Bull. Aust. Math. Soc. **81** (2010), 514–522
doi:10.1017/S0004972709001117

HEAT OPERATORS AND QUASIMODULAR FORMS

MIN HO LEE

(Received 5 October 2009)

Abstract

We introduce a differential operator on quasimodular polynomials that corresponds to the derivative operator on quasimodular forms. We then prove that such a differential operator is compatible with a heat operator on Jacobi-like forms in certain cases. These results show in those cases that the derivative operator on quasimodular forms corresponds to a heat operator on Jacobi-like forms.

2000 *Mathematics subject classification*: primary 11F11; secondary 11F50.

Keywords and phrases: quasimodular forms, modular forms, heat operators, Jacobi-like forms.

1. Introduction

Jacobi-like forms are formal Laurent series which generalize Jacobi forms in some sense, and they correspond to certain sequences of modular forms (see [2, 6]). Quasimodular forms, on the other hand, generalize modular forms (see [4]), and the coefficients of a Jacobi-like form are quasimodular forms. Consequently, there are natural projection maps sending a Jacobi-like form to its coefficients. Derivatives of modular forms are not modular forms in general, and similarly derivatives of Jacobi-like forms are not Jacobi-like forms. On the other hand, derivatives of quasimodular forms are quasimodular forms, and this paper is concerned with an operator on Jacobi-like forms corresponding to the derivative operator on quasimodular forms.

Quasimodular forms for a discrete subgroup Γ of $SL(2, \mathbb{R})$ can be identified with some polynomials, called quasimodular polynomials, that are invariant under certain actions of Γ (see [1]). There is a surjective map from Jacobi-like forms to quasimodular polynomials such that the coefficients of a quasimodular form of degree n are constant multiples of the first $n + 1$ coefficients of the corresponding Jacobi-like form.

In this paper we introduce a differential operator on quasimodular polynomials of a given degree that corresponds to the derivative operator on quasimodular forms. We then prove that such a differential operator is compatible with a heat operator on Jacobi-like forms studied in [5] under the above-mentioned projection map in certain cases. These results show in those cases that the derivative operator on quasimodular forms corresponds to a heat operator on Jacobi-like forms.

2. Formal Laurent series and polynomials

Let $\mathcal{H}$ be the Poincaré upper half-plane, and let R be the ring of holomorphic functions on $\mathcal{H}$. We denote by $R[[X]]$ the complex algebra of formal power series in X with coefficients in R. If δ is an integer, we set

$$R[[X]]_\delta = X^\delta R[[X]], \tag{2.1}$$

so that an element $\Phi(z, X) \in R[[X]]_\delta$ can be written in the form

$$\Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta} \tag{2.2}$$

with $\phi_k \in R$ for each $k \geq 0$. Thus, if we allow δ to be negative, elements of $R[[X]]_\delta$ may be regarded as formal Laurent series in X. We fix a nonnegative integer m and denote by $R_m[X]$ the complex algebra of polynomials in X over R of degree at most m.

The group $SL(2, \mathbb{R})$ acts on the Poincaré upper half-plane $\mathcal{H}$ as usual by linear fractional transformations. Thus we may write

$$\gamma z = \frac{az + b}{cz + d}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$. For the same z and γ, we set

$$\mathfrak{J}(\gamma, z) = cz + d, \quad \mathfrak{K}(\gamma, z) = c\mathfrak{J}(\gamma, z)^{-1} = \frac{c}{cz + d}. \tag{2.3}$$

The map $\mathfrak{J} : SL(2, \mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ determined by the first formula is a well-known automorphy factor satisfying the cocycle condition

$$\mathfrak{J}(\gamma\gamma', z) = \mathfrak{J}(\gamma, \gamma' z)\mathfrak{J}(\gamma', z) \tag{2.4}$$

for $\gamma, \gamma' \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$. The other map, on the other hand, can be shown to satisfy

$$\mathfrak{K}(\gamma\gamma', z) = \mathfrak{J}(\gamma', z)^{-2}\mathfrak{K}(\gamma, \gamma' z) + \mathfrak{K}(\gamma', z). \tag{2.5}$$

Given a function $f \in R$, a formal Laurent series $\Phi(z, X) \in R[[X]]_\delta$ with $\delta \in \mathbb{Z}$, a polynomial $F(z, X) \in R_m[X]$ and an integer λ, we set

$$(f \mid_\lambda \gamma)(z) = \mathfrak{J}(\gamma, z)^{-\lambda} f(\gamma z) \tag{2.6}$$

$$(\Phi \mid^J_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{K}(\gamma, z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X), \tag{2.7}$$

$$(F \parallel_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} F(\gamma z, \mathfrak{J}(\gamma, z)^2 (X - \mathfrak{K}(\gamma, z))) \tag{2.8}$$

https://doi.org/10.1017/S0004972709001117 Published online by Cambridge University Press

for all $\gamma \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$. From (2.4) we see easily that

$$f \mid_\lambda (\gamma\gamma') = (f \mid_\lambda \gamma) \mid_\lambda \gamma'$$

for all $\gamma, \gamma' \in SL(2, \mathbb{R})$. Using (2.4) and (2.5), it can also be shown that

$$\Phi \mid^J_\lambda (\gamma\gamma') = (\Phi \mid^J_\lambda \gamma) \mid^J_\lambda \gamma', \quad (F \parallel_\lambda \gamma) \parallel_\lambda \gamma' = F \parallel_\lambda (\gamma\gamma').$$

We consider the surjective map

$$\Pi^\delta_m : R[[X]]_\delta \to R_m[X] \tag{2.9}$$

with $\delta \in \mathbb{Z}$ defined by

$$(\Pi^\delta_m \Phi)(z, X) = \sum_{r=0}^{m} \frac{1}{r!} \phi_{m-r}(z) X^r \tag{2.10}$$

for an element $\Phi(z, X) \in R[[X]]_\delta$ of the form

$$\Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta}.$$

This map is $SL(2, \mathbb{R})$-equivariant with respect to the operations in (2.7) and (2.8). More precisely, given $\Phi(z, X) \in R[[X]]_\delta$ and $\lambda \in \mathbb{Z}$,

$$\Pi^\delta_m (\Phi \mid^J_\lambda \gamma) = \Pi^\delta_m (\Phi) \parallel_{\lambda+2m+2\delta} \gamma \tag{2.11}$$

for all $\gamma \in SL(2, \mathbb{R})$ (see [1])

Given $\nu \in \mathbb{Z}$, we now consider the formal differential operators

$$\mathcal{D}_\nu : R[[X]] \to R[[X]], \quad \widehat{\mathcal{D}}_\nu : R_m[X] \to R_{m+1}[X]$$

defined by

$$\mathcal{D}_\nu = \frac{\partial}{\partial z} - \nu \frac{\partial}{\partial X} - X \frac{\partial^2}{\partial X^2}, \tag{2.12}$$

$$\widehat{\mathcal{D}}_\nu = \frac{\partial}{\partial z} + X\left(\nu - X \frac{\partial}{\partial X}\right). \tag{2.13}$$

It was noted in [5] that operators of the form $\mathcal{D}_\nu$ correspond to heat operators on Jacobi forms considered by Eichler and Zagier in [3]. Thus $\mathcal{D}_\nu$ may be regarded as a heat operator on formal Laurent series, and it is $SL(2, \mathbb{R})$-equivariant in the sense of the following proposition.

PROPOSITION 2.1. *Given $\lambda, \delta \in \mathbb{Z}$ and a formal Laurent series $\Phi(z, X) \in R[[X]]_\delta$,*

$$(\mathcal{D}_\nu(\Phi) \mid^J_{\lambda+2} \gamma)(z, X) = \mathcal{D}_\nu(\Phi \mid^J_\lambda \gamma)(z, X) + (\lambda - \nu)\mathfrak{K}(\gamma, z)(\Phi \mid^J_\lambda \gamma)(z, X) \tag{2.14}$$

for all $\gamma \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$, where $\mid^J_\lambda$ and $\mid^J_{\lambda+2}$ are as in (2.7). In particular, we obtain

$$(\mathcal{D}_\lambda(\Phi) \mid^J_{\lambda+2} \gamma)(z, X) = \mathcal{D}_\lambda(\Phi \mid^J_\lambda \gamma)(z, X). \tag{2.15}$$

PROOF. This was proved in [5] for $\delta \geq 0$, and the proof of this proposition can be carried out in a similar manner. □

https://doi.org/10.1017/S0004972709001117 Published online by Cambridge University Press

3. Quasimodular and modular forms

Let R, $R[[X]]$, $R[[X]]_\delta$, $R_m[X]$ with $\delta \in \mathbb{Z}$ and $m \geq 0$ be as in Section 2, and let Γ be a discrete subgroup of $SL(2, \mathbb{R})$.

DEFINITION 3.1. Let $|_\xi$, $|_\xi^J$ and $\|_\xi$ with $\xi \in \mathbb{Z}$ be the operations in (2.6), (2.7) and (2.8).

(i) An element $f \in R$ is a *quasimodular form for* Γ *of weight* ξ *and depth at most* m if there are functions $f_0, \ldots, f_m \in R$ such that

$$(f \mid_\xi \gamma)(z) = \sum_{r=0}^{m} f_r(z)\mathfrak{K}(\gamma, z)^r \tag{3.1}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $\mathfrak{K}(\gamma, z)$ is as in (2.3).

(ii) A formal Laurent series $\Phi(z, X) \in R[[X]]_\delta$ with $\delta \in \mathbb{Z}$ is a *Jacobi-like form of weight* ξ *for* Γ if it satisfies

$$(\Phi \mid_\xi^J \gamma)(z, X) = \Phi(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

(iii) A polynomial $F(z, X) \in R_m[X]$ is a *quasimodular polynomial for* Γ *of weight* ξ *and degree at most* m if it satisfies

$$(F \|_\xi \gamma)(z, X) = F(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

If $\xi \in \mathbb{Z}$, we denote by $\mathcal{J}_\xi(\Gamma)_\delta$ the space of all Jacobi-like forms for Γ of weight ξ belonging to $R[[X]]_\delta$. We also denote by $QM_\xi^m(\Gamma)$ the space of quasimodular forms for Γ of weight ξ and depth at most m. The space of all quasimodular polynomials for Γ of weight ξ and degree at most m will be denoted by $QP_\xi^m(\Gamma)$.

If $f \in R$ is a quasimodular form belonging to $QM_\xi^m(\Gamma)$ satisfying (3.1), then we define the corresponding polynomial $(\mathcal{Q}_\xi^m f)(z, X) \in R_m[X]$ by

$$(\mathcal{Q}_\xi^m f)(z, X) = \sum_{r=0}^{m} f_r(z) X^r \tag{3.2}$$

for $z \in \mathcal{H}$. We note that $\mathcal{Q}_\xi^m f$ is well defined because f determines the functions f_k uniquely. Thus we obtain the complex linear map

$$\mathcal{Q}_\xi^m : QM_\xi^m(\Gamma) \to R_m[X]$$

for each $\xi \in \mathbb{Z}$. In fact, this map determines an isomorphism

$$\mathcal{Q}_\xi^m : QM_\xi^m(\Gamma) \to QP_\xi^m(\Gamma) \tag{3.3}$$

whose inverse is given by

$$(\mathcal{Q}_\xi^m)^{-1}(F(z, X)) = F(z, 0) \tag{3.4}$$

for all $F(z, X) \in QP_\xi^m(\Gamma)$ (see [1]).

https://doi.org/10.1017/S0004972709001117 Published online by Cambridge University Press

PROPOSITION 3.2. *Let $\widehat{\mathcal{D}}_\xi$ with $\xi \in \mathbb{Z}$ be as in (2.13), and let $\partial = d/dz : R \to R$ be the derivative operator on R. Then*

$$\mathcal{Q}_{\xi+2}^{m+1} \circ \partial = \widehat{\mathcal{D}}_\xi \circ \mathcal{Q}_\xi^m, \tag{3.5}$$

where $\mathcal{Q}_\xi^m$ and $\mathcal{Q}_{\xi+2}^{m+1}$ are as in (3.2).

PROOF. Let f be a quasimodular form belonging to $QM_\xi^m(\Gamma)$, so that there are holomorphic functions $f_0, f_1, \dots, f_m \in R$ satisfying

$$(f \mid_\xi \gamma)(z) = \mathfrak{J}(\gamma, z)^{-\xi} f(\gamma z) = \sum_{k=0}^m f_k(z)\mathfrak{K}(\gamma, z)^k$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. By taking the derivative of this relation we obtain

$$\begin{aligned} -\xi\mathfrak{J}(\gamma, z)^{-\xi-1}\left(\frac{d}{dz}\mathfrak{J}(\gamma, z)\right) f(\gamma z) + \mathfrak{J}(\gamma, z)^{-\xi} f'(\gamma z)\left(\frac{d}{dz}(\gamma z)\right) \\ = \sum_{k=0}^m \left[f_k'(z)\mathfrak{K}(\gamma, z)^k + k f_k(z)\mathfrak{K}(\gamma, z)^{k-1}\left(\frac{d}{dz}\mathfrak{K}(\gamma, z)\right)\right]. \end{aligned}$$

Using this and the relations

$$\frac{d}{dz}\mathfrak{J}(\gamma, z) = \mathfrak{K}(\gamma, z)\mathfrak{J}(\gamma, z),$$
$$\frac{d}{dz}\mathfrak{K}(\gamma, z) = -\mathfrak{K}(\gamma, z)^2, \qquad \frac{d}{dz}(\gamma z) = \mathfrak{J}(\gamma, z)^{-2},$$

we see that

$$\begin{aligned} -\xi\mathfrak{J}(\gamma, z)^{-\xi}\mathfrak{K}(\gamma, z) f(\gamma z) + \mathfrak{J}(\gamma, z)^{-\xi-2} f'(\gamma z) \\ = \sum_{k=0}^m [f_k'(z)\mathfrak{K}(\gamma, z)^k - k f_k(z)\mathfrak{K}(\gamma, z)^{k+1}]. \end{aligned}$$

Thus

$$\begin{aligned} ((\partial f) \mid_{\xi+2} \gamma)(z) &= \xi\mathfrak{K}(\gamma, z)(f \mid_\xi \gamma)(z) \\ &\quad + \sum_{k=0}^m [f_k'(z)\mathfrak{K}(\gamma, z)^k - k f_k(z)\mathfrak{K}(\gamma, z)^{k+1}] \\ &= \xi\mathfrak{K}(\gamma, z)\sum_{k=0}^m f_k(z)\mathfrak{K}(\gamma, z)^k \\ &\quad + \sum_{k=0}^m [f_k'(z)\mathfrak{K}(\gamma, z)^k - k f_k(z)\mathfrak{K}(\gamma, z)^{k+1}] \\ &= \sum_{k=1}^{m+1}(\xi - k + 1) f_{k-1}(z)\mathfrak{K}(\gamma, z)^k + \sum_{k=0}^m f_k'(z)\mathfrak{K}(\gamma, z)^k. \end{aligned}$$

https://doi.org/10.1017/S0004972709001117 Published online by Cambridge University Press

Hence we see that $\partial f \in QM^{r+1}_{\xi+2}(\Gamma)$ and, using (3.2), we obtain

$$((\mathcal{Q}^{m+1}_{\xi+2} \circ \partial) f)(z) = \sum_{k=0}^{m+1} h_k(z) X^k, \tag{3.6}$$

where

$$h_0 = f_0', \quad h_{m+1} = (\xi - m) f_m, \quad h_k = (\xi - k + 1) f_{k-1} + f_k' \tag{3.7}$$

for $1 \le k \le m$. On the other hand, since

$$(\mathcal{Q}^m_\xi f)(z, X) = \sum_{k=0}^{m} f_k(z) X^k \in R_m[[X]],$$

by using (2.13)

$$\begin{aligned}(\widehat{\mathcal{D}}_\xi \circ \mathcal{Q}^m_\xi) f(z, X) &= \sum_{k=0}^{m} f_k'(z) X^k + \xi \sum_{k=0}^{m} f_k(z) X^k - \sum_{k=0}^{m} k f_k(z) X^k \\ &= \sum_{k=0}^{m} f_k'(z) X^k + \sum_{k=1}^{m+1} (\xi - k + 1) f_{k-1}(z) X^k.\end{aligned}$$

Comparing this with (3.6), we obtain (3.5). □

The relation (3.5) provides us with the complex linear map

$$\widehat{\mathcal{D}}_\xi = \mathcal{Q}^{m+1}_{\xi+2} \circ \partial \circ (\mathcal{Q}^m_\xi)^{-1} : QP^m_\xi(\Gamma) \to QP^{m+1}_{\xi+2}(\Gamma).$$

On the other hand, using (2.11) and (2.15), we see that

$$\mathcal{D}_\lambda(\mathcal{J}_\lambda(\Gamma)) \subset \mathcal{J}_{\lambda+2}(\Gamma), \quad \Pi^\delta_m(\mathcal{J}_\lambda(\Gamma)_\delta) \subset QP^m_{\lambda+2m+2\delta}(\Gamma);$$

hence we obtain the two additional complex maps

$$\Pi^{\delta,\lambda}_m : \mathcal{J}_\lambda(\Gamma)_\delta \to QP^m_{\lambda+2m+2\delta}(\Gamma), \quad \mathcal{D}_\lambda : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathcal{J}_{\lambda+2}(\Gamma)_{\delta-1} \tag{3.8}$$

for each $\lambda \in \mathbb{Z}$, where $\Pi^{\delta,\lambda}_m$ is the restriction of Π^δ_m to $\mathcal{J}_\lambda(\Gamma)_\delta$.

THEOREM 3.3. *Given $\lambda, \delta \in \mathbb{Z}$, the diagram*

$$\begin{array}{ccc} \mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{\Pi^\delta_m} & QP^m_{\lambda+2m+2\delta}(\Gamma) \\ \Big\downarrow {\scriptstyle \mathcal{D}_\lambda} & & \Big\downarrow {\scriptstyle \widehat{\mathcal{D}}_{\lambda+2m+2\delta}} \\ \mathcal{J}_{\lambda+2}(\Gamma)_{\delta-1} & \xrightarrow{\Pi^{\delta-1}_{m+1}} & QP^{m+1}_{\lambda+2m+2\delta+2}(\Gamma) \end{array} \tag{3.9}$$

commutes if and only if $\delta = -m - 1$ or $\delta = -m - \lambda$.

https://doi.org/10.1017/S0004972709001117 Published online by Cambridge University Press

PROOF. Let $\Phi(z, X) \in R[[X]]_\delta$ be given by

$$\Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta}.$$

Then from (2.12) we obtain

$$\begin{aligned} \mathcal{D}_\lambda \Phi(z, X) &= \sum_{k=0}^{\infty} \phi_k'(z) X^{k+\delta} - \lambda \sum_{k=0}^{\infty} (k+\delta)\phi_k(z) X^{k+\delta-1} \\ &\qquad - X \sum_{k=0}^{\infty} (k+\delta)(k+\delta-1)\phi_k(z) X^{k+\delta-2} \\ &= \sum_{k=0}^{\infty} (\phi_{k-1}'(z) - (k+\delta)(k+\delta-1+\lambda)\phi_k(z)) X^{k+\delta-1} \\ &\in R[[X]]_{\delta-1} \end{aligned} \tag{3.10}$$

with $\phi_{-1}' = 0$. Using this and (2.10),

$$\begin{aligned} &((\Pi_{m+1}^{\delta-1} \circ \mathcal{D}_\lambda)\Phi)(z, X) \\ &\quad = \sum_{k=0}^{m+1} \frac{1}{k!} (\phi_{m-k}'(z) - (m+1-k+\delta)(m-k+\delta+\lambda)\phi_{m+1-k}(z)) X^k. \end{aligned} \tag{3.11}$$

On the other hand, from

$$(\Pi_m^\delta \Phi)(z, X) = \sum_{k=0}^{m} \frac{1}{k!} \phi_{m-k}(z) X^k$$

and (2.13) we see that

$$\begin{aligned} &((\widehat{\mathcal{D}}_{\lambda+2m+2\delta} \circ \Pi_m^\delta)\Phi)(z, X) \\ &\quad = \sum_{k=0}^{m} \frac{1}{k!} \phi_{m-k}'(z) X^k + \sum_{k=0}^{m} \frac{\lambda+2m+2\delta}{k!} \phi_{m-k}(z) X^{k+1} \\ &\qquad - \sum_{k=0}^{m} \frac{1}{(k-1)!} \phi_{m-k}(z) X^{k+1} \\ &\quad = \sum_{k=0}^{m} \frac{1}{k!} \phi_{m-k}'(z) X^k + \sum_{k=1}^{m+1} \frac{\lambda+2m+2\delta-k+1}{(k-1)!} \phi_{m-k+1}(z) X^k \\ &\quad = \sum_{k=0}^{m+1} \frac{1}{k!} (\phi_{m-k}'(z) + k(\lambda+2m+2\delta-k+1)\phi_{m+1-k}(z)) X^k. \end{aligned} \tag{3.12}$$

https://doi.org/10.1017/S0004972709001117 Published online by Cambridge University Press

Comparing (3.11) and (3.12),

$$\Pi_{m+1}^{\delta-1} \circ \mathcal{D}_\lambda = \widehat{\mathcal{D}}_{\lambda+2m+2\delta} \circ \Pi_m^\delta$$

if and only if

$$(m+\delta+1)(m+\delta+\lambda) = 0;$$

hence the theorem follows. □

If $\Pi_m^{\delta,\lambda}$ is the map in (3.8), we consider the corresponding linear map

$$\widehat{\Pi}_m^{\delta,\lambda} : \mathcal{J}_\lambda(\Gamma)_\delta \to QM_{\lambda+2m+2\delta}^m(\Gamma)$$

defined by

$$\widehat{\Pi}_m^{\delta,\lambda} = (\mathcal{Q}_{\lambda+2m+2\delta}^m)^{-1} \circ \Pi_m^{\delta,\lambda}, \tag{3.13}$$

where $\mathcal{Q}_{\lambda+2m+2\delta}^m$ is as in (3.3). Then from (2.10) and (3.4) we see that

$$(\widehat{\Pi}_m^{\delta,\lambda}\Phi)(z) = \phi_m(z)$$

for $\Phi(z) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta$.

COROLLARY 3.4. *Given $m, \lambda, \delta \in \mathbb{Z}$ with $m \geq 0$, if $\delta = -m-1$ or $\delta = -m-\lambda$, the diagram*

$$\begin{array}{ccc} \mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{\widehat{\Pi}_m^{\delta,\lambda}} & QM_{\lambda+2m+2\delta}^m(\Gamma) \\ \downarrow{\scriptstyle \mathcal{D}_\lambda} & & \downarrow{\scriptstyle \partial} \\ \mathcal{J}_{\lambda+2}(\Gamma)_{\delta-1} & \xrightarrow{\widehat{\Pi}_m^{\delta-1,\lambda+2}} & QM_{\lambda+2m+2\delta+2}^{m+1}(\Gamma) \end{array}$$

is commutative.

PROOF. From the relation (3.5) and the commutativity of the diagram (3.9) we see that the diagram

$$\begin{array}{ccccc} \mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{\Pi_m^{\delta,\lambda}} & QP_{\lambda+2m+2\delta}^m(\Gamma) & \xrightarrow{(\mathcal{Q}_{\lambda+2m+2\delta}^m)^{-1}} & QM_{\lambda+2m+2\delta}^m(\Gamma) \\ \downarrow{\scriptstyle \mathcal{D}_\lambda} & & \downarrow{\scriptstyle \widehat{\mathcal{D}}_{\lambda+2m+2\delta}} & & \downarrow{\scriptstyle \partial} \\ \mathcal{J}_{\lambda+2}(\Gamma)_{\delta-1} & \xrightarrow{\Pi_{m+1}^{\delta-1,\lambda+2}} & QP_{\lambda+2m+2\delta+2}^{m+1}(\Gamma) & \xrightarrow{(\mathcal{Q}_{\lambda+2m+2\delta+2}^{m+1})^{-1}} & QM_{\lambda+2m+2\delta+2}^{m+1}(\Gamma) \end{array}$$

commutes, assuming that $\delta = -m-1$ or $\delta = -m-\lambda$. Thus the corollary follows from this and the relations

$$\widehat{\Pi}_m^{\delta-1,\lambda+2} = (\mathcal{Q}_{\lambda+2m+2\delta+2}^{m+1})^{-1} \circ \Pi_{m+1}^{\delta-1,\lambda+2}$$

and (3.13). □

https://doi.org/10.1017/S0004972709001117 Published online by Cambridge University Press

References

[1] Y. Choie and M. H. Lee, 'Quasimodular forms, Jacobi-like forms, and pseudodifferential operators', Preprint.

[2] P. B. Cohen, Y. Manin and D. Zagier, 'Automorphic pseudodifferential operators', in: *Algebraic Aspects of Nonlinear Systems* (Birkhäuser, Boston, 1997), pp. 17–47.

[3] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, Vol. 55 (Birkhäuser, Boston, 1985).

[4] M. Kaneko and D. Zagier, *A Generalized Jacobi Theta Function and Quasimodular Forms*, Progress in Mathematics, Vol. 129 (Birkhäuser, Boston, 1995), pp. 165–172.

[5] M. H. Lee, 'Radial heat operators on Jacobi-like forms', *Math. J. Okayama Univ.* **104** (2009), 27–46.

[6] D. Zagier, 'Modular forms and differential operators', *Proc. Indian Acad. Sci. Math. Sci.* **104** (1994), 57–75.

MIN HO LEE, Department of Mathematics, University of Northern Iowa,
Cedar Falls, IA 50614, USA
e-mail: lee@math.uni.edu

https://doi.org/10.1017/S0004972709001117 Published online by Cambridge University Press