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COMMUTATIVE COHERENT RINGS

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Throughout this paper R will be a commutative ring with 1. The purpose of this paper is to provide two new characterizations of coherent rings. The first of these characterizations shows that the class of coherent rings is precisely the class of rings for which certain duality homomorphisms are isomorphisms. And the second of these characterizations shows that the class of coherent rings is precisely the class of rings for which the endomorphism ring of any injective module is a flat module. We can show as a consequence that the endomorphism ring of a universal injective R-module is a faithfully flat R-module whenever R is a coherent ring.

Definition. We shall say that an R-module E is a *universal injective* R-module if E is an injective R-module, and if given an R-module A and non-zero element $x \in A$ there exists an R-homomorphism $f : A \to E$ such that $f(x) \neq 0$. It is easily seen that the following statements are equivalent:

(1) E is a universal injective R-module.

(2) E is a injective R-module, and if * is the functor $\operatorname{Hom}_R(-, E)$, then the canonical map $A \to A^{**}$ is a monomorphism for all R-modules A.

(3) E is an injective R-module, and every R-module can be embedded in a direct product of copies of E.

(4) E is an injective R-module, and E contains a copy of every simple R-module.

It is easily verified that if Z is the ring of integers and Q is the field of rational numbers, then $\operatorname{Hom}_Z(R, Q/Z)$ is a universal injective R-module. Furthermore, it follows from (4) that the injective envelope of the direct sum of one copy of each of the simple R-modules is a universal injective R-module that is isomorphic to a direct summand of every other universal injective R-module.

PROPOSITION 1. *Let E be a universal injective R-module and $H = \operatorname{Hom}_R(E, E)$. Then:*

(1) *E is a faithful, R-module so that $R \subset H$.*

(2) *If I is an ideal of R, then $HI \cap R = I$.*

(3) *H is a flat R-module $\Leftrightarrow$ H is a faithfully flat R-module $\Leftrightarrow$ H/R is a flat R-module.*

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Proof. (1) Let a be a non-zero element of R. Since E is a universal injective R-module, there exists an R-homomorphism $f : R \to E$ such that $f(a) \neq 0$. Since $f(a) = af(1)$, we see that $aE \neq 0$. Thus E is a faithful R-module.

(2) Let I be an ideal of R, $r \in HI \cap R$, and suppose that $r \notin I$. Then we have an exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_R (R/(I, r), E) \to \operatorname{Hom}_R (R/I, E) \\ &\to \operatorname{Hom}_R ((I, r)/I, E) \to 0. \end{aligned}$$

It follows from this exact sequence that

$$\operatorname{Ann}_E (I, r) \subsetneqq \operatorname{Ann}_E (I).$$

Thus there exists $x \in E$ such that $Ix = 0$ and $rx \neq 0$.

Now

$$r = \sum_{i=1}^{n} a_i h_i,$$

where $a_i \in I$ and $h_i \in H$. Thus

$$0 \neq rx = \sum_{i=1}^{n} a_i h_i(x) = \sum_{i=1}^{n} h_i(a_i x) = 0.$$

This contradiction shows that $HI \cap R = I$.

(3) Let I be a proper ideal of R. Then $H \otimes_R (R/I) \cong H/HI$, and hence by (2), $H \otimes_R (R/I) \neq 0$. It follows that H is a flat R-module $\Leftrightarrow$ H is a faithfully flat R-module. Furthermore, if H is flat, then

$$\operatorname{Tor}_1^R (H/R, R/I) \cong \operatorname{Ker} (R/I \to H/HI) = (HI \cap R)/I = 0,$$

showing that H/R is also flat.

LEMMA 1. *Let $\{A_\alpha\}$, $\alpha \in \pi$, be a family of R-modules.*

(1) *If I is a finitely generated ideal of R, then*

$$I\left(\prod_\alpha A_\alpha\right) = \prod_\alpha (IA_\alpha).$$

(2) *Suppose that each A_α is a flat R-module, and that B_α is a submodule of A_α such that A_α/B_α is flat. Then $\prod_\alpha A_\alpha$ is flat $\Leftrightarrow$ $\prod_\alpha B_\alpha$ and $\prod_\alpha (A_\alpha/B_\alpha)$ are flat.*

Proof. (1) Let $I = Ra_1 + \ldots + Ra_n$, where $a_i \in R$. Let $y \in I(\prod_\alpha A_\alpha)$; then

$$y = a_1 x^1 + \ldots + a_n x^n, \quad \text{where } x^i = \langle x_\alpha{}^i \rangle \in \prod_\alpha A_\alpha.$$

Let

$$y_\alpha = a_1 x_\alpha{}^1 + \ldots + a_n x_\alpha{}^n \in IA_\alpha.$$

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Then

$$y = \langle y_\alpha \rangle \in \prod_\alpha (IA_\alpha).$$

Conversely, let $y \in \prod_\alpha (IA_\alpha)$; then $y = \langle y_\alpha \rangle$, where $y_\alpha \in IA_\alpha$. We have

$$y_\alpha = a_1 x_\alpha{}^1 + \ldots + a_n x_\alpha{}^n$$

where $x_\alpha{}^j \in A$. Hence

$$y = a_1\langle x^1 \rangle + \ldots + a_n \langle x^n \rangle \in I\left(\prod_\alpha A_\alpha\right).$$

(2) Assume that $\prod_\alpha A_\alpha$ is a flat R-module. Then it is sufficient to prove that $\prod_\alpha (A_\alpha/B_\alpha)$ is flat. Let I be a finitely generated ideal of R. Then

$$\begin{aligned}\operatorname{Tor}_1{}^R&\left(\prod_\alpha (A_\alpha/B_\alpha), R/I\right)\\ &\cong \left[\prod_\alpha B_\alpha \cap I\left(\prod_\alpha A_\alpha\right)\right] \Big/ I\left(\prod_\alpha B_\alpha\right) = (\text{by } (1))\\ &\cong \left[\prod_\alpha B_\alpha \cap \prod_\alpha (IA_\alpha)\right] \Big/ \prod_\alpha (IB_\alpha) = \prod_\alpha (B_\alpha \cap IA_\alpha)/\pi_\alpha(IB_\alpha)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong \prod_\alpha [(B_\alpha \cap IA_\alpha)/IB_\alpha].\end{aligned}$$

Hence it is sufficient to prove that $B_\alpha \cap IA_\alpha = IB_\alpha$. But

$$(B_\alpha \cap I_\alpha)/IB_\alpha \cong \operatorname{Tor}_1{}^R (A_\alpha/B_\alpha, R/I) = 0$$

because A_α and A_α/B_α are flat.

Definition. R is called a *coherent* ring if every finitely generated ideal of R is finitely presented. S. Chase has proved the theorem that the following statements are equivalent [**2**, Theorem 2.1]:

(1) R is a coherent ring.

(2) Every finitely generated submodule of a free R-module is finitely presented.

(3) Any direct product of flat R-modules is flat.

(4) Any direct product of copies of R is flat.

Definition. Let A, B, C be R-modules. Then there is a natural map:

$$\sigma_0 : \operatorname{Hom}_R (B, C) \otimes_R A \to \operatorname{Hom}_R (\operatorname{Hom}_R (A, B), C)$$

defined by $\sigma_0(f \otimes a)(g) = f(g(a))$, where $f \in \operatorname{Hom}_R (B, C)$, $a \in A$, and $g \in \operatorname{Hom}_R (A, B)$. It follows readily [**1**, Chapter VI, Proposition 5.2] that σ_0 induces natural homomorphisms:

$$\sigma_n : \operatorname{Tor}_R{}^n (\operatorname{Hom}_R (B, C), A) \to \operatorname{Hom}_R (\operatorname{Ext}_R{}^n (A, B), C).$$

THEOREM 1. *The following statements are equivalent:*

(1) *R is a coherent ring.*

(2) σ_n: $\mathrm{Tor}_n^R(\mathrm{Hom}_R(B, C), A) \rightarrow \mathrm{Hom}_R(\mathrm{Ext}_R^n(A, B), C)$ *is an isomorphism $\forall$ n whenever C is injective and A is finitely presented.*

(3) $\mathrm{Hom}_R(B, C)$ *is a flat R-module $\forall$ injective R-modules B and C.*

Proof. (1) $\Rightarrow$ (2). The proof in [**1**, Chapter VI, Proposition 5.3] that (2) is true for Noetherian rings only requires that A have a projective resolution of finitely generated free R-modules. But such a resolution exists for a finitely presented R-module A when R is a coherent ring by (2) Chase's Theorem.

(2) $\Rightarrow$ (3). Let B and C be injective R-modules and I a finitely generated ideal of R. Because σ_n is an isomorphism we have that

$$\mathrm{Tor}_n^R(\mathrm{Hom}_R(B, C), R/I) = 0.$$

Thus $\mathrm{Hom}_R(B, C)$ is a flat R-module.

(3) $\Rightarrow$ (1). Let E be a universal injective R-module and $H = \mathrm{Hom}_R(E, E)$. Since $R \subset H$ and, by assumption, H is flat, we have by Proposition 1 that H/R is flat. Let $\mathfrak{A}$ be an index set, $H_\alpha = H$, $R_\alpha = R$ and $E_\alpha = E$ $\forall$ $\alpha \in \mathfrak{A}$. Now

$$\prod_\alpha H_\alpha \cong \mathrm{Hom}_R\left(E_\alpha, \prod_\alpha E_\alpha\right)$$

is flat by assumption. Hence it follows from Lemma 1 that $\prod_\alpha R_\alpha$ is flat. Thus R is a coherent ring by Chase's Theorem [**2**, Theorem 2.1].

COROLLARY 1. *Let R be a coherent ring, E a universal injective R-module and $H = \mathrm{Hom}_R(E, E)$. Then*

(1) *E is a faithful R-module so that $R \subset H$.*

(2) *If I is an ideal of R, then $HI \cap R = I$.*

(3) *H is a faithfully flat R-module.*

Proof. Use Proposition 1 and Theorem 1.

Remarks. (1) Let R be a commutative Noetherian local ring with maximal ideal M; let E be the injective envelope of R/M; and let $H = \mathrm{Hom}_R(E, E)$. A straight forward and simple argument shows that H is ring isomorphic to $\tilde{R}$, the completion of R in the M-adic topology; (see also [**4**, Theorem 3.7]). Since E is a universal injective R-module, Corollary 1 provides a new and efficient way of showing that $\tilde{R}$ is a faithfully flat R-module, and that $\tilde{R}I \cap R = I$ for all ideals I of R. Since $\mathrm{Hom}_R(-, E)$ preserves finite lengths, we can use the isomorphism σ_0 of Theorem 1, and Corollary 1, to show that if I is an M-primary ideal of R, then $\tilde{R}/\tilde{R}I \cong R/I$. Many other elementary facts about $\tilde{R}$ can also be obtained by these methods.

(2) As a further example of the power contained in the homomorphisms σ_n we will demonstrate a short elegant proof of the following well known result (see [**3**, Corollary 11.30]): Let A be a finitely presented flat left module over a ring R (not necessarily commutative), then A is a projective R-module.

Since A is finitely presented it is easy to see that

$$\sigma_1: \operatorname{Tor}_1^R (\operatorname{Hom}_Z (B, Q/Z), A) \rightarrow \operatorname{Hom}_Z (\operatorname{Ext}_R^1 (A, B), Q/Z)$$

is an epimorphism for all left R-modules B. Since A is a flat R-module, the left hand side of this homomorphism vanishes. Hence so does the right side. Since Q/Z is a universal injective Z-module, it follows that $\operatorname{Ext}_R^1 (A, B) = 0$. Thus A is a projective R-module.

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