Those estimates enable us to recognize an almost diagonal system under new conditions which have not been encountered in the literature.

The moral of this analysis, induced from Gingold **[15]**, is that the diagonal elements of $[D(x)+\psi^{-1}(x)V(x)]$ play a different role than the off diagonal elements. Moreover, it is an *interrelation* between $v_{jk}(x)$ and $\psi(x)(d_j(x)-d_k(x))$, $j, k = 1 \ldots n$, $j \neq k$, which plays a crucial role in the asymptotic integration.

We also offer a formal generalization: The matrix $\psi(x)A(x)$ need not be bounded on $[a, \infty)$. The eigenvalues of $\psi(x)A(x)$ need not be distinct at $x=\infty$.

The order of this work is as follows. We will prove the main theorem in the following section. Conditions which guarantee that (1.2) is almost diagonal will be formulated. In Section 3, we will demonstrate that our result is "best possible" in a certain sense. We will also provide a few remarks and comparisons.

In the sequel, we will use an appropriate norm on matrices which will be denoted by $\| \ \|$. The matrix I will denote the identity matrix. Occasionally, we will replace the mapping $\psi(x)$ and matrix functions $V(x), R(x)$ etc. by ψ, V, R etc., respectively.

Let us proceed to the next section.

2. A theorem.

We formulate our theorem:

Theorem 2.1. *Let assumptions* 1, 3 *hold. Assume, also, that*

$$(\|(\psi^{-1}V)'\| + \|D'\| \, \|\psi^{-1}V\|) \in L^1[a, \infty) \tag{2.1}$$

Let

$$\psi^{-1}(x)\Sigma|v_{ik}(x)v_{lm}(x)|\in L^1[a,\infty), \tag{2.2}$$

where the summation is taken over the indices $i, k, l, m=1\ldots n$, $i\neq k$, $l\neq m$, $i\neq l$, $k\neq m$. *Then,* (2.1) *is an almost diagonal system.*

Proof. We will need a few lemmas.

Lemma A. *Denote by* $\lambda_j(x)$, $j=1\ldots n$ *the eigenvalues of* $(D(x)+\psi^{-1}(x)V(x))$. *Then, there exists a number* $\alpha\geqq a$ *such that for all* $j=1\ldots n$ *the eigenvalues* $\lambda_j(x)$ *are all distinct for* $x\geqq\alpha$. *Moreover,*

$$\lambda_j(x)-d_j(x)=\psi^{-2}(x)\Sigma v_{ik}(x)v_{lm}(x)\theta_{iklm}(x), i,k,l,m=1\ldots n. \tag{2.3}$$

The sum in (2.3) *is taken over the indices* $i\neq k$, $l\neq m$, $i\neq l$ *and* $k\neq m$. $\theta_{iklm}(x)$ *are certain mappings which are bounded on* $[a,\infty)$ *and also belong to* $C^1[a,\infty)$.

Proof. Consider the characteristic polynomial $c_n(\lambda,x)$

$$c_n(\lambda,x):=|D(x)+\psi^{-1}(x)V(x)-\lambda I|. \tag{2.4}$$

By using the basic definition of a determinant as a sum of $n!$ appropriate terms we have the following identity.

$$c_n(\lambda,x)=\prod_{j=1}^{j=n}(d_j(x)-\lambda)+c_{n-2}(\lambda,x), \tag{2.5}$$

where $c_{n-2}(\lambda,x)$ is the sum of $(n!-1)$ terms.

$$c_{n-2}(\lambda,x)=\sum_{v=1}^{n!-1}\zeta_v(x). \tag{2.6}$$

ζ_v have the form

$$\zeta_v(x)=\psi^{-2}(x)v_{ik}(x)v_{lm}(x)\hat{c}_{n-2}(\lambda,x). \tag{2.7}$$

v_{ik}, v_{lm} are certain entries of V. It is important to realize that the indices in (2.7) satisfy

$$i\neq k, l\neq m, i\neq l, k\neq m. \tag{2.8}$$

The mapping $\hat{c}_{n-2}(\lambda,x)$ is a polynomial of order $n-2$ of the form

$$\hat{c}_{n-2}(\lambda,x)=\sum_{v=0}^{v=n-2}g_v(x)\lambda^{n-v} \tag{2.9}$$

The coefficients $g_v(x)$ in (2.9) are related to the entries of $(D+\psi^{-1}V)$ in an obvious manner. By application of the implicit function theorem in a simple, straightforward manner, we obtain the conclusion of Lemma A.

Next we need the following lemma.

Lemma B. *There exists on* $[\alpha,\infty)$ *an* n *by* n *matrix function* $T(x)$ *with the following properties:*

$T(x)$ *is invertible and continuously differentiable on* $[\alpha,\infty)$.

$$T(x)=I+\Delta(x), \tag{2.10}$$

and

$$(I+\Delta(x))^{-1}(D(x)+\psi^{-1}(x)V(x))(I+\Delta(x))=\Lambda(x)=\{\operatorname{diag}\{\lambda_1(x)\ldots\lambda_n(x)\} \tag{2.11}$$

The following relations hold.

$$\|\Delta'(x)\|=O(m(x)),\|\Delta(x)\|=\int_x^\infty m(t)dt, x\to\infty \tag{2.12}$$

where

$$m(x)=O(\|(\psi^{-1}(x)V(x))'\|+\|D'(x)\|\,\|\psi^{-1}(x)V(x)\|) \tag{2.13}$$

Proof. We proved in Lemma A that for $x\geqq\alpha$ all eigenvalues of $(D+\psi^{-1}V)$ are distinct. Let $\Gamma_j, j=1\ldots n$ be a set of rectifiable closed Jordan curves in the λ-plane such that Γ_j contains $\lambda_j(x)$ in its interior and $\lambda_k(x)$, $k\neq j$ in its exterior, for $x\geqq\alpha$.

Construct

$$P_j(x):=\frac{1}{2\pi i}\oint_{\Gamma_j}[\lambda I-D-\psi^{-1}V]^{-1}d\lambda, j=1\ldots n. \tag{2.14}$$

Then, $P_j(x), j=1\ldots n$ are projection matrices. (See e.g. Riesz and Nagy [**25**] p. 419). Moreover, $P_j(x)$ are continuously differentiable matrix valued functions of x for $x\geqq\alpha$. This is verifiable using

$$\begin{aligned}P_j'(x)&=\frac{1}{2\pi i}\oint_{\Gamma_j}([\lambda I-D-\psi^{-1}V)]^{-1})'d\lambda\\&=\frac{1}{2\pi i}\oint_{\Gamma_j}[\lambda I-D-\psi^{-1}V]^{-1}[D'+(\psi^{-1}V)'][\lambda I-D-\psi^{-1}V]^{-1}d\lambda.\end{aligned} \tag{2.15}$$

In connection with projection matrices we use a differential system proposed by Kato [**22**] and utilized by Coppel [**6**], Gingold [**12**], and Gingold and Hsieh [**13**].

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Consider the initial value problems

$$W_j' = [P_j' P_j - P_j P_j'] W_j,\ W_j(\infty) = I,\ j = 1 \dots n. \tag{2.16}$$

We first intend to show that

$$\|(P_j' P_j - P_j P_j')\| = O(m(x)), x \to \infty, j = 1 \dots n. \tag{2.17}$$

From (2.14) and (2.15) we have

$$-4\pi^2 P_j' P_j = \oint_{\Gamma_j} \oint_{\Gamma_j} I_1 \, d\lambda \, d\mu$$

where

$$I_1 := ([\lambda I - D - \psi^{-1} V]^{-1})'(\mu I - D - \psi^{-1} V). \tag{2.18}$$

We notice that if $D + \psi^{-1} V$ would have been a diagonal matrix then $P_j' P_j - P_j P_j'$ would be identically zero. Therefore, we are going to "expand" the expressions of P_j, P_j' "about their diagonal term."

Put

$$R_{\lambda 0} = [\lambda I - D(x)]^{-1} \tag{2.19}$$

Then,

$$[\lambda I - D - \psi^{-1} V]^{-1} = [I + \sum_{\nu=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^{\nu}] R_{\lambda 0}. \tag{2.20}$$

$\|R_{\lambda 0}\|$ is bounded for λ on the contours of integration Γ_j, $j = 1 \dots n$ and $\psi^{-1}(\infty) V(\infty) = 0$. Therefore, the series in (2.20) are absolutely convergent for $x \geqq \alpha$ and α large enough. Moreover, without loss of generality, we may assume that the series below are absolutely convergent for $x \geqq \alpha$.

$$([\lambda I - D - \psi^{-1} V]^{-1})' = \left[I + \sum_{\nu=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^{\nu} \right] R_{\lambda 0}' + \left[\sum_{\nu=1}^{\infty} [\psi^{-1} R_{\lambda 0} V)^{\nu}]' \right] R_{\lambda 0} \tag{2.21}$$

Let us calculate I_1 in terms of the series in (2.20) and (2.21).

$$I_1 = \left\{ \left[I + \sum_{\nu=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^{\nu} \right] R_{\lambda 0}' + \left[\sum_{\nu=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^{\nu} \right]' R_{\lambda 0} \right\} \cdot \left[I + \sum_{\nu=1}^{\infty} (\psi^{-1} R_{\mu 0} V)^{\nu} \right] R_{\mu 0} = A + B \tag{2.22}$$

where

$$A := \left[I + \sum_{v=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^v\right] R'_{\lambda 0} \left[I + \sum_{v=1}^{\infty} (\psi^{-1} R_{\mu 0} V)^v\right] R_{\mu 0} \tag{2.23}$$

$$A = R'_{\lambda 0} R_{\mu 0} + \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^v\right] R'_{\lambda 0} R_{\mu 0} + R'_{\lambda 0} \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\mu 0} V)^v\right] R_{\mu 0}$$

$$+ \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^v\right] R'_{\lambda 0} \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\mu 0} V)^v\right] R_{\mu 0} \tag{2.24}$$

and

$$B := \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^v]'\right] R_{\lambda 0} \left[I + \sum_{v=1}^{\infty} (\psi^{-1} R_{\mu 0} V)^v\right] R_{\mu 0}. \tag{2.25}$$

In a similar manner we define

$$I_2 := \left[I + \sum_{v=1}^{\infty} (\psi^{-1} R_{\mu 0} V)^v\right] R_{\mu 0} \left\{\left[I + \sum_{v=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^v\right] R'_{\lambda 0} + \sum_{v=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^v]'] R_{\lambda 0}\right\} \tag{2.26}$$

where $I_2 = \hat{A} + \hat{B}$ and

$$\hat{A} := R_{\mu 0} R'_{\lambda 0} + R_{\mu 0} \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^v\right] R'_{\lambda 0} + \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\mu 0} V)^v\right] R_{\mu 0} R'_{\lambda 0}$$

$$+ \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\mu 0} V)^v\right] R_{\mu 0} \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^v\right] R'_{\lambda 0} \tag{2.27}$$

and

$$\hat{B} := \left[I + \sum_{v=1}^{\infty} (\psi^{-1} R_{\mu 0} V)^v\right] R_{\mu 0} \left[\sum_{v=1}^{\infty} (\psi^{-1} R_{\lambda 0} V)^v]'\right] R_{\lambda 0}. \tag{2.28}$$

We split B into two parts. The first, "a linear part" in $\psi^{-1} V$ is to be denoted by B_1.

$$B_1 := R_{\lambda 0} (\psi^{-1} V R_{\lambda 0})' R_{\mu 0} = R_{\lambda 0} (\psi^{-1} V)' R_{\lambda 0} R_{\mu 0} + \psi^{-1} R_{\lambda 0} V R'_{\lambda 0} R_{\mu 0}. \tag{2.29}$$

The second, the "remainder", is to be denoted by B_2,

$$B_2 := (B - B_1) \tag{2.30}$$

Similarly, we will have

$$\hat{B}_1 := R_{\mu 0} R_{\lambda 0} (\psi^{-1} V)' R_{\lambda 0} + \psi^{-1} R_{\mu 0} R'_{\lambda 0} V R_{\lambda 0} \tag{2.31}$$

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and

$$\hat{B}_2=\hat{B}-\hat{B}_1. \tag{2.32}$$

We notice that by the residuum theorem

$$\oint_{\Gamma_j} R'_{\lambda 0}\,d\lambda=0,\ j=1\ldots n. \tag{2.33}$$

Therefore, the contributions to $P'_jP_j-P_jP'_j$ from the first and third terms on the r.h.s. of (2.24) and (2.27) are 0.

We observe that for $x\geqq\alpha$, without loss of generality, we have for some constant $\rho<1$

$$\|\psi^{-1}R_{\lambda 0}V\|\leqq\|R_{\lambda 0}\|\,\|\psi^{-1}V\|\leqq\rho<1. \tag{2.34}$$

Then, it can easily be verified that for $x\geqq\alpha$

$$\left\|\sum_{v=1}^{\infty}(\psi^{-1}R_{\lambda 0}V)^v\right\|\leqq\|\psi^{-1}R_{\lambda 0}V\|(1-\rho)^{-1} \tag{2.35}$$

and

$$\left\|\sum_{v=1}^{\infty}[(\psi^{-1}R_{\lambda 0}V)^v]'\right\|\leqq\|(\psi^{-1}R_{\lambda 0}V)'\|(1-\rho)^{-1} \tag{2.36}$$

Thus we conclude that by virtue of (2.23), (2.27) and (2.33),

$$\left\|\oint_{\Gamma_j}\oint_{\Gamma_j}(A-\hat{A})\,d\lambda\,d\mu\right\|=\sigma(\|D'\|\,\|\psi^{-1}V\|),x\to\infty,j=1\ldots n. \tag{2.37}$$

We also conclude that

$$\oint_{\Gamma_j}\oint_{\Gamma_j}(\|B_2\|+\|\hat{B}_2\|)\,|d\lambda\,d\mu|=\sigma(\|(\psi^{-1}V)\|\,\|\psi^{-1}V\|),x\to\infty,j=1\ldots n. \tag{2.38}$$

The estimates on the second terms on the r.h.s. of (2.29) and (2.31) are

$$\begin{aligned}&\oint_{\Gamma_j}\oint_{\Gamma_j}\|\psi^{-1}R_{\lambda 0}VR'_{\lambda 0}R_{\mu 0}\|\,|d\lambda\,d\mu|+\|\psi^{-1}R_{\mu 0}R_{\lambda 0}VR_{\lambda 0}\|\,|d\lambda\,d\mu|\\&\qquad=\sigma(\|D'\|\,\|\psi^{-1}V\|),x\to\infty,j=1\ldots n.\end{aligned} \tag{2.39}$$

Thus, the validity of (2.13) is established.

Let us return to the initial value problems (2.16). Because of Assumption 1, they possess continouously differentiable solutions on $x\geqq\alpha$. It is an easy exercise to verify that for each $j, j=1\ldots n, P_j(\infty)$ is a projection matrix which possesses 1 in its (j,j) place and zero entries elsewhere. By repeating the arguments in Gingold **[12]** we obtain

$$W_j^{-1}(\infty)P_j(\infty)W_j(\infty)=P_j(\infty). \tag{2.40}$$

This implies that for $x \geqq \alpha$

$$W_j^{-1}(x)P_j(x)W_j(x)=P_j(\infty). \tag{2.41}$$

Consider the matrices

$$Z_j(x)=W_j^{-1}(x)[D(x)+\psi^{-1}(x)V(x)]W_j(x). \tag{2.42}$$

Let

$$Z_j(x)=(z_{jik}), i, k=1\dots n.$$

We will show that $Z_j(x)$ commutes with $P_j(\infty)$, $j=1\dots n$. By (2.42), we have

$$\begin{aligned}Z_j(x)P_j(\infty)&=W_j^{-1}(x)[D(x)+\psi^{-1}(x)V(x)]W_j(x)P_j(\infty)\\&=W_j^{-1}(x)[D(x)+\psi^{-1}(x)]P_j(x)W_j(x).\end{aligned} \tag{2.43}$$

Since $P_j(x)$ commutes with $[D(x)+\psi^{-1}(x)V(x)]$, we also have

$$\begin{aligned}Z_j(x)P_j(\infty)&=W_j^{-1}(x)P_j(x)[D(x)+\psi^{-1}(x)V(x)]W_j(x)\\&=P_j(\infty)W_j^{-1}(x)[D(x)+\psi^{-1}(x)V(x)]W_j(x)=P_j(\infty)Z_j(x).\end{aligned} \tag{2.44}$$

The relation (2.44) holds iff

$$z_{jij}=z_{jjk}=0, i, k=1\dots n, i\neq j, k\neq j. \tag{2.45}$$

This could easily be shown by partitioning the matrix Z_j into the following blocks.

$$Z_j=\begin{bmatrix} z_{11} & z_{12} & z_{13}\\ z_{21} & z_{22} & z_{23}\\ z_{31} & z_{32} & z_{32}\end{bmatrix} \tag{2.46}$$

$z_{22}:=z_{jjj}(x)$ is a one by one block and the remaining blocks are defined in an obvious manner. We then partition the matrix $P_j(\infty)$ into blocks, similar to the blocks into which Z_j is partitioned.

The matrices Z_j and $P_j(\infty)$ commute. From their commutation there result nine equations. They yield the desired conclusion.

We notice that $P_j(x)$ is the projection which takes the vector space on which $(D(x)+\psi^{-1}(x)V(x))$ operates into the subspace spanned by the eigenvector corresponding to $\lambda_j(x)$. Therefore,

$$z_{jjj}(x)=\lambda_j(x). \tag{2.47}$$

The j-th column of $W_j(x)$ is an eigenvector corresponding to $\lambda_j(x)$. This is manifested in the following relation.

$$[D(x)+\psi^{-1}(x)V(x)]W_j(x)P_j(\infty)=W_j(x)[Z_j(x)P_j(\infty)]. \tag{2.48}$$

Notice that the only possible non-zero element in $Z_j(x)P_j(\infty)$ is $z_{jjj}(x)$.

Consider the matrix $T(x)$ given by

$$T(x):=\sum_{j=1}^{j=n} W_j(x)P_j(\infty). \tag{2.49}$$

Then,

$$\begin{aligned}\{D(x)+\psi^{-1}(x)V(x)]T(x)&=\sum_{j=1}^{j=n}[D(x)+\psi^{-1}(x)V(x)]W_j(x)P_j(\infty)\\ &=\sum_{j=1}^{j=n} W_j(x)Z_j(x)P_j(\infty)=\sum_{j=1}^{j=n} W_j(x)P_j(\infty)\Lambda(x)=T(x)\Lambda(x).\end{aligned} \tag{2.50}$$

Since $\lambda_j(x)$ are distinct, $T(x)$ must be invertible and (2.11) is proved.

We now proceed to find estimates on $\Delta(x)$ defined by

$$\Delta(x):=T(x)-I \tag{2.51}$$

and on its derivative. We notice that

$$\Delta(x)=\sum_{j=1}^{j=n}[W_j(x)-I]P_j(\infty). \tag{2.52}$$

Denote by $g_j(x)$

$$g_j(x):=\|P_j'(x)P_j(x)-P_j(x)P_j'(x)\|. \tag{2.53}$$

Then, from (2.16) we obtain

$$\|W_j(x)-I\|\leqq\int_x^\infty g_j(t)\|W_j(t)-I\|\,dt+\int_x^\infty g_j(t)\,dt. \tag{2.54}$$

By Gronwall's generalized lemma (see e.g. Hille [**21**] p. 12), we obtain

$$\|W_j(x)-I\|\leqq\int_x^\infty g_j(t)\exp\int_x^\infty g_j(s)\,ds)\,dt=[\exp\int_x^\infty g_j(t)\,dt-1]=O(m(x)),\,x\to\infty,\,j=1\ldots n. \tag{2.55}$$

From (2.16) we have

$$\|(W_j(x)-I)'\|=o(m(x)),\,x\to\infty,\,j=1\ldots n. \tag{2.56}$$

Combining (2.55) and (2.56) in (2.52) yields the relation (2.12).

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For the final part of this theorem let

$$Y = T(x)Z. \tag{2.57}$$

Then, the transformation (2.57) takes the differential system (1.2) into

$$Z' = [\tilde{D} + \tilde{R}]Z. \tag{2.58}$$

$\tilde{D}$ is defined by

$$\tilde{D} := \operatorname{diag}[\psi\Lambda - T^{-1}T' + T^{-1}RT] = \operatorname{diag}\{\tilde{d}_1(x) \ldots \tilde{d}_n(x)\} \tag{2.59}$$

and

$$\tilde{R} := \Psi\Lambda - T^{-1}T + T^{-1}RT - \tilde{D}. \tag{2.60}$$

Because of (2.1) and of (2.2) we have

$$|d_j - \tilde{d}_j| \in L^1[\alpha, \infty), j = 1 \ldots n. \tag{2.61}$$

Therefore, Assumption 3 also holds for $\tilde{d}_j$ replacing d_j, $j = 1 \ldots n$. From Levinson [**24**] or some other method, it follows that (2.58) is an almost diagonal system.

Therefore,

$$\tilde{Y} = (I + \Delta(x))Z = (I + \Delta(x))(I + Q(x)) \exp \int_\alpha^x \tilde{D}(t)\, dt \tag{2.62}$$

is a fundamental solution of (1.2) with $Q(\infty) = 0$. Since

$$\int_x^\infty \|\tilde{D}(t) - D(t)\| = \sigma(1), x \to \infty \tag{2.63}$$

Y given by

$$Y = \tilde{Y} \exp \int_\alpha^\infty (D(t) - \tilde{D}(t))\, dt \tag{2.64}$$

is also a fundamental solution of (1.2). Define now

$$P(x) := (I + \Delta(x))(I + Q(x))(\exp \int_\alpha^\infty (D(t) - \tilde{D}(t))\, dt) - I \tag{2.65}$$

and conclude that (1.2) is an almost diagonal system.

Remarks. Let $A(x) \in C^1[a, \infty]$ possess distinct eigenvalues $\lambda_1(x) \ldots \lambda_n(x)$. Then the method of construction in Lemma B guarantees the existence of an invertible matrix function $T(x) \in C^1[a, \infty]$ such that

$$T^{-1}AT = \operatorname{diag}\{\lambda_1 \ldots \lambda_n\}. \tag{2.66}$$

It is worth noticing that the employment of a *nonlinear* equation, as suggested by [**16**], may *shrink* the domain of validity of the transformation $T(x)$ to an interval $[\hat{a}, \infty]$, with $\hat{a} > a$. Theoretically, this may be of little significance. Practically, it may become a handicap in applications. Therefore, we emphasize the contrast between our linear method and the nonlinear method in [**16**]. Moreover, thanks to the *linearity*, the *delicate estimates* (2.12), (2.13) became available in a relatively transparent way. Nonlinear equations are not always that easy to work with. At this stage, it is not trasnparent how (2.12), (2.13) follow from [**16**], [**18**]. The differential equation (2.16) *simultaneously* provides the properties of $W_j(x)$ and its derivative for $j = 1 \ldots n$. Moreover, the differential equation directed us to look for bounds on $P'_j P_j$. These turned out to yield a bound on T' sharper than the one we encountered in the literature. The bound on T' is crucial as it determined the bound on the perturbation term $T^{-1}T'$ in (2.58). This was essential in determining whether (2.58) is an almost diagonal system or not. There are many methods in the literature which provide the construction of a diagonalizing matrix like $T(x)$. To mention a few, see Levinson [**24**], Coppel [**5**], p. 111, Fedoryuk [**11**], Devinatz [**8**], Harris and Lutz [**16**] and Cassell [**30**]. We liked this method because of the reasons mentioned above and because the method clarifies how the properties of T and its derivative are inherited from $(D + \psi^{-1}V)$ and its derivative.

In the next section we plan to demonstrate the "sharpness" of Theorem 2.1.

3. Sharpness and concluding remarks

Consider the following special case of (1.2). Let $\psi \equiv 1$ and $R \equiv 0$. Let $D(x)$ be defined by $d_j(x) = i^j j(1 + x^{-\theta} \sin x)$, $(i = \sqrt{-1})$, $j = 1 \ldots n, 0 < \theta < 1$. Let $V(x) = (v_{jk}(x))$ be the following n by n matrix.

$$v_{11}(x) \equiv 0, \quad v_{lk}(x) = x^{-\delta}, \quad k = 1 \ldots n, \quad 0 < \delta < \tfrac{1}{2}, \quad \delta + \theta > 1. \tag{3.1}$$

$$v_{jj}(x) \equiv 0, \quad v_{jk}(x) = x^{-\beta}, \quad j, k = 1 \ldots n, \quad \delta + \beta > 1, \quad \theta + \beta > 1. \tag{3.2}$$

Then, Theorem 1.2 guarantees that (1.2) is an almost diagonal system. However, none of the asymptotic methods which I encountered in the literature recognizes the differential system of this example as being almost diagonal. It is not claimed here that *this example* cannot be treated by *modifying* other methods in the literature. In particular, an asymptotic decomposition can be obtained by utilizing an $(I + Q)$ transformation as suggested in [**16**], [**18**]. However, the purpose of our analysis is not just to obtain "some" asymptotic decomposition. Our purpose is to obtain an asymptotic decomposition "as early as possible". From practical considerations rather than from pure theoretical ones, this has a merit of its own. To this end, a "linear method" which is self-contained for the construction of a linear transformation was expounded.

The benefits of Theorem 2.1 can also be appreciated by considering the asymptotic decomposition of solutions of $y^{(n)}=c(x)y$ with $(\ln c(x))' \notin L^1$ $[a,\infty)$. See Gingold [**35**] for treatment of the location of zeros of solutions of $y^{(n)}=c(x)y$.

Next we intend to show that our result cannot be improved. To this end we will consider the differential system

$$Y=\begin{bmatrix}1 & x^{-\delta}\\ x^{-\beta} & -1\end{bmatrix}Y,\quad \delta>0,\beta>0. \tag{3.3}$$

Let $\theta:=\delta+\beta$.

By Levinson's theorem we know that a fundamental solution of (3.3) on $[\alpha,\infty)$ is given by

$$Y_1=(I+\tilde{P}(x))\left(\exp\int_\alpha^x \tilde{D}(t)\,dt\right)(I+P(\alpha))^{-1},\tilde{P}(\infty)=0,D(t):=\operatorname{diag}\{\sqrt{1+t^{-\theta}},-\sqrt{1+t^{-\theta}}\} \tag{3.4}$$

and $\|\tilde{P}(\alpha)\|<1$ for α large enough.

Our theorem guarantees that for $\theta>1$ (3.3) is an almost diagonal system with a fundamental solution

$$Y_2=(I+P(x))\left(\exp\int_\alpha^x D(t)\,dt\right)(I+P(\alpha))^{-1},P(\infty)=0,D(t)\doteq\operatorname{diag}\{1,-1\} \tag{3.5}$$

and $\|\tilde{P}(\alpha)\|<1$ for α large enough. We chose in (3.4) and (3.5) $Y_1(\alpha)=Y_2(\alpha)=I$. Therefore, for $x\geqq\alpha$

$$Y_1(x)=Y_2(x). \tag{3.6}$$

From (3.6) we obtain that

$$(I+M(x))=\left(\exp\int_\alpha^x D(t)\,dt\right)(I+M(\alpha))\left(\exp-\int_\alpha^x \tilde{D}(t)\,dt\right) \tag{3.7}$$

where

$$M(x):=(I+P(x))^{-1}(I+\tilde{P}(x))-I=(m_{jk}(x)),\ j,k=1,2. \tag{3.8}$$

Consider the diagonal element $(1+m_{11}(x))$ on the l.h.s. of (3.7). From the equality (3.7) we have

$$1+m_{11}(x)=\left(\exp-\int_\alpha^x \frac{t^{-\theta}}{1+\sqrt{1+t^{-\theta}}}\,dt\right)(1+m_{11}(\alpha)) \tag{3.9}$$

We let $x\to+\infty$ in (3.9) to obtain a contradiction for $\theta\leqq 1$.

It is worth observing that if (1.2) is such that $\psi(x)=x^{-\alpha}$ with $0<\alpha<1$, Theorem 2.1

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provides asymptotic integration of differential systems with coefficients matrices possessing coalescing eigenvalues. It is well-known that a transformation of the type

$$t=\int^{x}\psi(u)\,du \tag{3.10}$$

will reduce (1.2) to a new differential system with $\psi\equiv 1$. Then one may claim that the setting of (1.2) with arbitrary ψ is superfluous. Indeed, the theory and examples treated in this work could be attained by employing a transformation-like (3.10). However, there are reasons for not doing so.

First, notice that if $\psi^{-1}\equiv x^{q}$ where q is some integer and $A(x)$ has an asymptotic expansion in a sector as $x\to\infty$, then our setting is in harmony with the setting employed in the "analytic theory" of linear differential systems. See e.g. Wasow [**29**] Ch. I–IV.

Secondly, it is hoped that in the future, the linear theory of singular differential systems will be unified. This, to include linear differential systems of asymptotic integration, singularly perturbed linear differential systems and differential systems with moving singularities. For some thoughts in that direction, see Gingold [**31, 32, 33**] and Gingold and Rosenblat [**34**]. The preservation of a general ψ in systems like (1.1) helps to achieve that unification goal.

Thirdly, notice the following. Recently, a successful attack on general 2 by 2 linear differential systems with moving singularities was made. See Gingold [**15**]. The results obtained include as *particular cases* 2 by 2 first order linear differential systems with *multi coalescing turning points*. Part of the success is owed to refraining from transforming the independent variable at an early stage of an asymptotic decomposition. This is in contrast to what some researchers may be tempted to do.

One may single out two main steps in the asymptotic theory of linear singular differential systems. The first step consists of one or more linear transformations of the dependent variable. This is the stage upon which we focused in this work. The second stage is the final stage of asymptotic decomposition in which an attempt is made to prove that a certain linear differential system is almost diagonal.

In regards to the second stage, this work has the same shortcoming as the asymptotic methods of [**24**], [**36**], Vol. II, Ch. VI and e.g. [**7, 16, 30**]. The shortcoming is a result of the requirement that a differential system like (1.2) with $\psi\equiv 1$, be reduced to another one with a coefficient matrix whose off diagonal elements belong to $L^{1}[a,\infty)$ for some $a>0$. This requirement makes some results weaker than they should be. In particular, linear differential systems in asymptotic integration with "turning points" (more general than the ones considered in this article), are not amenable to methods which are sustained by such a requirement. It is hoped that this situation will be corrected in the future.

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Department of Mathematics
West Virginia University
Morgantown WV 26506