Can. J. Math., Vol. XLII, No. 4, 1990, pp. 709–730

ANALYTIC ISOMORPHISMS OF TRANSFORMATION GROUP C*-ALGEBRAS

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ABSTRACT. An analytic isomorphism of C*-algebras is a C*-isomorphism which maps one distinguished subalgebra, the analytic subalgebra, onto another. A strict partial order of a topological group acting on a topological space determines the analytic subalgebra of the transformation group C*-algebra as a certain non-self-adjoint subalgebra of the C*-algebra. When the group action is free and locally parallel, this analytic subalgebra is locally a subfield of compact operators contained in a reflexive algebra whose subspace lattice is determined by the group order. If in addition the group has the dominated convergence property, an analytic isomorphism of such transformation group C*-algebras induces a homeomorphism of the transformation spaces which maps orbits to orbits. In particular, the C*-algebras for two regular foliations of the plane are analytically isomorphic only if the foliations are topologically conjugate. In the case of parallel actions, a quotient of the group of analytic automorphisms is isomorphic to the second Čech cohomology of a transversal for the action.

In [**2**], W. B. Arveson describes a certain non-self-adjoint algebra of operators which characterizes an ergodic transformation of a measure space in the sense that two ergodic transformations are conjugate if and only if the corresponding algebras are unitarily equivalent. This algebra is concisely described as the norm-closed subalgebra of the C*-crossed product $L^\infty(M) \times_\alpha Z$ generated by $L^\infty(M)$ and a single unitary u which implements the transformation α on the measure space M and is denoted by $L^\infty(M) \times_\alpha Z^+$. The current work is motivated by an attempt to produce an analogous result for transformation groups; that is, the measure space is replaced by a topological space, the single ergodic transformation is replaced by a topological group of homeomorphisms of that space, and the C*-algebra in question is the crossed product $C_0(M) \times_\alpha G$.

That two actions of a group on a space should be topologically conjugate is a very strong condition, as even examples with the real line acting will show. A more reasonable equivalence is to require their orbits be topologically conjugate. That is, the action of a group on a topological space decomposes that space into a disjoint union of orbits; two such decompositions are topologically conjugate if there is a homeomorphism of the two spaces mapping orbits to orbits. The transformation group C*-algebra of the action is known *not* to characterize this orbit structure; for instance, two actions of the real line on the plane may give isomorphic C*-algebras, although there is no homeomorphism of the plane

Received by the editors September 21, 1989.

This research is partially supported by an NSERC Fellowship.

1980 Mathematics Subject Classification (1985 Revision). Primary 46L55, 22D25.

mapping one set of orbits onto the other. However, by using an order on the group of transformations to define an analytic subalgebra of the crossed product, it turns out that a C*-isomorphism of the transformation group C*-algebras which maps one analytic subalgebra onto the other determines a topological conjugacy of the corresponding orbit structures, given certain assumptions on the order and action.

To this end, in §2 a free, transitive action of a strictly ordered group is considered; this is essentially the case of the group acting on itself by left translation. Here, the analytic subalgebra of the transformation group C*-algebra is the subalgebra of compact operators contained in a reflexive algebra whose subspace lattice is determined by the order on the group. The multiplier algebra of the analytic subalgebra is the reflexive algebra itself, and the diagonal algebra is a m.a.s.a. of functions defined on the group. In §3 the case of a free, parallel action on a locally compact space is examined; the analytic subalgebra is then a norm-continuous field of compacts in the reflexive algebra, the multiplier algebra is a strong* continuous field of operators in the reflexive algebra, and the diagonal algebra is a strong* continuous field of operators in the m.a.s.a. and is represented as an algebra of functions on the transformation space.

To go further, additional structure on the group is required. The model is the real line, where the two important properties are that the intervals generate the topology for the group and that the group has the dominated convergence property. In the case of parallel actions, then, an analytic isomorphism induces a homeomorphism of the transformation spaces which maps orbits to orbits; thus the orbit structures are topologically conjugate. Near the end of §3, the more general case of a locally parallel action is considered; the situation is locally as in the parallel case and so analytic isomorphism again implies topological conjugacy of the orbits structures. These results apply to the case of regular foliations of the plane, implying that C*-algebras for two foliations are analytically isomorphic only if the foliations are topologically conjugate.

In §4 the cohomology arising from these algebras is studied, where it is shown that in the case of free, parallel actions, a certain quotient of the group of analytic automorphisms recovers the second cohomology group of the transverse space for the action. Finally, in §5, a number of examples are presented, to illustrate some applications of the theory.

This paper is based on part of the author's doctoral dissertation at the University of California at Berkeley. I would like to take this opportunity to thank Professor Bill Arveson for his supervision and helpful suggestions during the writing of this dissertation.

1. Preliminaries. For a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ the algebra of compact operators; Hilbert spaces in this paper are assumed separable. A field of operators will mean one of two things: either $C_0(X, A)$, the family of norm-continuous maps from locally compact space X into Banach algebra A which vanish at infinity, or $C_*(X, A)$, the family of bounded strong* continuous maps from X into A.

The multiplier algebra of a Banach algebra A is denoted by $\mathcal{M}(A)$; the strict topology on $\mathcal{M}(A)$ is the topology of pointwise-norm convergence for the action of $\mathcal{M}(A)$ on A. When A is a C*-algebra, so is $\mathcal{M}(A)$; if in addition A is a non-degenerate subalgebra of $\mathcal{B}(\mathcal{H})$, then so is $\mathcal{M}(A)$, consisting of the algebra of two-sided multipliers of A. A surjective homomorphism of separable C*-algebras extends to a surjective homomorphism of the multiplier algebras; we denote a homomorphism and its unique extension to the multiplier by the same symbol. Note that a norm-continuous action of a group on a C*-algebra extends to strictly-continuous action on the multiplier algebra.

A C*-dynamical system is a triple (A, G, α) consisting of a C*-algebra A, a topological group G and an action α which is a continuous group homomorphism from G into the automorphisms of A. We assume G is a locally compact, Hausdorff, second countable group with left Haar measure dt. In this paper, we only consider the case of a separable, abelian algebra A which is the algebra of continuous functions on a locally compact, separable space and so the action arises from a group of topological transformations of the topological space.

An isomorphism of C*-dynamical systems (A, G, α) and (A', G', α') is a pair of isomorphisms (ρ, γ) where ρ is a C*-isomorphism of A onto A' and γ is a group isomorphism of G onto G' such that $\alpha'_{\gamma(t)}(\rho(a)) = \rho(\alpha_t(a))$ for all a in A and t in G. A covariant representation of the C*-dynamical system is a triple $(\pi, u, \mathcal{H})$ consisting of a representation π of A on Hilbert space $\mathcal{H}$ and a unitary representation u of G on $\mathcal{H}$ such that $\pi(\alpha_t(a)) = u_t\pi(a)u_t^*$ for all a in A and t in G. Given a C*-dynamical system (A, G, α), the C*-crossed product $A \times_\alpha G$ is defined as the enveloping C*-algebra of the Banach *-algebra $L^1(G, A)$ of Bochner integrable A-valued functions on G with multiplication, involution and L^1-norm defined by

(i) $(f * g)_t = \int_G f_s\alpha_s(g_{s^{-1}t})\,ds$

(ii) $(f^*)_t = \partial ta(t)^{-1}\alpha_t(f_{t^{-1}})^*$

(iii) $\|f\|_{L^1} = \int_G \|f_s\|_A\,ds$

for all f, g in $L^1(G, A)$ and t in G, where ∂ta denotes the modular function on G. In general, A is not a subalgebra of the crossed product, but it embeds naturally into the multiplier algebra $\mathcal{M}(A \times_\alpha G)$ by considering A as a family of point masses at the identity of G; that is, an element a of A acts by left and right multiplication on function f in $L^1(G, A)$ via $(a*f)_t = a(f_t)$ and $(f*a)_t = (f_t)\alpha_t(a)$ for all t in G. This extends to an embedding of $\mathcal{M}(A)$ into $M(A \times_\alpha G)$.

Given a covariant representation $(\pi, u, \mathcal{H})$ of the C*-dynamical system (A, G, α), a representation of the cross product is obtained by extending to the enveloping C*-algebra the bounded representation $\pi \times u$ of $L^1(G, A)$ defined by the integral

$$\pi \times u(f) = \int_G \pi(f_s)u_s\,ds$$

for all f in $L^1(G, A)$. Indeed, there is a one-to-one correspondence between covariant representation of the C*-dynamical system and representations of the crossed product. For a representation $(\pi, \mathcal{H})$ of A, there is an induced representation $\mathrm{Ind}\pi$ of the crossed product on $L^2(G, \mathcal{H})$ given by

$$(\mathrm{Ind}\pi(f)\xi)(t) = \int_G \pi(\alpha_{t^{-1}}(f_s))\xi(s^{-1}t)\,ds$$

for all f in $L^1(G, A)$, ξ in $L^2(G, \mathcal{H})$ and t in G.

2. Ordered Groups and Compact Operators. Let G be a locally compact, second countable group with left Haar measure dt and fix Σ a closed subsemigroup of G. Σ induces a closed partial order on G by the relation $a \leqq b$ iff $ba^{-1} \in \Sigma$. We also assume $\Sigma \cap \Sigma^{-1} = \{e\}$, hence the order is strict. The graph of the partial order Γ is the closed set of pairs (x, y) in $G \times G$ with $y \leqq x$. Thus by Theorem 1.1.12 of [**1**], $(G, \leqq, dt)$ is a *standard partially ordered measure space*; that is, G is a standard Borel space, dt is σ-finite measure, and the order $\leqq$ is tractable in a certain well-defined sense; here it is enough to say the graph of the order is closed. Following Arveson in [**1**], we say that two partially ordered measure spaces $(G, \leqq, dt)$ and $(G', \leqq', dt')$ are *Borel order isomorphic* if there exists an a.e. defined Borel isomorphism from G onto almost all of G' taking null sets to null sets such that $s \leqq t$ iff $\phi(s) \leqq' \phi(t)$ for all s and t in the domain of ϕ. The map ϕ is called a *Borel order isomorphism*. Associated with the partially ordered measure space $(G, \leqq, dt)$ is a certain lattice of projections acting on $L^2(G)$. Specifically, given a Borel subset E of G, let P_E denote the projection in $L^\infty(G)$ acting on $L^2(G)$ which is given by multiplication by the characteristic function on E; the set E is said to be *increasing* if $x \in E$ and $x \leqq y$ implies $y \in E$. The strongly closed lattice of projections for the partial order is denoted by $\mathcal{L}(G, \leqq, dt)$ and consists of all projection P_E for increasing, Borel subsets E of G. As the order is strict, by Theorem 1.2.3 of [**1**] this lattice generates $L^\infty(G)$ as a von Neumann algebra. In fact, it characterizes the ordered space up to isomorphism: by Theorem 1.2.4 of [**1**], the two strictly ordered spaces $(G, \leqq, dt)$ and $(G', \leqq', dt')$ are Borel order isomorphic if and only if the corresponding subspace lattices $\mathcal{L}(G, \leqq, dt)$ and $\mathcal{L}(G', \leqq', dt')$ are unitarily equivalent. Indeed, from the proof in Theorem 1.2.4 of [**1**], the unitary U implementing the equivalence of lattices determines a Borel order isomorphism ϕ by the relation $UP_EU^{-1} = P_{\phi(E)}$ for every Borel set E of G. Since the lattice generates $L^\infty(G)$, ϕ determines U up to a factor in $L^\infty(G)$.

Along a different tack, consider the C*-dynamical system (A, G, α) with α the action of locally compact, second countable group G on separable C*-algebra A and let $A \times_\alpha G$ denote the C*-crossed product. Following M. J. McAsey and P. S. Muhly in [**9**], let $A \times_\alpha \Sigma$ denote the norm-closed subspace of $A \times_\alpha G$ generated by elements of $L^1(G, A)$ with support in Σ, the semigroup of positive elements in G. Assuming further that Σ is the closure of its interior and generates G as a group, then $A \times_\alpha \Sigma$ is a closed subalgebra of the crossed product which generates the whole crossed product as a C*-algebra. By Lemma 3.1 of [**9**], the subalgebra $A \times_\alpha \Sigma$ contains a (non-self-adjoint) bounded approximate identity for $A \times_\alpha G$

and thus by Corollary 2.5.1 of [**9**] the natural embedding of $A \times_\alpha \Sigma$ into $A \times_\alpha G$ extends to a unital isometric embedding of the multiplier algebra $\mathcal{M}(A \times_\alpha \Sigma)$ into $\mathcal{M}(A \times_\alpha G)$. Any closed subalgebra of a C*-algebra which generates the C*-algebra and contains a bounded approximate identity for the C*-algebra is called an *analytic* subalgebra; we distinguish $A \times_\alpha \Sigma$ as *the* analytic subalgebra of $A \times_\alpha G$. If B is an analytic subalgebra of a C*-algebra, the *diagonal* D is the C*-algebra $\mathcal{M}(B) \cap \mathcal{M}(B)^*$. An *analytic isomorphism* is a C*-isomorphism taking one analytic algebra onto another; note the extension of an analytic isomorphism takes the multiplier algebra of the first analytic subalgebra onto the multiplier algebra of the second, and thus takes the first diagonal onto the second diagonal.

In this section we consider only the action τ of left translation of the locally compact, second countable group G on the C*-algebra $C_0(G)$, with an order on G given by the semigroup Σ, where Σ is the closure of its interior, Σ generates G, and $\Sigma \cap \Sigma^{-1} = \{e\}$. With ρ_e the character on $C_0(G)$ given by evaluation at the identity e, let $\operatorname{Ind}\rho_e$ be the induced representation of the crossed product on $L^2(G)$; thus, for an L^1 map $t \longmapsto f_t$ in $C_0(G)$ we have

$$\operatorname{Ind}\rho_e(f)\xi(s) = \int_G f_t(s)\xi(t^{-1}s)\,dt$$

for all ξ in $L^2(G, dt)$ and s in G. By [**11**] or [**16**], $\operatorname{Ind}\rho_e$ is a faithful representation of $C_0(G) \times_\tau G$ onto the compact operators $\mathcal{K}(L^2(G))$. Moreover, the above integral is non-zero only if there is some t in the support of f with $t^{-1}s$ in the support of ξ; thus if $t \longmapsto f_t$ has support in Σ, ξ has support in Σr, and the integral is non-zero, then s is in Σr. In particular, $\operatorname{Ind}\rho_e(f)$ leaves invariant the subspace of functions ξ supported on the increasing set Σr, for all f supported in Σ, and all $r \in G$. By completion, we have that each element of $\operatorname{Ind}\rho_e(C_0(G) \times_\tau \Sigma)$ leaves invariant each subspace of the lattice $\mathcal{L}(G, \leqq, dt)$. In the case where G is abelian, McAsey and Muhly in [**9**] showed this property characterizes $C_0(G) \times_\tau \Sigma$; the first goal of this section is to show this result for the non-abelian case.

We begin with some technical lemmas. Σ is the semigroup of positive elements of G, and Γ the graph of the order relation for G, the set of pairs (x, y) in $G \times G$ with $y \leqq x$. The measure on Γ is the restriction of Haar measure on $G \times G$.

LEMMA 2.1. *(i) The boundary of Σ has measure zero in G;*
(ii) The boundary of Γ has measure zero in $G \times G$;
(iii) Continuous functions of compact support in the interior of Γ are dense in $L^2(\Gamma)$.

Proof. If $\partial\Sigma$ is not null, then $(\partial\Sigma)(\partial\Sigma)^{-1}$ contains a neighbourhood of the identity and so intersects the interior of Σ. Fix g in the interior of Σ with $g = ab^{-1}$ for some $a, b \in \partial\Sigma$. Let $a_n \longrightarrow a$ be a convergent sequence with each $a_n \notin \Sigma$, thus $g = \lim a_n b^{-1}$ and eventually $a_n b^{-1} \in \Sigma$, since g is in the interior. Hence eventually $a_n \in \Sigma b \subseteq \Sigma$, a contradiction.

The graph Γ can be written as the set of pairs (x, y) in $G \times G$ with $x \in \Sigma y$ so by building sequences $(x_n, y_n) \notin \Gamma$ converging to the boundary, its easy to

https://doi.org/10.4153/CJM-1990-037-9 Published online by Cambridge University Press

check that the boundary $\partial\Gamma$ is set of pairs (x, y) with $x \in \partial\Sigma y$. Since $\partial\Sigma$ has measure zero in G, so does each cross-section of $\partial\Gamma$; by Fubini's theorem $\partial\Gamma$ has measure zero in $G \times G$. The restriction of Haar measure on $G \times G$ to the interior of Γ is regular so the continuous functions with compact support in the interior Γ° are dense in $L^2(\Gamma^\circ)$ and so also dense in $L^2(\Gamma)$, since the boundary of Γ is a null set. □

The next lemma shows there are lots of Hilbert-Schmidt operators in the analytic subalgebra.

LEMMA 2.2. $\mathrm{Ind}\rho_e(C_0(G) \times_\tau \Sigma)$ *has a bounded approximate identity for the algebra of compacts, consisting of Hilbert-Schmidt operators.*

Proof. It suffices to show that $C_0(G) \times_\tau \Sigma$ has a bounded approximate identity whose image under $\mathrm{Ind}\rho_e$ is a sequence of Hilbert-Schmidt operators. Following Lemma 3.1 of [**9**], let $\{U_n\}$ be a decreasing sequence of relatively compact neighborhoods of the identity in Σ, intersecting to the identity. Let φ_n denote the normalized characteristic function on U_n, thus $\{\varphi_n\}$ is an approximate identity for $L^1(G)$. Let $\{\epsilon_n\}$ be a bounded approximate identity for $C_0(G)$ consisting of positive continuous functions on G with compact support. Then $\{f^n = \varphi_n\epsilon_n\}$ is a bounded approximate identity for $C_0(G) \times_\tau G$, and contained in $C_0(G) \times_\tau \Sigma$ since $f^n_t = \varphi_n(t)\epsilon_n$ vanishes for $t \notin \Sigma$. To see that $\mathrm{Ind}\rho_e(f^n)$ is Hilbert-Schmidt, note that its kernel k_n is given by

$$k_n(s, t) = f^n_{st^{-1}}(s) = \varphi_n(st^{-1})\epsilon_n(s)$$

which is a bounded measurable function supported on a compact set in $G \times G$, hence is an L^2 kernel. □

PROPOSITION 2.3. $C_0(G) \times_\tau \Sigma$ *is the set of x in* $C_0(G) \times_\tau G$ *such that* $\mathrm{Ind}\rho_e(x)$ *leaves invariant* $L^2(\Sigma t)$ *for every* $t \in G$.

Proof. From the remarks above Lemma 2.1, we have that every x in the analytic subalgebra leaves invariant $L^2(\Sigma t)$. For the converse, by the approximate identity constructed in the last lemma, it suffices to consider those x in $C_0(G) \times_\tau G$ whose image is Hilbert-Schmidt. With such an x whose image leaves invariant each $L^2(\Sigma t)$ we have that $\mathrm{Ind}\rho_e(x)$ lies in $\mathrm{alg}\mathcal{L}(G, \leqq, gt)$ so by Prop. 3.1 of [**7**], its Hilbert-Schmidt kernel k lives a.e. on the graph Γ. By Lemma 2.1, k can be approximated in L^2 norm by continuous kernels k_n of compact support in the interior of Γ, so the corresponding Hilbert-Schmidt operators converge in norm. Defining f^n in $C(G, C_0(G))$ by $f^n_t(s) = k_n(s, t^{-1}s)$ it is easy to check that f^n is in $L^1(G, C_0(G))$ and thus represents an element of the analytic subalgebra $C_0(G) \times_\tau \Sigma$. Moreover, $\mathrm{Ind}\rho_e(f^n)$ is Hilbert-Schmidt with kernel k_n and converges to $\mathrm{Ind}\rho_e(x)$, so x is the limit of the f^n, and thus x is in $C_0(G) \times_\tau \Sigma$. □

$\mathrm{Ind}\rho_e$ thus describes the analytic subalgebra as isomorphic to a certain subalgebra of the compacts determined by the group order; in the next proposition, it is also shown to describe the multiplier algebra and diagonal in terms of a certain reflexive algebra. Recall from [**1**] that $\mathrm{alg}\mathcal{L}(G, \leqq, dt)$ is the closed, non-self-adjoint algebra of operators in $L^2(G, dt)$ which leave invariant the lattice of projections $\mathcal{L}(G, \leqq, dt)$.

PROPOSITION 2.4. *Extending* $\mathrm{Ind}\rho_e$ *to the multiplier algebras gives the following isomorphisms:*

(i) $C_0(G) \times_\tau \Sigma \cong \mathcal{K} \cap \mathrm{alg}\mathcal{L}(G, \leqq, dt)$;

(ii) $\mathcal{M}(C_0(G) \times_\tau \Sigma) \cong \mathrm{alg}\mathcal{L}(G, \leqq, dt)$;

(iii) $\mathcal{M}(C_0(G) \times_\tau \Sigma) \cap \mathcal{M}(C_0(G) \times_\tau \Sigma)^* \cong L^\infty(G, dt)$.

Proof. The first statement is just a restatement of the last proposition, as $\mathrm{alg}\mathcal{L}(G, \leqq, dt)$ is the reflexive algebra determined by $\mathcal{L}(G, \leqq, dt)$. The second statement is a consequence of strict topology on the multiplier of $\mathcal{K}$ coinciding with the strong* operator topology $\mathcal{B}(L^2(G))$. The third statement is the observation that $\mathrm{alg}\mathcal{L}(G, \leqq, dt) \cap \mathrm{alg}\mathcal{L}(G, \leqq, dt)^*$ is the von Neumann algebra generated by $\mathcal{L}(G, \leqq, dt)$, which, by the strict order on G, is just $L^\infty(G, dt)$. □

It is now apparent that an analytic isomorphism will be closely connected to a Borel order isomorphism.

PROPOSITION 2.5. *Two ordered groups G and G' are Borel order isomorphic if and only if the analytic subalgebras algebras $C_0(G) \times_\tau \Sigma$ and $C_0(G') \times_\tau \Sigma'$ are analytically isomorphic. Moreover, the Borel order isomorphism ϕ and analytic isomorphism ψ may be chosen to agree on the diagonal; that is*

$$\mathrm{Ind}\rho'_e(\psi(x)) = (\mathrm{Ind}\rho_e(x)) \circ \phi$$

for all x in $\mathcal{M}(C_0(G) \times_\tau \Sigma) \cap \mathcal{M}(C_0(G) \times_\tau \Sigma)^*$, with $\mathrm{Ind}\rho_e(x)$ *considered as a function in $L^\infty(G)$.*

Proof. When G and G' are Borel order isomorphic, their lattices $\mathcal{L}(G, \leqq, dt)$ and $\mathcal{L}(G', \leqq', dt')$ are unitarily equivalent, and hence the compact subalgebras $\mathcal{K} \cap \mathrm{alg}\mathcal{L}(G, \leqq, dt)$ and $\mathcal{K}' \cap \mathrm{alg}\mathcal{L}(G', \leqq', dt')$ are also unitarily equivalent. Lifting via the isomorphisms $\mathrm{Ind}\rho_e$ and $\mathrm{Ind}\rho'_e$ gives an analytic isomorphism of the analytic subalgebras $C_0(G) \times_\tau \Sigma$ and $C_0(G') \times_\tau \Sigma'$.

Conversely, an analytic isomorphism induces an isomorphism of $\mathcal{K} \cap \mathrm{alg}\mathcal{L}(G, \leqq, dt)$ and $\mathcal{K}' \cap \mathrm{alg}\mathcal{L}(G', \leqq', dt')$ which is the restriction of a C*-isomorphism from $\mathcal{K}$ to $\mathcal{K}'$. Hence it is unitarily implemented and gives a unitary equivalence of the lattices.

That the order isomorphism of spaces and analytic isomorphism of algebras correspond on the diagonal follows from the observation that the unitary U implementing the lattice equivalence and the Borel order isomorphism ϕ are related by the equality $UM_fU^{-1} = M_{f\circ\phi}$, for all multiplication operator M_f, with $f \in L^\infty(G)$, as indicated at the beginning of this section. □

For a topological result, we desire that analytic isomorphisms induce homeomorphisms of the base spaces G and G' and not just Borel maps; in particular, we want order isomorphisms to be homeomorphisms. Thus, it is convenient to assume the collection of interiors of closed intervals $[a,b] = \{g \in G : a \leqq g \leqq b\}$ forms a basis for the topology of G. For instance, if G is totally ordered, or if the intervals are all compact, then it is easy to show that the intervals generate the topology. When the intervals generate the topology, G will be called a *topological ordered group*. For ordered groups with regularly closed subsemigroup of positive elements as above, we know of no examples where this is not the case; however, we are unable to prove it in general.

When the group is not discrete, an a.e. defined Borel order isomorphism may not be everywhere defined and need not be related to a homeomorphism in any way. For instance, two Borel order isomorphic groups are the real line R with the usual order and the direct product $Z \times R$ of the integers with the real line, with the lexicographical order; a Borel order isomorphism is obtained by mapping each interval $(n, n+1)$ in R onto the subset $n \times R$ in $Z \times R$, yet clearly these are not homeomorphic spaces. By the last proposition, the crossed products are also analytically isomorphic yet with non-homeomorphic base spaces. To overcome this difficulty, it is sufficient to require that the ordered groups have the *dominated convergence property*, or DCP for short: that is, every increasing sequence with upper bound in G has a limit point. Note that DCP groups include those groups wherein each closed interval is compact.

LEMMA 2.6. *An a.e. defined Borel order isomorphism of two ordered groups with the DCP extends uniquely to an everywhere defined order isomorphism. For topological ordered groups, the order isomorphism is a homeomorphism.*

Proof. Let $\phi : G \longrightarrow G'$ be the a.e. defined order isomorphism of groups G and G' with Σ and Σ' their positive subsets, and let $P_E \in \mathcal{L}(G, \leqq, dt)$ denote the projection supported on the increasing Borel set $E \subset G$, with P' the analogous projection map for G'. Observe that the map $g \longmapsto P_{\Sigma g}$ is a faithful, strongly continuous representation of G as a family of projections acting on $L^2(G)$, which inverts the partial order. With U a unitary implementing the lattice equivalence, we have $UP_{\Sigma g}U^{-1} = P'_{\Sigma'\phi(g)}$ for all g in the domain of ϕ.

To extend ϕ to some g not in the domain of ϕ, first fix $g_0 \geqq g$ in the domain of ϕ: for instance, take any $g_0 \in \Sigma g \cap Dom(\phi)$, a non-empty set as Σ has interior and the domain of ϕ is dense. Similarly, take an increasing sequence in the domain of ϕ such that $g_n \nearrow g$, hence $\phi(g_n)$ is an increasing sequence in G' with upper bound $\phi(g_0)$; by DCP, this sequence has a limit point g' in G' and by the strict order, the sequence converges to g'. Extend ϕ by setting $\phi(g) = g'$. By continuity of U, we have $UP_{\Sigma g}U^{-1} = P'_{\Sigma' g'}$ which shows the extension is order-preserving, and by faithfulness of the P's, the extension is independent of the choice of sequence.

To show uniqueness, note by convergence of the increasing sequence above that another order extension ρ will satisfy $\phi(g) \leqq \rho(g)$ for all g in the (extended) domain of ϕ; choosing a decreasing sequence gives the reverse inequality, and strict order gives equality.

For topological ordered groups, note that an order isomorphism maps intervals to intervals, and interiors of intervals to interiors, because the boundary can be approached by either increasing or decreasing convergent sequences, which therefore map to convergent sequences. As the interiors are a basis for the topology, the map is a homeomorphism. □

To put this in context of transformation groups C*-algebra, we state one essential result in the form that will appear later in a more general context.

PROPOSITION 2.7. *Let G and G' be topological ordered groups with a strict order and DCP such that the positive semigroups Σ and Σ' are regularly closed and generate their groups. Let G and G' act freely and transitively on locally compact spaces M and M'. Then the C*-crossed products $C_0(M) \times_\alpha G$ and $C_0(M') \times_{\alpha'} G'$ are analytically isomorphic if and only if there is a homeomorphism from M to M' which is a Borel order isomorphism along the single orbit for the order induced by the action. Moreover, the analytic map and the homeomorphism can be chosen to agree on $C_0(M)$, which is mapped onto $C_0(M')$.*

Proof. Fix x in M. Since the action is transitive, the map $t \longmapsto t \cdot x$ mapping G onto on M gives an isomorphism of the C*-dynamical systems $(C_0(G), G, \tau)$ and $(C_0(M), G, \alpha)$. This map also lifts the order of G onto M by the relation $M \leqq m'$ iff $m = t \cdot x$ and $m' = t' \cdot x$ for some $t \leqq t'$ in G, and in a similar manner lifts the Haar measure of G onto M. Thus the statement of the proposition is simply Proposition 2.5 translated from $C_0(G)$ to $C_0(M)$. □

3. Parallel Actions. Throughout this section, G is a locally compact, second countable Hausdorff group acting on a locally compact, second countable space M, where we assume in addition that G is a topological ordered group with the dominated convergence property and a strict order given by the regularly closed semigroup Σ which generates G, as in §2. With $t, m \longmapsto t \cdot m$ denoting an action of G on a topological space M, note that a free action lifts the partial order of G onto M. That is, two points m and m' in M are ordered as $m \leqq m'$ iff there is some x in M and $t \leqq t'$ in G with $m = t \cdot x$ and $m' = t' \cdot x$. Indeed, along each orbit in M, this is just the order of G lifted via the action α; the order on an orbit is independent of the choice of the point x since the order on G is invariant under *right* translation. Similarly, the free action of G on M lifts Haar measure on G to a measure on each orbit in M; although the measure depends on the choice of basepoint x, the measure class on the orbit is independent of the choice of x.

The action of G on M is *parallel* if there is a closed subset X in M such that the map $(x, t) \longmapsto t \cdot x$ is a homeomorphism of $X \times G$ onto M. That is, M is essentially a crossed product of the transversal X with G, with the action of left translation along each copy of G. It is known in this case that the C*-crossed product is isomorphic to a continuous field of compact operators over X; it turns out that the analytic subalgebra is simply a continuous field of analytic subalgebras of the compacts.

Let $\mathcal{K} = \mathcal{K}(L^2(G))$ denote the algebra of compact operators on $L^2(G)$, and $\Re(\Sigma) = \mathcal{K} \cap \operatorname{alg}\mathcal{L}(G, \leqq, dt)$ the analytic subalgebra of the compacts determined by the order on G. Given a transversal X of the space M, one obtains a representation ρ_X of the crossed product as a field of operators over X by inducing up the obvious representation of $C_0(M)$ into $C_0(X)$; that is, for any f in $L^1(G, C_0(M))$, $\rho_X(f)$ is a map from X into $\mathcal{K}$ given by

$$(\rho_X(f)(x)\xi)(s) = \int_G f_t(s \cdot x)\xi(t^{-1}s)\, dt$$

for all x in X, ξ in $L^2(G)$, and s in G.

PROPOSITION 3.1. *Let α be a parallel action of G on space M with transversal X. Then the representation ρ_X of the crossed product as a field of operators over X extends to an isomorphism of the multiplier algebra onto its image. Moreover,*

(i) $\rho_X(C_0(M) \times_\alpha G) = C_0(X, \mathcal{K}(L^2(G)))$;

(ii) $\rho_X(\mathcal{M}(C_0(M) \times_\alpha G)) = C_*(X, \mathcal{B}(L^2(G)))$;

(iii) $\rho_X(C_0(M) \times_\alpha \Sigma) = C_0(X, \Re(\Sigma))$;

(iv) $\rho_X(\mathcal{M}(C_0(M) \times_\alpha \Sigma)) = C_*(X, \operatorname{alg}\mathcal{L}(G, \leqq, dt))$;

(v) $\rho_X(\mathcal{M}(C_0(M) \times_\alpha \Sigma) \cap \mathcal{M}(C_0(M) \times_\alpha \Sigma)^*) = C_*(X, L^\infty(G))$;

(vi) $\rho_X(C_0(M)) = C_0(X, C_0(G))$.

Proof. That ρ_X is an isomorphism of the crossed product and $C_0(X, \mathcal{K})$ follows immediately from Corollary 2.9 of [**5**], thus the extension to the multipliers is also an isomorphism. The image of multiplier of the crossed product thus is the multiplier of $C_0(X, \mathcal{K})$, which is precisely $C_*(X, \mathcal{B}(L^2(G)))$, the algebra of bounded, strong* continuous maps from X into $\mathcal{B}(L^2(G))$, since the strict topology on the multiplier of $\mathcal{K}$ is just the strong* topology on $\mathcal{B}(L^2(G))$.

In (iii), it is easy to see that the image of the analytic crossed product sits inside $C_0(X, \Re(\Sigma))$ by the integral form for ρ_X. To show the image is all of $C_0(X, \Re(\Sigma))$, take any function $F : X \longrightarrow \Re(\Sigma)$ in $C_0(X, \Re(\Sigma))$. F is approximated by a finite sum of functions of the form $x \longmapsto g(x)V$ for some g in $C_0(X)$ and V in $\Re(\Sigma)$ (cf [**6**] pp. 809–811). V is the limit of Hilbert-Schmidt operators on $L^2(G)$ with support in the graph of the partial order; in fact, V may be approximated by operators of the form $\operatorname{Ind}\rho_e(h)$ for functions h in $C_c(\Sigma, C_0(G))$. Build an element f in $C_c(\Sigma, C_0(M))$ by $f_t(s \cdot x) = g(x)h_t(s)$ for all x in X and s in G. Then $\rho_X(f)(x) = g(x)\operatorname{Ind}\rho_e(h)$ so $\rho_X(f)$ is in the image of the analytic crossed product and is close to the function $x \longmapsto g(x)V$ used to approximate the given F. Thus the image is all of $C_0(X, \Re(\Sigma))$.

The remaining images are now clear by taking adjoints and intersections. □

It is interesting to observe that the diagonal algebra of the crossed product is thus characterized as the *-algebra of complex-valued functions on the space M which are "L^∞" along the orbits, yet "continuous" in the transverse direction. In the interesting cases it is an inseparable C*-algebra, larger than the multiplier algebra $C_b(M)$, yet not a von Neumann algebra.

For the remainder of this section, $C_0(X, \mathfrak{R}(\Sigma))$ is distinguished as *the* analytic subalgebra of $C_0(X, \mathcal{K})$ and so $C_0(X, \mathcal{K})$ is analytically isomorphic to $C_0(M) \times_\alpha G$. It will be convenient to find the analytic isomorphisms for algebras of the form $C_0(X, \mathcal{K})$ and then translate the results back to the crossed product form.

An isomorphism ψ of two fields of operators $C_0(X, \mathcal{K})$ and $C_0(X', \mathcal{K}')$ is described by a homeomorphism θ of X onto X' and a path of unitaries $x \longmapsto U_x$ from X into $\mathcal{U}(L^2(G), L^2(G'))$, the space of unitaries mapping $L^2(G)$ onto $L^2(G')$, such that $x \longmapsto \mathrm{Ad}(U_x)$ is a continuous path into the space of isomorphisms of $\mathcal{K}$ onto $\mathcal{K}'$; ψ is given by $\psi(F)(\theta(x)) = U_x F(x) U_x^{-1}$ for all x in X and F in $C_0(X, \mathcal{K})$ (cf **[10]** or **[14]**). It is easy to determine what the analytic isomorphisms must be:

PROPOSITION 3.2. *Let ψ be an isomorphism of $C_0(X, \mathcal{K})$ onto $C_0(X', \mathcal{K}')$ determined by an isomorphism θ of X onto X' and a path of unitaries $x \longmapsto U_x$. Then ψ is analytic if and only if*

$$U_x \mathcal{L}(G, \Sigma, dt) U_x^{-1} = \mathcal{L}(G', \Sigma', dt')$$

for all x in X.

Proof. ψ is analytic if and only if the operator $\psi(F)(\theta(x)) = U_x F(x) U_x^{-1}$ lies in the analytic subalgebra $\mathfrak{R}(\Sigma')$ for all $F \in C_0(X, \mathfrak{R}(\Sigma)$ and x in X, with a similar condition for ψ^{-1}. For fixed x, the set of operators $F(x)$ spans $\mathfrak{R}(\Sigma)$, while a consideration of ψ^{-1} shows the $\psi(F)(\theta(x))$ must generate $\mathfrak{R}(\Sigma')$ in the analytic case. Thus, ψ is analytic if and only if $U_x \mathfrak{R}(\Sigma) U_x^{-1} = \mathfrak{R}(\Sigma')$ for all x in X, which by Proposition 2.5 occurs if and only if U_x gives a unitary equivalence of the above lattices. □

Since the order is strict, the unitary U_x determines uniquely a Borel order isomorphism ϕ_x from G onto G' by the equation $U_x M_f U_x^{-1} = M_{f \circ \phi}$, for all f in $L^\infty(G)$, with M_f multiplication by f. Define a unitary V_x mapping $L^2(G)$ onto $L^2(G')$ by $V_x \xi(t) = \omega_x(t) \xi(\phi^{-1}(t))$ for all ξ in $L^2(G)$ and t in G, where ω_x is the positive square root of the Radon-Nikodym derivative for ϕ_x^{-1}. Clearly, $Ad(U_x) = Ad(V_x)$ when restricted to $L^2(G)$, so U_x and V_x differ by at most a factor in $L^\infty(G)$.

Although the map $x \longmapsto U_x$ need not be strongly continuous or even measurable, the interesting fact is that $x \longmapsto V_x$ is.

LEMMA 3.3. *Let $\psi : C_0(X, \mathcal{K}) \longrightarrow C_0(X', \mathcal{K}')$ be an analytic isomorphism, with V_x the path of unitaries defined above. Then $x \longmapsto V_x$ is strongly continuous.*

Proof. Notice first that given any two complex number a, b of modulus 1, and non-negative real numbers c, d, we have $|c-d| \leqq |ac-bd|$. Writing $U_x = V_x W_x$ for the path of unitaries determined by ψ, with W_x in $L^\infty(G)$, let ξ in $L^2(G)$ be a non-negative real function of L^2 norm 1, and a a complex number of modulus 1. Considering W_x as an L^∞ unitary function, $W_x(t)$ is a complex number of modulus 1 for almost all t in G, as with the complex conjugate $W_x^*(t)$, thus for almost all t in G and all a of modulus 1,

$$\begin{aligned} |V_x\xi(t) - V_y\xi(t)| &= |\omega_x(t)\xi(\phi_x^{-1}(t)) - \omega_y(t)\xi(\phi_y^{-1}(t))| \\ &\leqq |aW_x^*(t)\omega_x(t)\xi(\phi_x^{-1}(t)) - W_y^*(t)\omega_y(t)\xi(\phi_y^{-1}(t))| \\ &= |aU_x\xi(t) - U_y\xi(t)|. \end{aligned}$$

Squaring and integrating over G' yields $\|V_x\xi - V_y\xi\|^2 \leqq \|aU_x\xi - U_y\xi\|^2$ for all a of modulus 1. Letting Q_ξ denote the rank one projection onto the span of ξ, we have

$$\begin{aligned} \|V_x\xi - V_y\xi\|^2 &\leqq \inf_{|a|=1} \|aU_x\xi - U_y\xi\|^2 \\ &\leqq 2\|Q_{U_x\xi} - Q_{U_y\xi}\|^2 \\ &\leqq 2\|U_xQ_\xi U_x^{-1} - U_yQ_\xi U_y^{-1}\|^2. \end{aligned}$$

As $x \longmapsto \mathrm{Ad}(U_x)$ is continuous in the point-norm topology for $Iso(\mathcal{K}, \mathcal{K}')$, the last term in the inequality goes to zero as x goes to y, hence $x \longmapsto V_x\xi$ is norm continuous. Now, the set of non-negative functions ξ of norm one spans $L^2(G)$, so $x \longmapsto V_x$ is strongly continuous. □

The above lemma thus gives a recipe for constructing analytic isomorphisms; for each x in X, choose a Borel order isomorphism of G onto G' such that the corresponding "change of variables" unitaries V_x gives a strongly continuous path of unitaries from X into $\mathcal{U}(L^2(G), L^2(G'))$. Modifying by a path of unitaries W_x in $L^\infty(G)$ such that $x \longmapsto \mathrm{Ad}(W_x)$ is continuous gives another analytic isomorphism. It turns out that this strong continuity condition on the unitaries is very significant: it forces the order isomorphisms to glue together into a homeomorphism of $X \times G$. First we need another technical lemma.

LEMMA 3.4. *Fix space X and topological ordered groups G and G' with DCP and strict orders. Let $x \longmapsto \phi_x$ be a path on X of Borel order isomorphisms from G to G' such that $x \longmapsto V_x$ is a strongly continuous path of unitaries, where $V_x : L^2(G) \longrightarrow L^2(G')$ is the change-of-variables unitary for ϕ_x. Then*

(i) $x \longmapsto \phi_x(t)$ is continuous for each t in G.

(ii) $x, t \longmapsto \phi_x(t)$ is jointly continuous.

Proof. (i): Fix t in G and x in X; we show $\phi_y(t)$ goes to $\phi_x(G)$ as y goes to x. Recall that as *topological* ordered groups, the topologies of G and G' are generated by the interiors of intervals. Thus, let $[a, b]$ be a compact interval containing $t' = \phi_x(t)$ in its interior. Then $[t', b]$ is a compact interval with non-empty interior and $\xi = 1_{[t',b]}$, the characteristic function on $[t', b]$, is a non-zero element of $L^2(G')$. With $P(\Sigma t)$ denoting the projection in $L^\infty(G)$ supported on the increasing set Σt and P' the analogous projection in $L^\infty(G')$, we have

$$P'(\Sigma'\phi_y(t)) = V_y^{-1}P(\Sigma t)V_y \longrightarrow V_x^{-1}P(\Sigma t)V_x = P'(\Sigma'\phi_x(t)) \quad \text{strongly}$$

as y tends to x. Thus for y sufficiently close to x, the support of ξ intersects $\Sigma'\phi_y(t)$, so $\phi_y(t) \leqq b$. By a similar arguments with decreasing sets, y sufficiently close to x implies that $a \leqq \phi_y(t)$ so $\phi_y(t)$ lies in $[a, b]$. By choice of $[a, b]$, this is true for any compact interval containing $\phi_x(t)$ in the interior, so

$$\phi_y(t) \longrightarrow \phi_x(t) \quad \text{as } y \longrightarrow x$$

as desired.

(ii) Fix t in G and x in X. To show joint continuity, it suffices to find for each neighbourhood interval $[a, b]$ of $\phi_x(t)$, a neighbourhood of (x, t) so that each pair (y, s) in the neighbourhood maps to $\phi_y(s)$ in $[a, b]$. Let $[c, d]$, contained in the interior of $[a, b]$, be a smaller neighbourhood interval of $\phi_x(t)$. By continuity of the map $s \longmapsto \phi_x(s)$ at $s = t$, there is a neighbourhood interval $[t_0, t_1]$ of t such that $t_0 \leqq s \leqq t_1$ implies $c \leqq \phi_x(s) \leqq d$. By continuity of the map $y \longmapsto \phi_y(t_1)$ at $y = x$, there is a neighbourhood U_1 of x such that $a \leqq \phi_y(t_1)$ for all y in U_1. Similarly, there is a neighbourhood U_2 of x such that $\phi_y(t_2) \leqq b$ for all y in U_2. Thus for any y in $U_1 \cap U_2$ and s in $[t_1, t_2]$, we have

$$a \leqq \phi_y(t_1) \leqq \phi_y(s) \leqq \phi_y(t_2) \leqq b.$$

That is, any element (y, s) of the neighbourhood $(U_1 \cap U_2) \times [t_1, t_2]$ maps to $\phi_y(s)$ in $[a, b]$, as required. □

PROPOSITION 3.5. *Let G and G' be topological ordered groups with DCP and strict orders, and let X and X' be locally compact, second countable spaces. If ψ is an analytic isomorphism of $C_0(X, \mathcal{K})$ onto $C_0(X', \mathcal{K}')$, then its extension to the multiplier satisfies*

$$\psi(C_0(X, C_0(G))) = C_0(X', C_0(G')).$$

In particular, the spectrum map θ and the family of Borel order isomorphisms ϕ_x of G onto G' determined by ψ yield a homeomorphism of $X \times G$ onto $X' \times G'$.

Proof. ψ is described by the spectrum map θ and a path of unitaries U_x by $\psi(F)(\theta(x)) = U_xF(x)U_x^{-1}$ for all F in $C_0(X, \mathcal{K})$ and x in X. Let ψ_x be the Borel order map determined by U_x; by the DCP, ψ is a homeomorphism and the change of variables unitary V_x for ψ_x gives a strongly continuous path of unitaries on X. By the last lemma, the map

$$(x, t) \longmapsto (\theta(x), \phi_x(t))$$

is a continuous map of $X \times G$ onto $X' \times G'$, as is its inverse, hence it is a homeomorphism. Now for F in $C_*(X, L^\infty(G))$, we have $\psi(F)(\theta(x)) \circ \phi_x = F(x)$ for all x in X, and restricting to $C_0(X, C_0(G))$ gives the required equality. □

We now restate the above theorem in terms of the crossed product algebras, using the analytic isomorphism between crossed product algebras and the function algebras $C_0(X, \mathcal{K})$ described in Proposition 3.1.

PROPOSITION 3.6. *Let G and G' be topological ordered groups with DCP and strict orders, with parallel actions on locally compact, separable spaces M and M'. If ψ is an analytic isomorphism of $C_0(M) \times_\alpha G$ onto $C_0(M') \times_{\alpha'} G'$, then its extension to the multiplier satisfies*

$$\psi(C_0(M)) = C_0(M').$$

Moreover, the homeomorphism of M onto M' determined by ψ maps orbits to orbits and preserves measure class and order on each orbit.

Proof. This follows immediately from the analytic isomorphism of $C_0(M) \times_\alpha G$ onto $C_0(X, \mathcal{K})$, which maps $C_0(M)$ onto $C_0(X, C_0(G))$ and maps orbits in M onto copies of G. □

We may now extend this result to an interesting class of free actions. The action of a group on a space is *locally parallel* if each point in the space has an open neighbourhood on which the action is parallel; that is, for each m in G-space M, there is an open set Q containing m and a relatively closed subset X in Q such that the map $(x, t) \longmapsto t \cdot x$ is a homeomorphism of $X \times G$ onto Q. It is worth noting that a locally parallel action of a Lie group defines a foliation, as seen in some of the examples in §5.

With Q any open, G-invariant subset of M, and $C_0(Q)$ identified as the ideal in $C_0(M)$ consisting of functions on M which vanish on the compliment of Q, then $C_0(Q) \times_\alpha G$ is an ideal in $C_0(M) \times_\alpha G$. The analytic part behaves in a straightforward manner:

LEMMA 3.7. *Let Q be an open, G-invariant subset of M. Then*

$$C_0(Q) \times_\alpha \Sigma = (C_0(Q) \times_\alpha G) \cap (C_0(M) \times_\alpha \Sigma).$$

Proof. This follows immediately by noting that $L^1(\Sigma, C_0(Q))$ is the intersection of $L^1(G, C_0(Q))$ with $L^1(\Sigma, C_0(M))$. □

PROPOSITION 3.8. *Let G and G' be topological ordered groups with DCP and strict orders, with locally parallel actions on topological spaces M and M'. If ψ is an analytic isomorphism of $C_0(M) \times_\alpha G$ onto $C_0(M') \times_{\alpha'} G'$, then its extension to the multiplier satisfies*

$$\psi(C_0(M)) = C_0(M').$$

Moreover, the homeomorphism of M onto M' determined by ψ maps orbits to orbits and preserves measure class and order on each orbit.

Proof. As each action is locally parallel, it is free and orbits are closed, so by Corollary 5.12 of [**18**], the crossed product is CCR and the spectrum is isomorphic to the orbit space. Thus the isomorphism ψ gives a homeomorphism θ of the orbit spaces M/G and M'/G'. That is, with $C_0(M \backslash G \cdot m)$ the ideal in $C_0(M)$ of functions vanishing on the orbit $G \cdot m$, ψ maps the primitive ideal $C_0(M \backslash G \cdot m) \times_\alpha G$ onto $C_0(M' \backslash \theta(G \cdot m)) \times_{\alpha'} G'$, where $\theta(G \cdot m)$ is an orbit in M'.

Fix m in M, Q an open neighbourhood of m on which the action is parallel, and a relatively closed subset X in Q with $(x, t) \longmapsto t \cdot x$ a homeomorphism of $X \times G$ onto Q. Similarly, for some m' in the corresponding orbit $\theta(G \cdot m)$ in M', find analogous Q' and X'. Considering $\theta^{-1}(Q')$ as a union of orbits in M, let Q_0 be the intersection of Q with $\theta^{-1}(Q')$ and similarly, let Q'_0 be the intersection of Q' with $\theta(Q)$. Then $\theta(Q_0) = Q'_0$ and since θ is a homeomorphism of the orbit spaces it is easy to check that Q_0 is an open neighbourhood of m and $(x, t) \longmapsto t \cdot m$ is a homeomorphism of $(X \cap Q_0) \times G$ onto Q_0, with the similar result for Q'_0. By the correspondence between ideals and open sets in the spectra, ψ maps the ideal $C_0(Q_0) \times_\alpha G$ onto $C_0(Q'_0) \times_{\alpha'} G'$; moreover, by the last lemma, ψ maps the analytic subalgebra $C_0(Q_0) \times_\alpha \Sigma$ onto $C_0(Q'_0) \times_{\alpha'} \Sigma'$. Thus ψ is an analytic isomorphism of these parallel crossed products, so by Proposition 3.6, ψ maps $C_0(Q_0)$ onto $C_0(Q'_0)$ and the homeomorphism from Q_0 to Q'_0 determined by ψ maps orbits to orbits, and preserves order and measure class along each orbits.

Now let m vary over all of M; the various $C_0(Q_0)$ and $C_O(Q'_0)$ constructed above generate all of $C_0(M)$ and $C_0(M')$. Thus ψ maps $C_0(M)$ into $C_0(M')$; considering ψ^{-1} shows it is an isomorphism. The homeomorphism of M onto M' determined by ψ is pieced together from the homeomorphisms of the Q_0's and Q'_0, so it too maps orbits to orbits, preserving order and measure class along each orbit. □

In general, if G and G' act freely on M and M' with closed orbits, the arguments above may be extended to show that an analytic isomorphism of the crossed products determines a pointwise correspondence between the spaces which maps orbits to orbits, and is an order-preserving, measure class-preserving homeomorphism along each orbit. However, it is not at all clear how to "glue together" these homeomorphisms along the orbits into a homeomorphism of the spaces.

We conclude with an application of the previous proposition, wherein local parallel actions arise naturally. We say an action of a Lie group on a differential manifold is *smooth* if the action is smooth in both variables; that is, the map $(m, t) \longmapsto \alpha_t(m)$ is C^∞ in (m, t). Smooth actions of the real line on the two-dimensional plane are especially nice.

LEMMA 3.9. *Let α be a smooth action of the real line on the plane, with no fixed points. Then α is locally parallel.*

Proof. By Poincaré-Bendixson theory for solutions of ordinary differential equations in the plane, α is a free action, for otherwise it would have a closed loop for an orbit, inside which there must be a fixed point. Let β be a smooth action of the real line on the plane which is normal to α; for instance, solve the system of ordinary differential equations for a smooth vector field on the plane normal to the vector field given by α. As everything is smooth, for each point m in the plane, there is an open interval I in the real line about the origin such that the map $(s,t) \longmapsto \alpha_t(\beta_s(m))$ is a diffeomorphism of $I \times I$ onto an open neighbourhood of m. Again by Poincaré-Bendixson, each orbit of α hits this neighbourhood at most once, otherwise there is a closed loop consisting of part of an α-orbit and part of the β-orbit passing through m, inside of which would lie a fixed point. Thus the map $(s,t) \longmapsto \alpha_t(\beta_s(m))$ extends to a diffeomorphism of $I \times R$ onto an open set in the plane. That is, m sits inside an open set on which α is a parallel action, so α is locally parallel. □

PROPOSITION 3.10. *Let α and α' be smooth actions of the real line on the plane with no fixed points. If $C_0(R^2) \times_\alpha R$ is analytically isomorphic to $C_0(R^2) \times_{\alpha'} R$, then there is a homeomorphism of R^2 mapping α-orbits to α'-orbits.*

Proof. By the lemma, these actions are locally parallel in R^2, so apply Proposition 3.8. □

In the language of foliations, this says that an analytic isomorphism of the C*-algebras for C^∞ foliations of the plane implies topological conjugacy of the foliations. It is interesting to recall that every regular foliation of the plane is topologically conjugate to a C^∞ foliation (cf. [**17**]); however, the above results do not give a description of how to construct an analytic isomorphism from a topological conjugacy. Indeed, for general foliations, it is not even clear what is the analytic subalgebra of the C*-algebra of the foliation. We leave this question aside for future work.

4. Cohomological Results. Even in the case of parallel actions, the analytic isomorphisms reveal interesting topological information. In this section we uncover some cohomology of the underlying spaces.

When A is an analytic subalgebra of C*-algebra B, denote the group of analytic automorphisms of B by $AAut(B)$. Any analytic automorphism maps the diagonal $D = \mathcal{M}(A) \cap \mathcal{M}(A)^*$ to itself; let $AAut_D(B)$ denote the normal subgroup of analytic automorphisms that fix D pointwise. For any unitary U in the diagonal one obtains an analytic automorphism of B simply by conjugation: $a \longmapsto UaU^*$; denote this subgroup of inner automorphisms by $AInn_D(B)$. When D is abelian, $AInn_D(B)$ is a normal subgroup of $AAut_D(B)$. Thus we have a decomposition series

$$AInn_D(B) \lhd AAut_D(B) \lhd AAut(B).$$

When B is the crossed product $C_0(M) \times_\alpha G$ for G with a locally parallel action on M, with A its analytic subalgebra given by the order on G, then by Proposition 3.8, the quotient group $AAut(B)/AAut_D(B)$ is identified with a subgroup of homeomorphisms of M; indeed, it is contained in the subgroup of homeomorphisms mapping orbits to orbits and preserving order and measure class along each orbit. It is the other quotient $AAut_D(B)/AInn_D(B)$ which is of interest here.

For parallel actions, Proposition 3.1 identifies the diagonal as the algebra of strong* continuous maps from the transversal X into $L^\infty(G)$. The group of unitaries in the diagonal is then the space $C(X, \mathcal{U}(L^\infty(G)))$ of continuous functions from X into the group of unitaries in $L^\infty(G)$ endowed with the strong* topology. Thus $AInn_D(B)$ is identified as $Ad_* C(X, \mathcal{U}(L^\infty(G)))$, where Ad_* is the map taking a path of unitaries $x \longmapsto U_x$ to a path of automorphisms of the compacts $x \longmapsto Ad(U_x)$.

Also, the crossed product is identified as the algebra $C_0(X, \mathcal{K})$ of norm continuous maps from X into the compact operators on $L^2(G)$, so an automorphism is given by a path of unitaries $x \longmapsto U_x$ from X into $\mathcal{B}(L^2(G))$ such that $x \longmapsto Ad(U_x)$ is continuous from X into the group of automorphisms of $\mathcal{K}$. If the automorphism is analytic and fixes the diagonal, each U_x must lie in $\mathcal{U}(L^\infty(G))$. Thus $AAut_D(B)$ is identified with the space $C(X, Ad(\mathcal{U}(L^\infty(G))))$ of continuous functions from X into the space of automorphisms of the compacts given by conjugation with a unitary in $\mathcal{U}(L^\infty(G))$.

Let $\mathcal{U}$ denote $\mathcal{U}(L^\infty(G))$. The quotient $AAut_D(B)/AInn_D(B)$ is thus identified with $C(X, Ad(\mathcal{U}))/Ad_* C(X, \mathcal{U})$. Identifying this quotient is a lifting problem: when can a continuous map $x \longmapsto Ad(U_x)$ be lifted to a continuous map $x \longmapsto V_x$ with $Ad(V_x) = Ad(U_x)$. There are two extreme cases: one when G is discrete, in which case it can always be done, and the other when G is non-discrete, where the cohomology presents an obstruction.

PROPOSITION 4.1. *Let G be a discrete group. Then*

$$C(X, \mathrm{Ad}(\mathcal{U})) = \mathrm{Ad}_* C(X, \mathcal{U}),$$

so for parallel actions of G, every analytic automorphism fixing D is inner.

Proof. Let $\{g_1, g_2, g_3, \ldots\}$ be the elements of G, $\{\xi_1, \xi_2, \xi_3, \ldots\}$ the basis for $L^2(G)$ consisting of characteristic functions on the points $\{g_1, g_2, g_3, \ldots\}$, and ϵ_{ij} the rank one operator that takes ξ_j to ξ_i. Each element $F \in C(X, \mathrm{Ad}(\mathcal{U}))$ is represented by a map $x \longmapsto U_x$ of X into $\mathcal{U}$ such that $x \longmapsto \mathrm{Ad}(U_x) = F(x)$ in $Aut(\mathcal{K})$ is continuous, where each U_x can be considered as a function in $L^\infty(G)$ taking complex values of modulus 1. Let V_x in $\mathcal{U}$ be defined by

$$V_x(g_i) = U_x(g_i) U_x(g_1)^*$$

for $i = 1, 2, 3, \ldots$. Thus $\mathrm{Ad}(V_x) = \mathrm{Ad}(U_x)$ for all x in X, so V represents the same element F in $C(X, \mathrm{Ad}(\mathcal{U}))$. However, $V_x \xi_1 = \xi_1$ and an easy calculation shows

$$\mathrm{Ad}(V_x)\epsilon_{j1} = V_x \epsilon_{j1} V_x^* = V_x(g_j)\epsilon_{j1}.$$

Since ϵ_{j1} is compact, the map $x \longmapsto V_x(g_j)$ is continuous for each j, thus $x \longmapsto V_x$ is strongly continuous. That is, V is in $C(X, \mathcal{U})$ and

$$F = \mathrm{Ad}_* V \in \mathrm{Ad}_* C(X, \mathcal{U}). \qquad \square$$

We do not get a similar result when G is not discrete because there are no atoms around and so $\mathcal{U}(L^\infty(G))$ is contractible.

LEMMA 4.2. *Let G be a locally compact, separable, non-discrete group. Then the unitary group $\mathcal{U}(L^\infty(G))$ is contractible in the strong operator topology.*

Proof. G is a non-atomic separable σ-finite measure space and so the groups of unitaries in $L^\infty(G)$ and $L^\infty[0,1]$ are topologically isomorphic. Thus it suffices to show that $\mathcal{U} = \mathcal{U}(L^\infty[0,1])$ is contractible. We define a homotopy $h_t : \mathcal{U} \longrightarrow \mathcal{U}$ by

$$h_t(U) = 1_{[0,t]} + U 1_{[t,1]}$$

for all U in $\mathcal{U}$ and t in $[0,1]$, where $1_{[s,t]}$ is the characteristic function on the interval $[s,t]$. It is clear that $t \longmapsto h_t(U)$ is weakly continuous for any U in $\mathcal{B}(L^2([0,1]))$ and since $h_t(U)$ is in $\mathcal{U}$ for all U in $\mathcal{U}$, the path is strongly continuous for all U in $\mathcal{U}$. Thus $h_0(U) = U$ and $h_1(U) = I$ so this is a homotopy between U and I as desired. $\square$

PROPOSITION 4.3. *Let X be a compact space, G non-discrete, and $\mathcal{U} = \mathcal{U}(L^\infty(G))$. Then $C(X, Ad(\mathcal{U}))/Ad_*C(X, \mathcal{U})$ is isomorphic to $\check{H}^2(X;Z)$, the second Čech cohomology of X with coefficients in the integers. Thus, when G acts in parallel on space M with transversal X, the crossed product $B = C_0(M) \times_\alpha G$ gives a quotient of automorphism groups*

$$AAut_D(B)/AInn_D(B) \cong \check{H}^2(X;Z).$$

Proof. The results follows from the fact that Π, the set of complex scalars of modulus one, sits inside $\mathcal{U}$ as a closed normal subgroup and so the map Ad : $\mathcal{U} \longrightarrow \mathrm{Ad}(\mathcal{U}) \subseteq Aut(\mathcal{K})$ is a principal Π-bundle, where Π is homeomorphic to the circle S^1, $\mathcal{U}$ has the strong operator topology, and $Aut(\mathcal{K})$ has the topology of pointwise convergence. There is a well-developed theory on principal fiber bundles (cf. [**15**]) and rather than describe them here, note that in Theorem 4.1 of [**14**], there is a proof of the result we have here except that in [**14**], $\mathcal{U}$ is replaced by the full group of unitaries on a Hilbert space $\mathcal{H}$ in the uniform topology, which is a contractible group in that topology. We will summarize the proof for our case, noting that only the objects have changed. There is an exact homotopy sequence

$$\cdots \longrightarrow \pi_i(S^1) \longrightarrow \pi_i(\mathcal{U}) \xrightarrow{\mathrm{Ad}} \pi_i(\mathrm{Ad}(\mathcal{U})) \xrightarrow{\partial} \pi_{i-1}(S^1) \longrightarrow \cdots.$$

Since $\mathcal{U}$ is contractible, $\pi_i(\mathcal{U}) = 0$ for all i and ∂ is an isomorphism between $\pi_i(\mathrm{Ad}(\mathcal{U}))$ and $\pi_{i-1}(S^1)$ for $i \geqq 1$. Thus

$$\pi_i(\mathrm{Ad}(\mathcal{U})) = \begin{cases} Z & \text{if } i = 2 \\ 0 & \text{if } i \neq 2 \end{cases}$$

so $\mathrm{Ad}(\mathcal{U})$ is called an Eilenberg-MacLane space of type $(Z, 2)$ and has the property that there is a natural isomorphism of abelian groups

$$\hat{H}^2(X; Z) \cong [X, \mathrm{Ad}(\mathcal{U})]$$

where $[X, \mathrm{Ad}(\mathcal{U})]$ denotes the groups of homotopy classes of maps $F : X \longrightarrow \mathrm{Ad}(\mathcal{U})$. Now, with $C_n(X, \mathrm{Ad}(\mathcal{U}))$ the normal subgroup of $C(X, \mathrm{Ad}(\mathcal{U}))$ consisting of null-homotopic maps, there is a natural isomorphism of discrete groups

$$[X, \mathrm{Ad}(\mathcal{U})] \cong C(X, \mathrm{Ad}(\mathcal{U}))/C_n(X, \mathrm{Ad}(\mathcal{U})).$$

Finally, by the covering homotopy theorem, every null-homotopic map can be lifted, which shows that $C_n(X, \mathrm{Ad}(\mathcal{U})) \subseteq \mathrm{Ad}_* C(X, \mathcal{U})$, while the fact that $\mathcal{U}$ is contractible implies every element of $\mathrm{Ad}_* C(X, \mathcal{U})$ is null-homotopic, so the reverse inclusion holds. Thus $C_n(X, \mathrm{Ad}(\mathcal{U})) = \mathrm{Ad}_* C(X, \mathcal{U})$ and combining the last three equations gives

$$\hat{H}^2(X; Z) \cong C(X, \mathrm{Ad}(\mathcal{U}))/\mathrm{Ad}_* C(X, \mathcal{U}).$$

The final statement of the proposition is just identifying the automorphism groups of the crossed product with the maps on X. □

There is nothing mysterious about these analytic automorphisms coming from the cohomology; for example the real line R acting on the space of parallel $S^2 \times R$ for X the two-sphere, such an automorphism is obtained by a map of the crossed product in the form

$$\int_R g_t u_t dt \longmapsto \int_R g_t \omega_t u_t dt$$

where these integrals are considered as the universal representation of the crossed product, with $t \longmapsto \omega_t$ a non-trivial cocycle from R into the group of unitaries in the diagonal.

5. Examples. The motivating example for topological ordered groups is of course the real line with the usual order. More generally, an n-dimensional real vector space G with a regularly closed convex cone Σ such that $\Sigma \cap (-\Sigma) = \{0\}$ gives an example of a topological ordered group with the dominated convergence property, where the order is given by $s \leqq t$ if $t - s$ is an element of Σ. For instance, such a group is $G = R^n$ with Σ equal to the positive orthant of vectors with

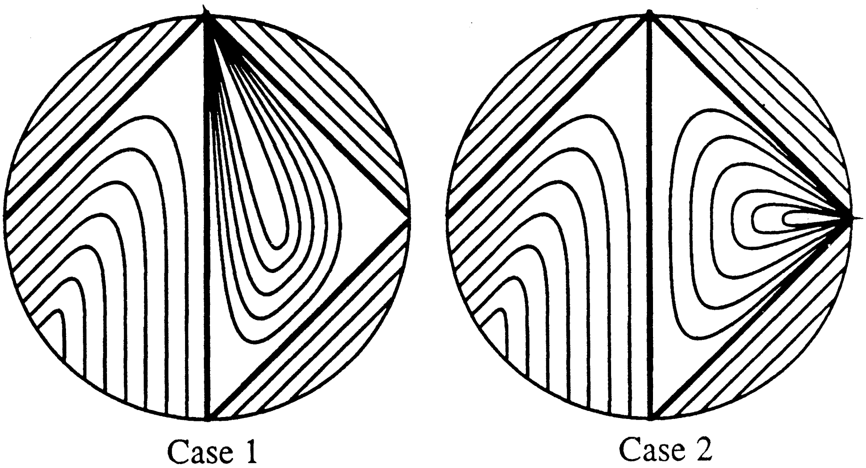

Figure 1. Two non-conjugate foliations.

non-negative coordinates, as is Minkowski space R^4 with Σ equal to the forward light cone. An example of a topological ordered group without the DCP is the direct product of the integers with the real line, under the lexicographic order; that is, two pairs in $Z \times R$ are ordered $(n, t) \leqq (n', t')$ iff $n \leqq n'$, and $t \leqq t'$ when $n = n'$. The ordered group $Z \times R$ is order isomorphic to a single copy of the real line, with a discrete countable set removed; thus $Z \times R$ is Borel order isomorphic to the real line, yet not homeomorphic to it.

A non-abelian example of a topological ordered group is the so-called "$ax+b$" group; that is, the group of affine transformations of the real line given by pairs of real numbers (a, b), with a positive. Take Σ to be the semigroup of elements (a, b) with $a \geqq 1$ and $b \geqq 0$. It is easy to check for this group that intervals are compact, thus the interiors generate the topology and the group has the dominated convergence property.

For an instance of two isomorphic C*-crossed products which are not analytically isomorphic, consider Wang's example 4.3.3 in [**17**] of two smooth foliations of the unit disk as shown here in Figure 1. It is not hard to see that these foliations are not conjugate by observing the difference in the orbits in the triangular region on the right half of each disk, or by calculating the Kaplan diagram. Thus, no matter what choice of smooth actions one takes for defining the C*-crossed product, by Proposition 3.10 the algebras are not analytically isomorphic. Nevertheless, by Wang, the C*-algebras of the foliation are isomorphic. Following the algorithm developed in [**17**] for describing the C*-algebra of a foliation, it is possible to describe the analytic algebras as the algebra of C_0-functions from the half-open interval $[0, 1)$ into a direct product of lower triangular matrix algebras, with a boundary condition at 0. With $\mathcal{K}$ the algebra of compacts on $L^2(R)$ and $\mathfrak{R} = \mathfrak{R}(\Sigma)$ the analytic subalgebra of $\mathcal{K}$ determined by the order on the real line, one choice for the orientation of the foliations gives the analytic algebras as functions f on $[0,1]$ vanishing at 1, and taking values in the direct product

$$\mathfrak{N} \oplus \begin{pmatrix} \mathfrak{N} & 0 \\ \mathcal{K} & \mathfrak{N} \end{pmatrix} \oplus \begin{pmatrix} \mathfrak{N} & 0 & 0 \\ \mathcal{K} & \mathfrak{N} & 0 \\ \mathcal{K} & \mathcal{K} & \mathfrak{N} \end{pmatrix} \oplus \mathfrak{N} \oplus \mathfrak{N}$$

with $f(0)$ of the form

$$T_1 \oplus \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \oplus \begin{pmatrix} T_2 & 0 & 0 \\ 0 & T_3 & 0 \\ 0 & 0 & T_4 \end{pmatrix} \oplus T_3 \oplus T_4$$

in the first foliation, and in the second foliation, of the form

$$T_1 \oplus \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \oplus \begin{pmatrix} T_4 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix} \oplus T_3 \oplus T_4.$$

The C*-crossed products are just the above algebras with the lower triangular terms replaced by the full algebras of compacts, including replacing the lower triangular algebras $\mathfrak{N}$ with $\mathcal{K}$, and retaining the same conditions on $f(0)$. It is easy to see why the C*-algebras are isomorphic: take an isomorphism which shuffles the rows and columns in the center 3 by 3 matrix as appropriate to transform the first condition on $f(0)$ to the second. This is not an analytic isomorphism since it destroys the lower triangular form of the 3 by 3 matrix, so it cannot map one analytic algebra onto the other. It is also interesting to note that for both of these foliations, there is no automorphism of the crossed product $C_0(\Omega) \times_\alpha R$ taking $C_0(\Omega) \times_\alpha R^+$ onto $C_0(\Omega) \times_\alpha R^-$. Thus, the orientation given by the order, as determined by the choice of the action, is significant.

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