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Algebraic properties of Shintani's generating functions: Dedekind sums and cocycles on $\mathrm{PGL}_2(\mathbb{Q})$

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Abstract. We introduce certain quotients of two-variable formal power series, called 'Shintani Functions'. They generalise the generating functions of [Sh] for special values of partial ζ-functions over real quadratic fields. A study of their formal properties leads to quick, non-analytic proofs of some results on generalised Dedekind sums (Reciprocity Law, etc.). It also gives an algebraic construction of certain 1-cocycles on the group $\mathrm{PGL}_2(\mathbb{Q})$ similar to those constructed by R. Sczech and G. Stevens using analytic methods.

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1. Introduction and notations

This paper seeks to make a new contribution to a circle of ideas linking special values of zeta-functions with generalised Dedekind sums. The special values in question are those taken at non-positive integers by the partial zeta-function $\zeta_K(s, \mathfrak{a})$ associated to a ray-class $\mathfrak{a}$ of a real quadratic field K. Siegel was the first to obtain explicit general formulae for these values in [Si1] and [Si2]. His formulae involve sums of products of values of Bernoulli polynomials, sums which can be recognised as generalisations of the sums '$s(h,k)$' of Dedekind appearing in the transformation law for the logarithm of the η-function. Siegel's formulae show in particular that these special values are *rational,* a result extended by Klingen and Siegel to an arbitrary totally-real number field F of degree d over $\mathbb{Q}$. In his paper [Sh], Shintani gave a new proof of this fact by means of some remarkable explicit formulae for $\zeta_F(-k, \mathfrak{a})$, $k = 0, 1, 2, \ldots$. His method is entirely different from Siegel's. It involves the construction of cones in $\mathbb{R}^d$ to which one associates certain quotients of d-variable power-series. Shintani proved by complex-analytic methods that, roughly speaking, these quotients act as generating functions for the above-mentioned special values. When d equals 1 they are the generating functions of the Bernoulli polynomials themselves. These polynomials are well known to give the special values of Hurwitz' zeta-functions (the case $d = 1$, $F = \mathbb{Q}$). In the case $d = 2$, Shintani showed how Siegel's explicit formulae for $\zeta_K(-k, \mathfrak{a})$ could be recovered from his and the Dedekind-type sums reappear.

A new and unifying element enters the circle in the work of Glenn Stevens [St] and Robert Sczech [Sc1]. Both of these authors construct *'universal'* 1-*cocycles* on the group $\mathrm{PGL}_2(\mathbb{Q})$. On the one hand their cocycles can be written out explicitly in terms of generalised Dedekind sums. On the other, by specialising them at the appropriate matrix (representing the action of a certain unit), they can be used to calculate the partial zeta-values for *any* real quadratic field K. Both constructions are analytic in nature. (Stevens uses certain 'periods' of Eisenstein series and his theory of 'modular caps' and modular symbols, making the connection with zeta-values by means of Siegel's work. Sczech constructs his cocycles by real-analytic methods). The importance of the cocycle interpretation can be seen from its applications both to the calculation of zeta-values – where it gives rise to highly efficient continued-fraction algorithms – and to generalised Dedekind sums, where it produces very general versions of Dedekind's 'Reciprocity Law' for the sums $s(h,k)$. These applications are explained in [St] and in [Sc1] (see also [Hay] for the zeta-values at $s=0$) although the generalised reciprocity laws are 'predicted' rather than written out precisely.

The aim of this paper is to present 1-cocycles on $\mathrm{PGL}_2(\mathbb{Q})$ of a similar nature to those of Sczech and Stevens but with two significant differences. Firstly, our 'Shintani Cocycles' are *'parabolic'*: they vanish when evaluated at (the images of) matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ in $\mathrm{PGL}_2(\mathbb{Q})$. Those of Sczech and Stevens are not parabolic; but then the spaces in which they take values, though similar, are not the same either. Secondly, our construction is completely elementary and essentially algebro-combinatorial. The fundamental tool is a modification of the formal generating functions which were introduced by Shintani and can be attached to certain geometrical data involving cones and lattices in $\mathbb{R}^2$. The object of Section 2 is to define these 'Shintani functions' and then to elucidate their fundamental properties, not as generators of zeta-values but rather as formal algebraic objects in their own right. A precise connection with Dedekind sums is established in Section 3. It transpires that the coefficients of the Shintani functions are, essentially, the elements of a doubly-infinite sequence of highly general sums which were defined explicitly by Halbritter in [Hal], although they were already present in Siegel's formulae. The functional properties of Section 3 consequently lead to new and elementary proofs of certain identitites for these sums, not only Halbritter's generalisation of the Reciprocity Law but also the so-called 'Generalised Petersson–Knopp Identities' (see e.g. [A-V]). One could probably devise further applications of a similar nature. The final section of this paper deals with the construction of cocycles by means of Shintani functions. Thanks to the results of Section 2, very little actually needs to be done once the appropriate framework of $\mathrm{PGL}_2(\mathbb{Q})$-modules of distributions on $\mathbb{R}^2/\mathbb{Z}^2$ has been set up.

The fact that the Shintani Cocycles also calculate partial zeta-values is guaranteed by their origins in Shintani's generating functions and follows therefrom with no further analytic work. We have chosen nevertheless to postpone details of this connection to a sequel to this paper in which we shall have room to discuss several further aspects of the two-way relationship between the cocycle formulation and properties of zeta-values. In one direction, there are continued fraction algorithms of the type mentioned above. In the other, the non-vanishing of certain L-functions implies non-triviality results for the Shintani cocycles. The cocycle interpretation also suggests further modifications of Shintani's functions which can be applied, for instance, to give a new variant of the construction of the p-adic partial zeta-functions by p-adic interpolation. (For another p-adic application related to the cocycle property, see [So]).

Two natural questions concerning the cocycle formulation are: *'How, precisely, are its three variants (Stevens', Sczech's and ours) related?'* and *'Do they generalise to dimensions d greater than 2?'* No precise comparison between Sczech's and Stevens' cocycles seems to have been carried out and, apart from above remark concerning parabolicity, we shall not compare our cocycle directly with either of the other two versions in the present paper. However, Stevens' cocycle is believed to be essentially the same as a certain cohomologous variant of ours which will be introduced in a sequel, [So2] As regards higher dimensions, Sczech has already generalised his analytic construction to arbitrary d in a later paper [Sc2] to [Sc1]. If the construction of the present paper is to be generalised, the main (or at least the first) task must be to gain a proper understanding of the combinatorics of lattices, configurations of cones and their degenerations in higher dimensions. (*Added in proof*: See [Hu] for the case $d = 3$ and a partial generalisation for all d).

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ will have their usual meanings in this paper (note that $0 \in \mathbb{N}$). $\mathbb{R}^\times$ will denote the multiplicative group of the nonzero real numbers and sgn: $\mathbb{R}^\times \rightarrow \{\pm 1\}$ the sign homomorphism whose kernel we shall denote $\mathbb{R}^\times_+$. The multiplicative groups $\mathbb{Q}^\times$ and $\mathbb{Q}^\times_+$ are defined analogously. The function 'sgn' will also be extended to $\mathbb{R}$ by setting $\mathrm{sgn}(0) = 0$. Bold face upper-case letters $\mathbf{A}$, $\mathbf{\Omega}$ etc. will be used to denote matrices (usually elements of the general linear group $\mathrm{GL}_2(\mathbb{R})$). These act on the left on (column) vectors which will be notated in bold lower-case ($\mathbf{a}$, $\mathbf{x}$, $\mathbf{0}$ etc.) and will usually represent elements of $\mathbb{R}^2$ or of some associated quotient or subgroup. The symbol $\mathbf{z}$, however, will be reserved for a pair of formal variables (z_1, z_2) to be regarded as a *row* vector. By sticking to these conventions, expressions such as $\mathbf{zM}$, $\mathbf{Ma}$ and $\mathbf{z}.\mathbf{a}$ should be quite unambiguous. We shall use the symbols $\mathbb{R}(\mathbf{z})$ and $\mathbb{R}((\mathbf{z}))$ to denote the fraction-fields of the rings $\mathbb{R}[\mathbf{z}]$ and $\mathbb{R}[[\mathbf{z}]]$ of polynomials and formal power-series respectively, all naturally included in $\mathbb{R}((\mathbf{z}))$. The (total) degree function 'deg' defined on $\mathbb{R}[\mathbf{z}]\backslash\{0\}$ provides a grading on the $\mathbb{R}$-algebra $\mathbb{R}[\mathbf{z}]$. For each $l \in \mathbb{Z}$ we shall write $\mathbb{R}(\mathbf{z})_l$ for the $\mathbb{R}$-vector subspace of $\mathbb{R}(\mathbf{z})$ consisting of the homogeneous rational functions of degree l, plus 0. That is

$$\begin{aligned}\mathbb{R}(\mathbf{z})_l \;:=\; \{0\} \cup \{F/G \in \mathbb{R}(\mathbf{z}) \;:\; &F, G \in \mathbb{R}[\mathbf{z}]\backslash\{0\},\\ &F, G \text{ homogeneous}, \deg(F) - \deg(G) = l\}.\end{aligned}$$

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We shall also employ the special notation $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ for the $\mathbb{R}$-subalgebra of $\mathbb{R}((\mathbf{z}))$ consisting of those elements which have a homogeneous denominator

$$\mathbb{R}((\mathbf{z}))^{\mathrm{hd}} := \{F/G \in \mathbb{R}((\mathbf{z})) \,:\, F \in \mathbb{R}[[\mathbf{z}]], G \in \mathbb{R}[\mathbf{z}]\backslash\{0\}, G \text{ homogeneous}\}.$$

For any power-series F in $\mathbb{R}[[\mathbf{z}]]$ and $l \in \mathbb{N}$ we write $F_l \in \mathbb{R}[\mathbf{z}]$ for its lth homogeneous part, the formal sum (possibly 0) of its component monomials of degree l. This notation has an obvious extension to $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$: the '$l$th homogeneous part' H_l of $H \in \mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ is unambiguously defined to be F_{l+k}/G where F/G is any representation of H with G a homogeneous polynomial of degree k. The mapping $H \mapsto H_l$ thus defines an $\mathbb{R}$-linear surjection π_l from $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ onto its $\mathbb{R}$-vector subspace $\mathbb{R}(\mathbf{z})_l$ in such a way that the product map

$$\prod_{l\in\mathbb{Z}} \pi_l \colon \mathbb{R}((\mathbf{z}))^{\mathrm{hd}} \to \prod_{l\in\mathbb{Z}} \mathbb{R}(\mathbf{z})_l$$

is injective. The fact that it is obviously not surjective won't prevent us from referring to $\mathbb{R}(\mathbf{z})_l$ as the 'lth homogeneous component' of $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$.

2. Shintani functions

2.1. Definitions and basic properties

The basic objects under study are certain elements of $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ which we call 'Shintani functions' and denote $\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}) = \mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z})$. We first explain how they are defined. The data Λ, $\mathbf{x}$, $\mathfrak{r}$ and $\mathfrak{s}$ are as follows: Λ is any (rank-2) lattice in $\mathbb{R}^2$ and $\mathbf{x} \in \mathbb{R}^2/\Lambda$ is any equivalence class modulo Λ (so $\mathbf{x}$ is a subset of $\mathbb{R}^2$). The symbols $\mathfrak{r}$ and $\mathfrak{s}$ denote 'Λ-rational' rays emanating from the origin in $\mathbb{R}^2$. More precisely, they are equivalence classes for the multiplicative action of $\mathbb{Q}_+^\times$ on $\mathbb{Q}\Lambda\backslash\{\mathbf{0}\}$. We shall denote the set of such rays by $\mathbb{P}_+(\mathbb{Q}\Lambda)$. Given such a quadruple $(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$ with $\mathfrak{r} \neq \pm\mathfrak{s}$, we first define an element of $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ by

$$\tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}) = \tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z}) := \frac{\sum_{\mathbf{a}\in\mathbf{x}\cap P(\mathbf{r},\mathbf{s})} \mathrm{e}^{\mathbf{z}.\mathbf{a}}}{(1-\mathrm{e}^{\mathbf{z}.\mathbf{r}})(1-\mathrm{e}^{\mathbf{z}.\mathbf{s}})}. \tag{1}$$

Here, we have chosen any $\mathbf{r} = \in \Lambda \cap \mathfrak{r}$ and $\mathbf{s} \in \Lambda \cap \mathfrak{s}$ (these sets are clearly non-empty) and $P(\mathbf{r}, \mathbf{s})$ denotes the half-open parallelogram (see Figure 1)

$$P(\mathbf{r}, \mathbf{s}) := \{\mu\mathbf{r} + \nu\mathbf{s} \,:\, \mu, \nu \in \mathbb{R},\ 0 < \mu \leqslant 1,\ 0 \leqslant \nu < 1\}$$

whose intersection with $\mathbf{x}$ is of cardinality equal to the (finite) index $[\Lambda\colon \mathbb{Z}\mathbf{r} + \mathbb{Z}\mathbf{s}]$. The expressions $\mathrm{e}^{\mathbf{z}.\mathbf{a}}$ etc. of course represent formal exponential series. Thus we can take $(z_1r_1 + z_2r_2)(z_1s_1 + z_2s_2)$ as a degree 2, homogeneous denominator for $\tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$. Note that the sets $\Lambda \cap \mathfrak{r}$ and $\Lambda \cap \mathfrak{s}$ consist of positive integral multiples of 'minimal' elements $\mathbf{r}_0$ and $\mathbf{s}_0$ respectively. Nevertheless, to have specified $\mathbf{r} = \mathbf{r}_0$

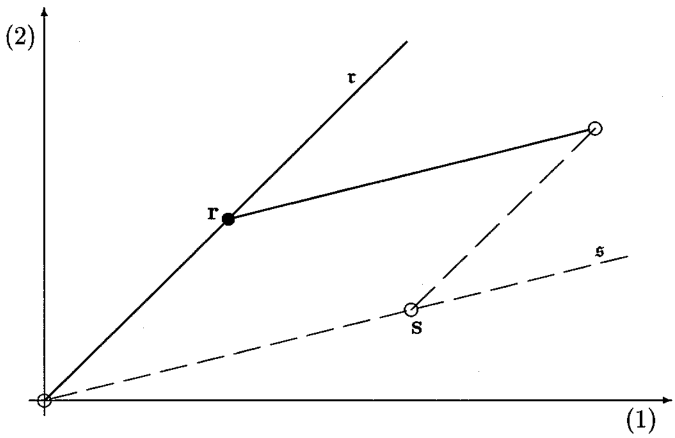

Figure 1. $P(\mathbf{r},\mathbf{s})$ and $C(\mathfrak{r},\mathfrak{s})$.

and $\mathbf{s}=\mathbf{s}_0$ would have been pointlessly restrictive since the right-hand side of (1) is independent of the choices of $\mathbf{r}$ and $\mathbf{s}$: Replacing $\mathbf{r}$ by a positive integral multiple $n\mathbf{r}$ simply multiplies both the numerator and denominator by $\Sigma_{i=0}^{n-1}\,\mathrm{e}^{\mathbf{z}.\mathbf{r}}$, and similarly for a change in $\mathbf{s}$.

REMARK 1. The reader should keep in mind the following meaningless identity obtained by the wholly illegitimate procedure of 'expanding the denominators' in (1)

$$`\tilde{\mathcal{P}}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s},\mathbf{z}) = \sum_{\mathbf{a}\in\mathbf{x}\cap C(\mathfrak{r},\mathfrak{s})} \mathrm{e}^{\mathbf{z}.\mathbf{a}}\text{'}. \tag{2}$$

Here, $C(\mathfrak{r},\mathfrak{s})$ denotes $C(\mathfrak{r},\mathfrak{s}) := (\mathbb{R}_+^\times\mathfrak{r}+\mathbb{R}_+^\times\mathfrak{s})\cup\mathbb{R}_+^\times\mathfrak{r}$, the half-open, positive, real cone on $\mathfrak{r}$ and $\mathfrak{s}$. It is the disjoint union of the translates of $P(\mathbf{r},\mathbf{s})$ by the elements of $\mathbb{N}\mathbf{r}+\mathbb{N}\mathbf{s}$. The sum on the right of (2) is of course infinite and does not converge as a power series in $\mathbb{R}[[\mathbf{z}]]$. The idea should nevertheless be helpful.

For $\mathfrak{r}=\pm\mathfrak{s}$ we set $\tilde{\mathcal{P}}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s};\mathbf{z})=0$ for all Λ and $\mathbf{x}$. Next given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ we define $\mathfrak{S}(\mathbf{u},\mathbf{v})$ to be $\mathrm{sgn}\begin{vmatrix} u_1 & v_1\\ u_2 & v_2\end{vmatrix}$ and for rays $\mathfrak{r},\mathfrak{s}\in\mathbb{P}_+(\mathbb{Q}\Lambda)$, we shall write $\mathfrak{S}(\mathfrak{r},\mathfrak{s})$ for the common value of $\mathfrak{S}(\mathbf{r},\mathbf{s})$ for any $\mathbf{r}\in\mathfrak{r}$ and $\mathbf{s}\in\mathfrak{s}$. Using the traditional picture of $\mathbb{R}^2$ in the plane, $\mathfrak{S}(\mathfrak{r},\mathfrak{s})$ is equal to $+1$ or -1 (for $\mathfrak{r}\neq\pm\mathfrak{s}$), according as one turns anticlockwise or clockwise in passing from $\mathfrak{r}$ to $\mathfrak{s}$ *within* $C(\mathfrak{r},\mathfrak{s})$. It is zero if $\mathfrak{r}=\pm\mathfrak{s}$. We can now make the

DEFINITION 2.1. For each quadruple $(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s})$ as above, we define corresponding the Shintani function in $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ to be

$$\begin{aligned}\mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s}) &:= \tfrac{1}{2}\mathfrak{S}(\mathfrak{r},\mathfrak{s})(\tilde{\mathcal{P}}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s};\mathbf{z}) + \tilde{\mathcal{P}}(\Lambda,\mathbf{x},\mathfrak{s},\mathfrak{r};\mathbf{z}))\\ &= \mathfrak{S}(\mathfrak{r},\mathfrak{s})\frac{\sum'_{\mathbf{a}\in\mathbf{x}\cap\overline{P(\mathbf{r},\mathbf{s})}} \mathrm{e}^{\mathbf{z}.\mathbf{a}}}{(1-\mathrm{e}^{\mathbf{z}.\mathbf{r}})(1-\mathrm{e}^{\mathbf{z}.\mathbf{s}})},\end{aligned} \tag{3}$$

for any $\mathbf{r}\in\mathfrak{r}$ and $\mathbf{s}\in\mathfrak{s}$. Here $\overline{P(\mathbf{r},\mathbf{s})}$ denotes the common closure of $P(\mathbf{r},\mathbf{s})$ and $P(\mathbf{s},\mathbf{r})$ and the symbol Σ' will indicate by convention that any terms corresponding to points $\mathbf{a}$ which lie on the *boundary* of this parallelogram should be included in the sum with a coefficient of $\frac{1}{2}$, *except* those (if any) corresponding to the two vertices $\mathbf{0}$ and $\mathfrak{r}+\mathfrak{s}$, which are always to be excluded from the sum.

We now investigate the behaviour of $\mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s})$ as a function of its four arguments. First we consider the dependence on the rays $\mathfrak{r},\mathfrak{s}$. The *Dirac function* $\delta_\Lambda(\mathbf{x})$ on $\mathbb{R}^2/\Lambda$ is defined to be 1 or 0 according as $\mathbf{x}$ is or is not the zero class.

PROPOSITION 2.1. *For all quadruples* $(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s})$ *we have*

(i) $\mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s}) = -\mathcal{P}(\Lambda,\mathbf{x},\mathfrak{s},\mathfrak{r})$,
(ii) $\mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s}) + \mathcal{P}(\Lambda,\mathbf{x},-\mathfrak{s},\mathfrak{r}) = -\frac{1}{2}\mathfrak{S}(\mathfrak{r},\mathfrak{s})\delta_\Lambda(\mathbf{x})$,
(iii) $\mathcal{P}(\Lambda,\mathbf{x},-\mathfrak{r},-\mathfrak{s}) = \mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s})$ *and*
(iv) $\mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},-\mathfrak{s}) = \mathcal{P}(\Lambda,\mathbf{x},-\mathfrak{r},\mathfrak{s}) = \mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s}) + \frac{1}{2}\mathfrak{S}(\mathfrak{r},\mathfrak{s})\delta_\Lambda(\mathbf{x})$.

Proof. Part (i) is obvious, (iii) follows from (ii) by iteration and (iv) follows from (i), (ii) and (iii). It remains to prove part (ii) of which only the case $\mathfrak{r}\neq\pm\mathfrak{s}$ is non-trivial. For this, we select $\mathbf{r}\in\mathfrak{r}$ and $\mathbf{s}\in\mathfrak{s}$ and apply the second formula in (3) using the pairs $(\mathbf{r},\mathbf{s})$ and $(-\mathbf{s},\mathbf{r})$. This gives

$$\mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s}) + \mathcal{P}(\Lambda,\mathbf{x},-\mathfrak{s},\mathfrak{r}) = \mathfrak{S}(\mathfrak{r},\mathfrak{s})\frac{\sum'_{\mathbf{a}\in\mathbf{x}\cap\overline{P(\mathbf{r},\mathbf{s})}} \mathrm{e}^{\mathbf{z}.\mathbf{a}} - \mathrm{e}^{\mathbf{z}.\mathbf{s}}\sum'_{\mathbf{a}\in\mathbf{x}\cap\overline{P(-\mathbf{s},\mathbf{r})}} \mathrm{e}^{\mathbf{z}.\mathbf{a}}}{(1-\mathrm{e}^{\mathbf{z}.\mathbf{r}})(1-\mathrm{e}^{\mathbf{z}.\mathbf{s}})}.$$

A simple argument using the definition of Σ' implies that there is complete cancellation in the numerator except for any possible 'vertex terms'. These latter arise only if $\mathbf{x}$ is the zero class and then give a numerator of $(\mathrm{e}^{\mathbf{z}.\mathbf{r}}+\mathrm{e}^{\mathbf{z}.\mathbf{s}}-\mathrm{e}^{\mathbf{z}.(\mathbf{r}+\mathbf{s})}-1)/2 = -(1-\mathrm{e}^{\mathbf{z}.\mathbf{r}})(1-\mathrm{e}^{\mathbf{z}.\mathbf{s}})/2$. □

Next, we consider the effects of changing Λ and $\mathbf{x}$. Let $\mathbf{M} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ be a matrix in $\mathrm{GL}_2(\mathbb{R})$. For each $F\in\mathbb{R}[[\mathbf{z}]]$, we write $F\circ\mathbf{M}$ for the power series $F(\mathbf{z}\mathbf{M}) = F(az_1+cz_2, bz_1+dz_2)$. The 'change-of-variable-map' $F\mapsto F\circ\mathbf{M}$ is clearly an $\mathbb{R}$-algebra automorphism of $\mathbb{R}[[\mathbf{z}]]$ and so extends to an automorphism of the quotient field which preserves $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ and which we also denote by '$\circ\mathbf{M}$'. We define an $\mathbb{R}$-linear, left action, '$\star$', of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ by setting

$\mathbf{M} \star F := \operatorname{sgn}(\det \mathbf{M}) F \circ \mathbf{M}$, for all $\mathbf{M} \in \mathrm{GL}_2(\mathbb{R})$ and $F \in \mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$. Any matrix $\mathbf{M} \in \mathrm{GL}_2(\mathbb{R})$ also acts naturally on the left on $\mathbb{R}^2$, taking a lattice Λ to a lattice $\mathbf{M}\Lambda$, a class $\mathbf{x} \in \mathbb{R}^2/\Lambda$ to a class $\mathbf{M}(\mathbf{x}) \in \mathbb{R}^2/\mathbf{M}\Lambda$ and each ray $\mathfrak{r} \in \mathbb{P}_+(\mathbb{Q}\Lambda)$ to a ray $\mathbf{M}\mathfrak{r} \in \mathbb{P}_+(\mathbb{Q}\mathbf{M}\Lambda)$. With these notations, the following proposition should be evident from Definition 2.1.

PROPOSITION 2.2. *For any quadruple* $(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$ *and any* $\mathbf{M} \in \mathrm{GL}_2(\mathbb{R})$ *we have the identities* $\tilde{\mathcal{P}}(\mathbf{M}\Lambda, \mathbf{M}(\mathbf{x}), \mathbf{M}\mathfrak{r}, \mathbf{M}\mathfrak{s}) = \tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}) \circ \mathbf{M}$ *and* $\mathcal{P}(\mathbf{M}\Lambda, \mathbf{M}(\mathbf{x}), \mathbf{M}\mathfrak{r}, \mathbf{M}\mathfrak{s}) = \mathbf{M} \star \mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$. □

EXAMPLE 1. Suppose that $\{\boldsymbol{\omega}, \boldsymbol{\omega}'\}$ is a base for the lattice Λ. Then $\Lambda = \mathbf{\Omega}\mathbb{Z}^2$ where

$$\mathbf{\Omega} = \begin{pmatrix} \omega_1 & \omega_1' \\ \omega_2 & \omega_2' \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

Proposition 2.2 tells us that for any $\mathbf{x} \in \mathbb{Q}\Lambda/\Lambda$ and $\mathfrak{r}, \mathfrak{s} \in \mathbb{P}_+(\mathbb{Q}\Lambda)$ we can write $\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$ as $\mathbf{\Omega} \star \mathcal{P}(\mathbb{Z}^2, \mathbf{\Omega}^{-1}(\mathbf{x}), \mathbf{\Omega}^{-1}\mathfrak{r}, \mathbf{\Omega}^{-1}\mathfrak{s})$ with $\mathbf{\Omega}^{-1}(\mathbf{x}) \in (\mathbb{R}/\mathbb{Z})^2$ and $\mathbf{\Omega}^{-1}\mathfrak{r}$, $\mathbf{\Omega}^{-1}\mathfrak{s} \in \mathbb{P}_+(\mathbb{Q}^2)$. Thus, for many purposes it is sufficient to study the Shintani functions associated to the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. In this case we shall often abbreviate $\mathcal{P}(\mathbb{Z}^2, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$ to $\mathcal{P}(\mathbf{x}, \mathfrak{r}, \mathfrak{s}) = \mathcal{P}(\mathbf{x}, \mathfrak{r}, \mathfrak{s}, \mathbf{z})$. Indeed, by insisting (as we may) that ω be the 'minimal' element $\mathbf{r}_0$ of $\mathfrak{r} \cap \Lambda$, this linear change of variable reduces us to the consideration of functions of type

$$\mathcal{P}\left(\mathbb{Z}^2, \mathbf{x}, \mathbb{Q}_+^\times \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathfrak{s}\right) \text{ for } \mathfrak{s} \in \mathbb{P}_+(\mathbb{Q}^2).$$

Now, for each sublattice Λ' of Λ, we have $\mathbb{Q}\Lambda = \mathbb{Q}\Lambda'$ and we denote by $\pi_{\Lambda',\Lambda}$ the natural, $[\Lambda : \Lambda']$-to-1 surjection from $\mathbb{R}^2/\Lambda'$ onto $\mathbb{R}^2/\Lambda$.

PROPOSITION 2.3. *Given any lattice* Λ *with sublattice* Λ', *a class* $\mathbf{x} \in \mathbb{R}^2/\Lambda$ *and two rays* $\mathfrak{r}, \mathfrak{s} \in \mathbb{P}_+(\mathbb{Q}\Lambda) = \mathbb{P}_+(\mathbb{Q}\Lambda')$, *we have*

$$\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}) = \sum_{\substack{\mathbf{x}' \in \mathbb{R}^2/\Lambda' \\ \pi_{\Lambda',\Lambda}(\mathbf{x}') = \mathbf{x}}} \mathcal{P}(\Lambda', \mathbf{x}', \mathfrak{r}, \mathfrak{s}).$$

Proof. It suffices to prove the corresponding equality for $\tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$ in the case $\mathfrak{r} \neq \pm\mathfrak{s}$: Use $\mathbf{r} \in \mathfrak{r} \cap \Lambda'$, $\mathbf{s} \in \mathfrak{s} \cap \Lambda'$ on both sides and note that, as a subset of $\mathbb{R}^2$, $\mathbf{x}$ is the disjoint union of the $\mathbf{x}'$ in the sum on the right. □

EXAMPLE 2. Let n be a nonzero integer. Applying the Proposition with $\Lambda' = n\Lambda$ we find

$$\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}) = \sum_{\substack{\mathbf{x}' \in \mathbb{R}^2/n\Lambda \\ \pi_{n\Lambda,\Lambda}(\mathbf{x}') = \mathbf{x}}} \mathcal{P}(n\Lambda, \mathbf{x}', \mathfrak{r}, \mathfrak{s}) = \sum_{\substack{\mathbf{x}' \in \mathbb{R}^2/n\Lambda \\ \pi_{n\Lambda,\Lambda}(\mathbf{x}') = \mathbf{x}}} \mathcal{P}(n\Lambda, \mathbf{x}', n\mathfrak{r}, n\mathfrak{s}), \qquad (4)$$

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by Proposition 2.1, part (iii), since $(n\mathfrak{r}, n\mathfrak{s}) = \pm(\mathfrak{r}, \mathfrak{s})$. We shall frequently write $\mathbf{n}$ for the matrix $\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ whose action on $\mathbb{R}^2$ induces an isomorphism $\mathbf{n}\colon \mathbf{y} \mapsto \mathbf{n}(\mathbf{y})$ from $\mathbb{R}^2/\Lambda$ onto $\mathbb{R}^2/n\Lambda$ such that the resulting composite map $\mathbb{R}^2/\Lambda \xrightarrow{\mathbf{n}} \mathbb{R}^2/n\Lambda \xrightarrow{\pi_{n\Lambda,\Lambda}} \mathbb{R}^2/\Lambda$ is multiplication by n in the Abelian group $\mathbb{R}^2/\Lambda$. Thus the right-hand side of (4) can be rewritten as

$$\sum_{\substack{\mathbf{y}\in\mathbb{R}^2/\Lambda \\ n\mathbf{y}=\mathbf{x}}} \mathcal{P}(\mathbf{n}\Lambda, \mathbf{n}(\mathbf{y}), \mathbf{n}\mathfrak{r}, \mathbf{n}\mathfrak{s})$$

and Proposition 2.2 gives the distribution relation

$$\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}) = \mathbf{n} \star \sum_{\substack{\mathbf{y}\in\mathbb{R}^2/\Lambda \\ n\mathbf{y}=\mathbf{x}}} \mathcal{P}(\Lambda, \mathbf{y}, \mathfrak{r}, \mathfrak{s}), \qquad \forall n \in \mathbb{Z}\backslash\{0\}. \tag{5}$$

EXAMPLE 3. In the case $\Lambda = \mathbb{Z}^2$ we can generalise the previous Example, replacing $\mathbf{n}$ by an arbitrary matrix $\mathbf{A}$ in $\mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$. The action of $\mathbf{A}$ on $\mathbb{R}^2$ induces the isomorphism $\mathbf{A}\colon \mathbf{y} \mapsto \mathbf{A}(\mathbf{y})$ from $\mathbb{R}^2/\mathbb{Z}^2$ onto $\mathbb{R}^2/\mathbf{A}\mathbb{Z}^2$ and the composite map $\pi_{\mathbf{A}\mathbb{Z}^2,\mathbb{Z}^2} \circ \mathbf{A}$ is the natural endomorphism $\mathbf{y} \mapsto \mathbf{A}\mathbf{y}$ of $\mathbb{R}^2/\mathbb{Z}^2$. (Note the distinction between $\mathbf{A}(\mathbf{y})$ and $\mathbf{A}\mathbf{y}$ in this context). Replacing the rays $\mathfrak{r}$ and $\mathfrak{s}$ by $\mathbf{A}\mathfrak{r}$ and $\mathbf{A}\mathfrak{s}$ and arguing exactly as before, we obtain

$$\mathcal{P}(\mathbf{x}, \mathbf{A}\mathfrak{r}, \mathbf{A}\mathfrak{s}) = \sum_{\substack{\mathbf{y}\in\mathbb{R}^2/\mathbb{Z}^2 \\ \mathbf{A}\mathbf{y}=\mathbf{x}}} \mathcal{P}(\mathbf{A}\mathbb{Z}^2, \mathbf{A}(\mathbf{y}), \mathbf{A}\mathfrak{r}, \mathbf{A}\mathfrak{s}) = \mathbf{A} \star \sum_{\substack{\mathbf{y}\in\mathbb{R}^2/\mathbb{Z}^2 \\ \mathbf{A}\mathbf{y}=\mathbf{x}}} \mathcal{P}(\mathbf{y}, \mathfrak{r}, \mathfrak{s}). \tag{6}$$

2.2. A FORMAL ANALOGUE OF CAUCHY'S THEOREM

Throughout this subsection we fix a lattice $\Lambda \subset \mathbb{R}^2$. We need some terminology to describe triples of (Λ-rational) rays in $\mathbb{R}^2$.

DEFINITION 2.2 (Dichotomy). A triple $(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2)$ of rays in $\mathbb{P}_+(\mathbb{Q}\Lambda)^3$ will be called 'degenerate' if and only if $|\{\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2\}| < 3$. (That is, if and only if there exist i and j with $i \neq j$ and $\mathfrak{r}_i = \mathfrak{r}_j$). Otherwise it will be called 'non-degenerate'.

DEFINITION 2.3 (Trichotomy). A triple $(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2)$ of rays in $\mathbb{P}_+(\mathbb{Q}\Lambda)^3$ will be called 'critical' if and only if $|\{\mathbb{Q}^\times \mathfrak{r}_0, \mathbb{Q}^\times \mathfrak{r}_1, \mathbb{Q}^\times \mathfrak{r}_2\}| < |\{\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2\}|$. (That is, if and only if there exist i and j with $\mathfrak{r}_i = -\mathfrak{r}_j$). A non-critical triple will be called 'splayed' if $0 \in \mathfrak{r}_0 + \mathfrak{r}_1 + \mathfrak{r}_2$, otherwise it will be called 'folded'. (See Figure 2).

A consequence of these definitions is that critical and folded triples can be either degenerate or non-degenerate but a splayed triple is automatically non-degenerate.

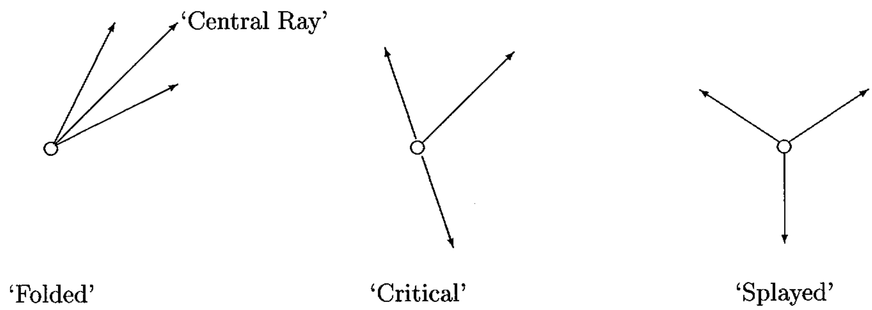

Figure 2. Non-degenerate triples.

In fact:

LEMMA 2.1. *A triple* $(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{P}_+(\mathbb{Q}\Lambda^3$ *is splayed if and only if it is non-critical, non-degenerate and satisfies* $\mathfrak{S}(\mathfrak{r}_0, \mathfrak{r}_1) = \mathfrak{S}(\mathfrak{r}_1, \mathfrak{r}_2) = \mathfrak{S}(\mathfrak{r}_2, \mathfrak{r}_0)$.

Proof. 'If': Choose $\mathbf{r}_i \in \mathfrak{r}_i$ for $i = 0, 1, 2$ then there exist $\mu_i \in \mathbb{Q}$, not all zero, such that $\mu_0\mathbf{r}_0 + \mu_1\mathbf{r}_1 + \mu_2\mathbf{r}_2 = 0$, which implies $\mathfrak{S}(\mu_0\mathbf{r}_0, \mu_1\mathbf{r}_1) = \mathfrak{S}(\mu_1\mathbf{r}_1, \mu_2\mathbf{r}_2) = \mathfrak{S}(\mu_2\mathbf{r}_2, \mu_0\mathbf{r}_0)$. Since $(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2)$ is non-degenerate and non-critical, every μ_i is necessarily nonzero as is each $\mathfrak{S}(\mathfrak{r}_i, \mathfrak{r}_j)$, so comparison with the equality of the cyclically ordered $\mathfrak{S}(\mathfrak{r}_i, \mathfrak{r}_j)$'s yields: $\mathrm{sgn}(\mu_0\mu_1) = \mathrm{sgn}(\mu_1\mu_2) = \mathrm{sgn}(\mu_2\mu_0)$. The μ_i are therefore all of the same sign, w.l.o.g. positive, so that $0 \in \mathfrak{r}_0 + \mathfrak{r}_1 + \mathfrak{r}_2$. Details of the (similar) converse argument are left to the reader. □

COROLLARY 2.1. *If* $(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2)$ *is splayed then* $(-\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2)$, $(\mathfrak{r}_0, -\mathfrak{r}_1, \mathfrak{r}_2)$ *and* $(\mathfrak{r}_0, \mathfrak{r}_1, -\mathfrak{r}_2)$ *are all non-degenerate and folded.* □

COROLLARY 2.2. *If* $(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2)$ *is non-degenerate and folded then there is a unique* $i_c \in \{0, 1, 2\}$ *such that replacing* $\mathfrak{r}_{i_c}$ *by* $-\mathfrak{r}_{i_c}$ *makes the triple splayed. If* $\{0, 1, 2\} = \{i_c, i_1, i_2\}$ *then* $\mathfrak{S}(\mathfrak{r}_{i_1}, \mathfrak{r}_{i_c}) = \mathfrak{S}(\mathfrak{r}_{i_c}, \mathfrak{r}_{i_2}) = -\mathfrak{S}(\mathfrak{r}_{i_2}, \mathfrak{r}_{i_1})$. □

In the situation of Corollary 2.2, $\mathfrak{r}_{i_c}$ will be called the '*central ray*' of the non-degenerate, folded triple. Geometrically, it is the unique ray which is contained in the (open) positive cone on the other two, as follows from the definition of splayedness.

LEMMA 2.2 (The Juxtaposition Lemma). *Let* $(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{P}_+(\mathbb{Q}\Lambda)^3$ *be a non-degenerate, folded triple with* $\mathfrak{r}_1$ *as the central ray. Then, for all* $\mathbf{x} \in \mathbb{R}^2/\Lambda$, *we have*

$$\tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}_0, \mathfrak{r}_1) + \tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}_1, \mathfrak{r}_2) = \tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}_0, \mathfrak{r}_2). \tag{7}$$

Proof. Since $\mathfrak{r}_1$ is contained in interior of the half-open cone $C(\mathfrak{r}_0, \mathfrak{r}_2)$, we clearly have

$$C(\mathfrak{r}_0, \mathfrak{r}_2) = C(\mathfrak{r}_0, \mathfrak{r}_1) \,\dot{\cup}\, C(\mathfrak{r}_1, \mathfrak{r}_2). \tag{8}$$

This lemma is therefore a natural and intuitive consequence of the regrettably meaningless identity (2). A 'proper' (i.e. rigorous) proof can be obtained by dressing this idea in respectable clothes: First consider the case where **x** is a torsion or 'Λ-rational' class, so that $\mathbf{x} \in (1/N)\Lambda$ for some $N \in \mathbb{N}$. Choose any $\mathbf{u}, \mathbf{v} \in \Lambda$, linearly independent over $\mathbb{Q}$ and such that $\mathfrak{r}_0$, $\mathfrak{r}_2$ (and hence $\mathfrak{r}_1$) are contained in the closed positive cone on **u** and **v** (e.g. take $\mathbf{u} \in \mathfrak{r}_0 \cap \Lambda$, $\mathbf{v} \in \mathfrak{r}_2 \cap \Lambda$). Set

$$\mathbf{A} = N[\Lambda \colon \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v}] \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}^{-1} \in \mathrm{GL}_2(\mathbb{R}).$$

If we can prove that

$$\tilde{\mathcal{P}}(\mathbf{A}\Lambda, \mathbf{A}(\mathbf{x}), \mathbf{A}\mathfrak{r}_0, \mathbf{A}\mathfrak{r}_1) + \tilde{\mathcal{P}}(\mathbf{A}\Lambda, \mathbf{A}(\mathbf{x}), \mathbf{A}\mathfrak{r}_1, \mathbf{A}\mathfrak{r}_2) = \tilde{\mathcal{P}}(\mathbf{A}\Lambda, \mathbf{A}(\mathbf{x}), \mathbf{A}\mathfrak{r}_0, \mathbf{A}\mathfrak{r}_2), \tag{9}$$

then (7) will follow on applying $\circ\mathbf{A}^{-1}$, by Proposition 2.2. Now, the definition of **A** implies that $\mathbf{A}\Lambda$ is contained in $N\mathbb{Z}^2$, $\mathbf{A}(\mathbf{x})$ in $\mathbb{Z}^2$ and $C(\mathbf{A}\mathfrak{r}_i, \mathbf{A}\mathfrak{r}_j)$ in the set $\{(\mu, \nu) \in \mathbb{R}^2 : \mu, \nu \geqslant 0\}$, for each (i, j). Hence, by choosing $\mathbf{b}^{(i)}$ in $\mathbf{A}\mathfrak{r}_i \cap \mathbf{A}\Lambda$, for $i = 0, 1, 2$ and setting $T_1 = \mathrm{e}^{z_1}$, $T_2 = \mathrm{e}^{z_2}$, we can rewrite Equation (9) as

$$\begin{aligned} &\frac{\sum_{\mathbf{a} \in \mathbf{A}(\mathbf{x}) \cap P(\mathbf{b}^{(0)}, \mathbf{b}^{(1)})} T_1^{a_1} T_2^{a_2}}{(1 - T_1^{\mathbf{b}_1^{(0)}} T_2^{\mathbf{b}_2^{(0)}})(1 - T_1^{\mathbf{b}_1^{(1)}} T_2^{\mathbf{b}_2^{(1)}})} + \frac{\sum_{\mathbf{a} \in \mathbf{A}(\mathbf{x}) \cap P(\mathbf{b}^{(1)}, \mathbf{b}^{(2)})} T_1^{a_1} T_2^{a_2}}{(1 - T_1^{\mathbf{b}_1^{(1)}} T_2^{\mathbf{b}_2^{(1)}})(1 - T_1^{\mathbf{b}_1^{(2)}} T_2^{\mathbf{b}_2^{(2)}})} \\ &\quad = \frac{\sum_{\mathbf{a} \in \mathbf{A}(\mathbf{x}) \cap P(\mathbf{b}^{(0)}, \mathbf{b}^{(2)})} T_1^{a_1} T_2^{a_2}}{(1 - T_1^{\mathbf{b}_1^{(0)}} T_2^{\mathbf{b}_2^{(0)}})(1 - T_1^{\mathbf{b}_1^{(2)}} T_2^{\mathbf{b}_2^{(2)}})}. \end{aligned} \tag{10}$$

Since all the exponents lie in $\mathbb{N}$ (and the $\mathbf{b}_i$ in particular are not equal to $(0, 0)$), this last equation can be viewed as taking place in the *localisation* $\mathbb{Q}[T_1, T_2]_{\mathfrak{M}}$ *of the formal polynomial ring* $\mathbb{Q}[T_1, T_2]$ *at the maximal ideal* $\mathfrak{M} := (T_1, T_2)$. Going one stage further, we can embed this local ring in its completion, canonically identified with $\mathbb{Q}[[T_1, T_2]]$. This manipulation amounts to 'expanding denominators', and so, using the geometrically obvious decomposition of $C(\mathbf{A}\mathfrak{r}_i, \mathbf{A}\mathfrak{r}_j)$ as the disjoint union $\bigcup_{s,t \in \mathbb{N}} (P(\mathbf{b}^{(i)}, \mathbf{b}^{(j)} A) + s\mathbf{b}^{(i)} + t\mathbf{b}^{(j)})$ and the fact that the $\mathbf{b}_i$ lie in $\mathbf{A}\Lambda$, it is easy to see that the target Equation (10) becomes the following equality of formal power-series

$$\sum_{\mathbf{a} \in \mathbf{A}(\mathbf{X}) \cap C(\mathbf{A}\mathfrak{r}_0, \mathbf{A}\mathfrak{r}_1)} T_1^{a_1} T_2^{a_2} + \sum_{\mathbf{a} \in \mathbf{A}(\mathbf{x}) \cap C(\mathbf{A}\mathfrak{r}_1, \mathbf{A}\mathfrak{r}_2)} T_1^{a_1} T_2^{a_2} = \sum_{\mathbf{a} \in \mathbf{A}(\mathbf{x}) \cap C(\mathbf{A}\mathfrak{r}_0, \mathbf{A}\mathfrak{r}_2)} T_1^{a_1} T_2^{a_2}.$$

To complete the proof in the case when $\mathbf{x}$ is torsion, one simply observes that $C(\mathbf{A}\mathfrak{r}_0, \mathbf{A}\mathfrak{r}_2)$ is the disjoint union of the juxtaposed half-open cones $C(\mathbf{A}\mathfrak{r}_0, \mathbf{A}\mathfrak{r}_1)$ and $C(\mathbf{A}\mathfrak{r}_1, \mathbf{A}\mathfrak{r}_2)$, (apply $\mathbf{A}$ to Equation (8)) so that the last equation is obviously satisfied.

The case $\mathbf{x} \notin \mathbb{Q}\Lambda/\Lambda$ now follows by a simple continuity argument which we shall only sketch. Clearing the denominators, Equation (7) is equivalent to the following equality in $\mathbb{R}[[\mathbf{z}]]$

$$(1 - \mathrm{e}^{\mathbf{z}.\mathbf{r}_2}) \sum_{0,1}(\mathbf{x}) + (1 - \mathrm{e}^{\mathbf{z}.\mathbf{r}_0}) \sum_{1,2}(\mathbf{x}) = (1 - \mathrm{e}^{\mathbf{z}.\mathbf{r}_1}) \sum_{0,2}(\mathbf{x}), \tag{11}$$

where $\mathbf{r}_i \in \mathfrak{r}_i \cap \Lambda$ and $\Sigma_{i,j}(\mathbf{x})$ denotes the finite sum $\Sigma_{\mathbf{a}\in\mathbf{x}\cap P(\mathbf{r}_i,\mathbf{r}_j)} \mathrm{e}^{\mathbf{z}.\mathbf{a}}$ for each pair (i, j). The present case will follow from the previous one if we can construct a sequence $\{\mathbf{x}_m\}_{m\in\mathbb{N}}$ of Λ-*rational* classes such that, for each (i, j), the power series $\Sigma_{i,j}(\mathbf{x}_m)$ tends coefficientwise to $\Sigma_{i,j}(\mathbf{x})$ as $m \to \infty$. If $\mathbf{x} \cap \mathbb{R}_+^\times \mathfrak{r}_i$ is empty for $i = 0, 1, 2$, then this construction is easy. Since the vectors $\mathbf{a}$ appearing in each $\Sigma_{i,j}(\mathbf{x})$ then all lie in the *interior* of the half-open parallelogram $P(\mathbf{r}_i, \mathbf{r}_j)$ and since $\mathbb{Q}\Lambda$ is dense in $\mathbb{R}^2$, we can take $\mathbf{x}_m$ to be $\mathbf{c}_m + \Lambda$, where $\{\mathbf{c}_m\}_{m\in\mathbb{N}}$ is any sequence in $\mathbb{Q}\Lambda$ tending to any given $\mathbf{c} \in \mathbf{x}$. If, on the contrary, there exists $i_0 \in \{0, 1, 2\}$ such that $\mathbf{x} \cap \mathbb{R}_+^\times \mathfrak{r}_{i_0} \neq \emptyset$ then such an i_0 is unique (otherwise the pairwise $\mathbb{R}$-linear independence of the rays would force $\mathbf{x} \in \mathbb{Q}\Lambda$). It follows that all the vectors $\mathbf{a}$ appearing in $\Sigma_{i,j}(\mathbf{x})$ lie *either* in the interior of $P(\mathbf{r}_i, \mathbf{r}_j)$ *or* on an edge (but not at a vertex) which is contained in a translate of $\mathbb{R}_+^\times \mathfrak{r}_{i_0}$. In order to ensure the convergence of $\Sigma_{i,j}(\mathbf{x}_m)$ to $\Sigma_{i,j}(\mathbf{x})$ for each (i, j) in this case, it therefore suffices to choose $\mathbf{c} \in \mathbf{x} \cap \mathbb{R}_+^\times \mathfrak{r}_{i_0}$ and insist that the $\mathbf{c}_n$ tending to $\mathbf{c}$ *all lie* in $\mathfrak{r}_{i_0}$. Again, this is possible by density. $\square$

REMARK. 2. This is the only point at which anything resembling an analytic argument enters our proofs. Were we to insist systematically that $\mathbf{x}$ lie in $\mathbb{Q}\Lambda/\Lambda$, then purely algebro-combinatorial methods would suffice and, in fact, such a restriction is quite natural from a number of viewpoints. For example, it is obviously 'stable' under passage to a sublattice and the action of $\mathrm{GL}_2(\mathbb{R})$ (cf. Propostions 2.2 and 2.3). It also ensures that the Shintani function $\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$ lies in $\mathbb{Q}(\Lambda)((\mathbf{z}))^{\mathrm{hd}}$, where, $\mathbb{Q}(\Lambda)$ denotes the subfield of $\mathbb{R}$ generated over $\mathbb{Q}$ by the co-ordinates of (a $\mathbb{Z}$-base for) the points of Λ. What's more, many of the applications of Shintani functions (e.g. to zeta-functions) take place within the context of torsion classes $\mathbf{x}$.

With the aid of the Juxtaposition Lemma, we shall now establish an interesting and fundamental property of Shintani functions with far-reaching consequences in the applications.

THEOREM 2.1. *Let* $(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{P}_+(\mathbb{Q}\Lambda)^3$ *be any triple of* Λ-*rational rays. Then, for all* $\mathbf{x} \in \mathbb{R}^2/\Lambda$, *we have*

$$\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}_0, \mathfrak{r}_1) + \mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}_1, \mathfrak{r}_2) + \mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}_2, \mathfrak{r}_0) = -\delta_\Lambda(\mathbf{x}) W(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2), \tag{12}$$

where $W\colon \mathbb{P}_+(\mathbb{Q}\Lambda)^3 \to \{0, \pm\frac{1}{2}, \pm 1\}$ *is the unique function invariant under cyclic permutation of its arguments defined by*

$$W(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2)$$

$$:= \begin{cases} 0 & \textit{if } (\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2) \textit{ is folded,} \\ \frac{1}{2}\mathfrak{S}(\mathfrak{r}_0, \mathfrak{r}_1) = \frac{1}{2}\mathfrak{S}(\mathfrak{r}_2, \mathfrak{r}_0) & \textit{if } (\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2) \textit{ is critical, with } \mathfrak{r}_2 = -\mathfrak{r}_1, \\ \mathfrak{S}(\mathfrak{r}_0, \mathfrak{r}_1) = \mathfrak{S}(\mathfrak{r}_1, \mathfrak{r}_2) = \mathfrak{S}(\mathfrak{r}_2, \mathfrak{r}_0) & \textit{if } (\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2) \textit{ is splayed.} \end{cases}$$

Proof. For a degenerate triple, we can assume by cyclic permutation that $\mathfrak{r}_2 = \mathfrak{r}_0$ and the result follows from Proposition 2.1, part (i). In the folded, non-degenerate case we can similarly assume that $\mathfrak{r}_1$ is the central ray. The Juxtaposition Lemma then gives

$$\tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}_0, \mathfrak{r}_1) + \tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}_1, \mathfrak{r}_2) - \tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}_0, \mathfrak{r}_2) = 0$$

and also (swapping $\mathfrak{r}_0$ with $\mathfrak{r}_2$)

$$\tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}_2, \mathfrak{r}_1) + \tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}_1, \mathfrak{r}_0) - \tilde{\mathcal{P}}(\Lambda, \mathbf{x}, \mathfrak{r}_2, \mathfrak{r}_0) = 0.$$

Taking the arithmetic mean of these two equations, multiplying by $\mathfrak{S}(\mathfrak{r}_0, \mathfrak{r}_1)$ and using Corollary 2.2 gives (12) in this case. In the critical non-degenerate case there is a unique cyclic permutation of the $\mathfrak{r}_i$ which makes $\mathfrak{r}_2 = -\mathfrak{r}_1$, so that $\mathfrak{r}_0 \neq \pm\mathfrak{r}_1$ and $\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}_1, \mathfrak{r}_2) = 0$. This case therefore follows from Proposition 2.1 part (ii). In the splayed case, Corollary 2.1 tells us that $(\mathfrak{r}_0, -\mathfrak{r}_1, \mathfrak{r}_2)$ is folded, so that (dropping Λ and $\mathbf{x}$): $\mathcal{P}(\mathfrak{r}_2, \mathfrak{r}_0) = -\mathcal{P}(\mathfrak{r}_0, -\mathfrak{r}_1) - \mathcal{P}(-\mathfrak{r}_1, \mathfrak{r}_2) = \mathcal{P}(-\mathfrak{r}_1, \mathfrak{r}_0) + \mathcal{P}(\mathfrak{r}_2, -\mathfrak{r}_1)$ by Proposition 2.1, part (i). Therefore

$$\begin{aligned} &\mathcal{P}(\mathfrak{r}_0, \mathfrak{r}_1) + \mathcal{P}(\mathfrak{r}_1, \mathfrak{r}_2) + \mathcal{P}(\mathfrak{r}_2, \mathfrak{r}_0) \\ &\quad = (\mathcal{P}(\mathfrak{r}_0, \mathfrak{r}_1) + \mathcal{P}(-\mathfrak{r}_1, \mathfrak{r}_0)) + (\mathcal{P}(\mathfrak{r}_2, -\mathfrak{r}_1) + \mathcal{P}(\mathfrak{r}_1, \mathfrak{r}_2)) \\ &\quad = -\tfrac{1}{2}\delta_\Lambda(\mathbf{x})(\mathfrak{S}(\mathfrak{r}_0, \mathfrak{r}_1) + \mathfrak{S}(\mathfrak{r}_2, -\mathfrak{r}_1)), \end{aligned}$$

by Proposition 2.1, part (ii), and the result follows from Lemma 2.1. □

Notice that W is really a winding-number! To be more precise, suppose that we are given a finite sequence of points $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1}, \mathbf{r}_n = \mathbf{r}_0$ in $\mathbb{Q}\Lambda \backslash \{\mathbf{0}\}$ for some integer $n \geqslant 3$ we shall denote by $\mathfrak{r}_i$ the Λ-rational ray $\mathbb{Q}_+^\times \mathbf{r}_i \in \mathbb{P}_+(\mathbb{Q}\Lambda)$ and we assume that $\mathfrak{r}_i \neq \pm\mathfrak{r}_{i+1}$ for $i = 0, 1, \ldots, n-1$. We can therefore define $\Gamma(\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1})$ to be the 'piecewise-linear, oriented, closed path in $\mathbb{Q}\Lambda \backslash \{\mathbf{0}\}$' whose ith edge is the 'Λ-rational line segment' going from $\mathbf{r}_i$ to $\mathbf{r}_{i+1}$ for $i = 0, 1, \ldots, n-1$. We let $\bar{\Gamma}(\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1})$ denote the closure of this path, which is contained in $\mathbb{R}^2 \backslash \{\mathbf{0}\}$. Up to homotopy in $\mathbb{R}^2 \backslash \{\mathbf{0}\}$, it depends only on the sequence of rays $(\mathfrak{r}_0, \mathfrak{r}_1, \ldots, \mathfrak{r}_{n-1})$. In the case $n = 3$ we are dealing with non-degenerate, non-critical triples $(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2)$ and $W(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2)$ as defined in the theorem is then obviously equal to the standard (anticlockwise) winding-number of the real path $\bar{\Gamma} = \bar{\Gamma}(\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2) \subset \mathbb{R}^2 \backslash \{\mathbf{0}\}$. (Explicitly, for any path γ in $\mathbb{R}^2 \backslash \{\mathbf{0}\}$, we define this winding number to be $w(\gamma) := (1/2\pi i) \int_\gamma (1/z)\,\mathrm{d}z$, where $\mathbb{R}^2$ is identified with $\mathbb{C}$ in the conventional way). Using these notations, we can deduce the following result from Theorem 2.1.

COROLLARY 2.3 (The Formal Cauchy Theorem). *Let n be an integer greater than* 2 *and let $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1}, \mathbf{r}_n = \mathbf{r}_0$ be any sequence in $\mathbb{Q}\Lambda \backslash \{\mathbf{0}\}$ which satisfies $\mathfrak{r}_i \neq \pm \mathfrak{r}_{i+1}$ for $i = 0, 1, \ldots, n-1$. Then*

$$\sum_{i=0}^{n-1} \mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}_i, \mathfrak{r}_{i+1}) = -\delta_\Lambda(\mathbf{x}) w(\bar{\Gamma}(\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1})). \tag{13}$$

Proof (Sketch). In order to ensure that $\mathfrak{r}_0 \neq \pm \mathfrak{r}_i$, $i = 1, \ldots, n-1$, we can, if necessary, insert an extra point $\mathbf{r}$ of $\mathbb{Q}\Lambda \backslash \{\mathbf{0}\}$ 'between' two successive points of the sequence and cyclically relabel to make it $\mathbf{r}_0$. This does not change the left-hand side of (13) (by Theorem 2.1), or the right-hand side. Now break the path up into a sum of triangles $\bar{\Gamma}(\mathbf{r}_0, \mathbf{r}_i, \mathbf{r}_{i+1})$ for $i = 1, \ldots, n-2$ and conclude by induction on n, using Theorem 2.1 and an obvious 'additivity' property both sides of (13). $\square$

Of course, this process could be put into reverse. By taking $\Lambda = \mathbb{Z}^2$ one could give a *purely rational definition* of the winding number of a piecewise-linear, oriented, rational, closed path $\Gamma(\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1}) \subset \mathbb{Q}^2 \backslash \{\mathbf{0}\}$ as being $-\Sigma_{i=0}^{n-1} \mathcal{P}(\mathbf{0}, \mathfrak{r}_i, \mathfrak{r}_{i+1})$, (*a priori* an element of $\mathbb{Q}((\mathbf{z}))^{\mathrm{hd}}$, but actually an integer).

2.3. AN EXPLICIT FORMULA

It is relatively easy to give an explicit expression for the Shintani functions in terms of Bernoulli polynomials. These latter appear (almost by definition) as the coefficients of certain 1-dimensional analogues of the $\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$. In order to stress this analogy, we start by considering an arbitrary, rank-1 lattice λ embedded in $\mathbb{R}$. We identify the quotient set $(\mathbb{Q}\lambda \backslash \{0\})/\mathbb{Q}_+^\times$ of λ-rational rays with $\{\pm 1\}$ by means of the 'sign' function and for each $\varepsilon \in \{\pm 1\}$ and $x \in \mathbb{R}/\lambda$ we set

$$p(\lambda, x, \varepsilon) = p(\lambda, x, \varepsilon; z) := \varepsilon \frac{\sum_{a \in x \cap s(l)} \mathrm{e}^{az}}{\mathrm{e}^{lz} - 1}, \tag{14}$$

for any $l \in \lambda \backslash \{0\}$ such that $\mathrm{sgn}(l) = \varepsilon$. (Here, $s(l)$ denotes the half-open interval $(0, l]$ or $[l, 0)$ according as l is positive or negative). Note that, as in the definition of $\tilde{\mathcal{P}}$, the choice of l satisfying these conditions is immaterial and that $p(\lambda, x, \varepsilon; z)$ lies in $(1/z)\mathbb{R}[[z]] \subset \mathbb{R}((z))$.

PROPOSITION 2.4.

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(i) $p(\lambda, x, -\varepsilon) = p(\lambda, x, \varepsilon) - \begin{cases} \varepsilon & \textit{if } x \textit{ is the zero class} \\ 0 & \textit{if } x \textit{ is nonzero.} \end{cases}$

(ii) *For all* $\alpha \in \mathbb{R}^\times$ *and* $x \in \mathbb{R}/\lambda$ *we have* $p(\alpha\lambda, \alpha(x), \mathrm{sgn}(\alpha)\varepsilon; z) = \mathrm{sgn}(\alpha)p(\lambda, x, \varepsilon; \alpha z)$, *where '*$\alpha(x)$*' indicates the class* αx *considered as an element of* $\mathbb{R}/\alpha\lambda$.

(iii) *If* $\lambda' \subset \lambda$ *is any sublattice and* $\pi_{\lambda',\lambda}: \mathbb{R}/\lambda' \to \mathbb{R}/\lambda$ *denotes the quotient map, then* $p(\lambda, x, \varepsilon) = \sum_{\substack{x' \in \mathbb{R}/\lambda' \\ \pi_{\lambda',\lambda}(x') = x}} p(\lambda', x', \varepsilon)$.

These statements are analogous to Propositions 2.1 (part (ii)), and Propositions 2.2 and 2.3 respectively and have entirely analogous proofs. □

Part (ii) above shows that we may as well reduce to the case $\lambda = \mathbb{Z}$ and we shall write $p(x, \varepsilon; z)$ for $p(\mathbb{Z}, x, \varepsilon; z)$ for all $x \in \mathbb{R}/\mathbb{Z}$. For such a class we shall denote by $\langle x \rangle$ its unique representative in $(0, 1]$ and we write $B_m(T) \in \mathbb{Q}[T]$ for the mth Bernoulli polynomial for each $m \in \mathbb{N}$. By the very definition of these polynomials, taking $l = 1$ in (14) gives

$$p(x, 1; z) = \frac{\mathrm{e}^{\langle x \rangle z}}{\mathrm{e}^z - 1} = \frac{1}{z} \sum_{m \in \mathbb{N}} B_m(\langle x \rangle) \frac{z^m}{m!},$$

for each $x \in \mathbb{R}/\mathbb{Z}$, and part (i) of Proposition 2.4 suggests the definition

$$\begin{aligned} p(x; z) &:= \tfrac{1}{2}(p(x, 1; z) + p(x, -1; z)) \\ &= \begin{cases} \frac{1}{2}\left(\frac{\mathrm{e}^z}{\mathrm{e}^z - 1} + \frac{1}{\mathrm{e}^z - 1}\right) = p(x, 1; z) - \frac{1}{2} & \text{if } \langle x \rangle = 1 \\ & \quad \text{(i.e. } x \text{ is the zero class)} \\ \frac{\mathrm{e}^{\langle x \rangle z}}{\mathrm{e}^z - 1} = p(x, 1; z) & \text{if } \langle x \rangle \neq 1 \\ & \quad \text{(i.e. } x \text{ is nonzero).} \end{cases} \\ &= \frac{1}{z} \sum_{m \in \mathbb{N}} \hat{B}_m(\langle x \rangle) \frac{z^m}{m!}, \end{aligned} \tag{15}$$

where $\hat{B}_m$ denotes the restriction of B_m to $(0, 1]$ except that we define $\hat{B}_m(1) = \frac{1}{2}(B_m(1) + B_m(0))$. Thus $\hat{B}_m(1) = B_m(0) = b_m$, (the mth Bernoulli number) if $m \neq 1$, while $\hat{B}_1(1) = 0$.

REMARK 3. We denote by $\tilde{B}_m$ the periodic extension of $\hat{B}_m$ to $\mathbb{R}$. This is the so-called 'mth periodified Bernoulli polynomial' which, for $m \geqslant 1$ is given by the Fourier series

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$$\begin{aligned}\tilde{B}_m(t) &= \hat{B}_m(\langle t+\mathbb{Z}\rangle)\\ &= -\frac{m!}{(2\pi i)^m}\sum_{j=1}^{\infty}\frac{1}{j^m}(\mathrm{e}^{2\pi ijt}+(-1)^m\,\mathrm{e}^{-2\pi ijt}), \qquad \forall\, t\in\mathbb{R}. \end{aligned} \tag{16}$$

Finally, let r be any nonzero integer. Taking $\alpha = r$, $\lambda = \mathbb{Z}$ and $\lambda' = r\mathbb{Z}$ in Proposition 2.4 parts (ii) and (iii), arguing exactly as in Example 2 and using the definition of $p(x;z)$, we get the following parity/distribution relations for each $r\in\mathbb{Z}\backslash\{0\}$ and $\mathbf{x}\in\mathbb{R}/\mathbb{Z}$

$$\begin{aligned} p(x;z) &= \operatorname{sgn}(r)\sum_{\substack{y\in\mathbb{R}/\mathbb{Z}\\ ry=x}} p(y;rz) \ \text{ or, equivalently:}\\ \hat{B}_m(\langle x\rangle) &= \operatorname{sgn}(r) r^{m-1}\sum_{\substack{y\in\mathbb{R}/\mathbb{Z}\\ ry=x}}\hat{B}_m(\langle y\rangle), \qquad \forall\, m\in\mathbb{N}. \end{aligned}\tag{17}$$

We can now prove

THEOREM 2.2. *Let Λ be a (rank-2) lattice in $\mathbb{R}^2$ and $\mathfrak{r},\mathfrak{s}\in\mathbb{P}_+(\mathbb{Q}\Lambda)$, $\mathfrak{r}\neq\pm\mathfrak{s}$ two Λ-rational rays. Then, for all $\mathbf{x}\in\mathbb{R}^2/\Lambda$ we have the formula*

$$\begin{aligned}&\mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s};\mathbf{z})\\ &\quad= \begin{pmatrix} r_1 & s_1\\ r_2 & s_2\end{pmatrix}\star\left[\frac{1}{z_1z_2}\sum_{m,n\in\mathbb{N}}\left(\sum_{t=1}^{N}\hat{B}_m(\mu_t)\hat{B}_n(\nu_t)\right)\frac{z_1^m z_2^n}{m!n!} - \tfrac{1}{4}\delta_\Lambda(\mathbf{x})\right],\end{aligned}\tag{18}$$

in $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$, for any $\mathbf{r}\in\mathfrak{r}\cap\Lambda$ and $\mathbf{s}\in\mathfrak{s}\cap\Lambda$, the integer $N\in\mathbb{N}$ and the distinct pairs (μ_t,ν_t) in $(0,1]^2$ being such that $\mathbf{x}\cap\{\mu\mathbf{r}+\nu\mathbf{s}:(\mu,\nu)\in(0,1]^2\} = \{\mu_t\mathbf{r}+\nu_t\mathbf{s}\}_{t=1}^N$.

Proof. The formal sum $\Sigma_{m,n\in\mathbb{N}}$ representing an element of $\mathbb{R}[[\mathbf{z}]]$ in (18) will be abbreviated to Σ in the following. We can rewrite Equation (3) as

$$\mathcal{P}(\Lambda,\mathbf{x},\mathfrak{r},\mathfrak{s};\mathbf{z}) = \begin{pmatrix} r_1 & s_1\\ r_2 & s_2\end{pmatrix}\star\sideset{}{'}\sum_{\mu\mathbf{r}+\nu\mathbf{s}\in\mathbf{x}\cap\overline{P(\mathbf{r},\mathbf{s})}}\left(\frac{\mathrm{e}^{\mu z_1}}{\mathrm{e}^{z_1}-1}\right)\left(\frac{\mathrm{e}^{\nu z_2}}{\mathrm{e}^{z_2}-1}\right).$$

The only delicate point is now to account for the precise summation procedure for the terms in the sum Σ' (see Definition 2.1). Firstly there are the terms corresponding to points $\mu\mathbf{r}+\nu\mathbf{s}$ in the *interior* of the parallelogram $P(\mathbf{r},\mathbf{s})$, in other words such that $\mu=\mu_t\neq 1$ and $\nu=\nu_t\neq 1$ for some unique $t\in\{1,\ldots,N\}$. These appear in the sum Σ' with a coefficient of 1 so that their contribution to it is precisely the contribution to $(1/z_1z_2)\Sigma$ for this value of t, by Equation (15). Next, each term in Σ' such that $(\mu,\nu)=(\mu_t,\nu_t)$ for some $t\in\{1,\ldots,N\}$ with $\mu_t=1$ and $\nu_t\in(0,1)$ corresponds to a point $\mathbf{r}+\nu_t\mathbf{s}$ lying on an edge of $P(\mathbf{r},\mathbf{s})$, but not

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at a vertex. Each such term can be paired with a term of Σ' corresponding to the point $\nu_t\mathbf{s}$ on the opposite edge. Both terms are counted with a coefficient $\frac{1}{2}$ and so make a combined contribution of

$$\frac{1}{2}\left(\frac{\mathrm{e}^{z_1}}{\mathrm{e}^{z_1}-1}+\frac{1}{\mathrm{e}^{z_1}-1}\right)\left(\frac{\mathrm{e}^{\nu_t z_2}}{\mathrm{e}^{z_2}-1}\right)$$

to Σ', which again equals the contribution to $(1/z_1z_2)\Sigma$ for this value of t, by (15). The other edge terms $(\mu,\nu)=(\mu_t,\nu_t)$ (such that $\mu_t\in(0,1)$ and $\nu_t=1$) are similarly paired off with their corresponding terms and treated identically. If $\delta_\Lambda(\mathbf{x})=0$ then there are no 'vertex terms' in Σ' and no pair (μ_t,ν_t) is equal to $(1,1)$, so we are done. Otherwise, $\mathbf{x}$ is the zero class, $\delta_\Lambda(\mathbf{x})=1$ and the two remaining terms in Σ' are

$$\begin{aligned}&\frac{1}{2}\left(\frac{\mathrm{e}^{z_1}}{\mathrm{e}^{z_1}-1}\right)\left(\frac{1}{\mathrm{e}^{z_2}-1}\right)+\frac{1}{2}\left(\frac{1}{\mathrm{e}^{z_1}-1}\right)\left(\frac{\mathrm{e}^{z_2}}{\mathrm{e}^{z_2}-1}\right)\\&\quad=\frac{1}{4}\left(\frac{\mathrm{e}^{z_1}}{\mathrm{e}^{z_1}-1}+\frac{1}{\mathrm{e}^{z_1}-1}\right)\left(\frac{\mathrm{e}^{z_2}}{\mathrm{e}^{z_2}-1}+\frac{1}{\mathrm{e}^{z_2}-1}\right)-\frac{1}{4},\end{aligned}$$

which accounts for the remaining contribution to (18) from the pair $(\mu_t,\nu_t)=(1,1)$. $\square$

3. Dedekind sums

We shall be considering the following generalisation of the 'classical' Dedekind sums.

DEFINITION 3.1. For all $m,n\in\mathbb{N}$, $a,c\in\mathbb{Z}$, $c\neq 0$ and

$$\mathbf{x}=\begin{pmatrix}x_1\\x_2\end{pmatrix}+\mathbb{Z}^2\in(\mathbb{R}/\mathbb{Z})^2,$$

we set

$$\begin{aligned}&S_{m,n}(a,c,\mathbf{x})\\&\quad:=\frac{1}{m!n!}\sum_{t=1}^{|c|}\hat{B}_m\left(\left\langle x_1-\frac{a}{c}(x_2+\xi_t)\right\rangle\right)\hat{B}_n\left(\left\langle\frac{1}{c}(x_2+\xi_t)\right\rangle\right)\in\mathbb{R},\end{aligned}\qquad(19)$$

where $\{\xi_1,\ldots,\xi_{|c|}\}$ is any complete set of representatives for $\mathbb{Z}$ modulo $c\mathbb{Z}$.

(Here, by a minor abuse of notation which we intend to perpetuate, we have written $\langle u\rangle$ instead of $\langle u+\mathbb{Z}\rangle$, with the result that $\hat{B}_m(\langle u\rangle)$ and $\tilde{B}_m(u)$ are synonymous for $u\in\mathbb{R}$). It is easy to see that this definition does not depend on the choice of the

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set of representatives $\{\xi_t\}_{t=1}^{|c|}$ and it follows that it is also independent of the choice of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{x}$. Our sums are identical to the sums $C(r,s,h,k,u,v)$ of Halbritter ([Hal]) *except* for a relabelling of the variables and for the constant factor $1/m!n!$ which we have introduced for our own convenience and to simplify the formulae. (For the purposes of translation, the precise correspondence is: $S_{m,n}(a,c,\mathbf{x}) = (1/m!n!)C(n,m,-a,c,x_2,x_1)$). The sum $S_{m,n}(a,c,\mathbf{x})$ already appears in Satz 1 of [Si1], which can be used to express the value of a partial zeta-function over a real-quadratic field at a non-positive integer (granted the appropriate functional equation, see e.g. [Sh, Sect. 6] for the details). These are among the most general Dedekind sums to have been defined. The most basic sum $s(h,k)$ mentioned in the Introduction (see also [R-G]) is given in our notation by $S_{1,1}(-h,k,\mathbf{0})$, and many of the more general versions considered by Apostol, Carlitz, Meyer and several other authors are still special cases of $S_{m,n}(a,c,\mathbf{x})$.

We start by showing that a certain Shintani function is effectively a generating function for the sums $S_{m,n}(a,c,\mathbf{x})$ for fixed a and c.

THEOREM 3.1. *Let* $\begin{pmatrix} a \\ c \end{pmatrix}$ *, be an element of* $\mathbb{Z}^2$ *with* $c \neq 0$ *and* $\mathbf{x}$ *an element of* $(\mathbb{R}/\mathbb{Z})^2$ *then*

$$\mathcal{P}\left(\mathbb{Z}^2, \mathbf{x}, \mathbb{Q}_+^\times \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbb{Q}_+^\times \begin{pmatrix} a \\ c \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \star \left[\frac{1}{z_1 z_2}\left(\sum_{m,n\in\mathbb{N}} S_{m,n}(a,c,\mathbf{x}) z_1^m z_2^n\right) - \tfrac{1}{4}\delta_{\mathbb{Z}^2}(\mathbf{x})\right]. \qquad (20)$$

Of course, if $c = 0$ (and $a \neq 0$) then

$$\mathcal{P}\left(\mathbb{Z}^2, \mathbf{x}, \mathbb{Q}_+^\times \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbb{Q}_+^\times \begin{pmatrix} a \\ c \end{pmatrix}\right) = 0,$$

by definition.

Proof. One simply applies Theorem 2.2: For any fixed $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{x}$, the condition on $(\mu,\nu) \in (0,1]^2$ that $\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} a \\ c \end{pmatrix}$ belong to $\mathbf{x}$ is that $\nu c \equiv x_2$ and $\mu + \nu a \equiv x_1 \pmod{\mathbb{Z}}$. So a complete set of solutions is obtained by setting

$$\nu_t = \left\langle \frac{1}{c}(x_2 + \xi_t) \right\rangle \quad \text{and} \quad \mu_t = \left\langle x_1 - \frac{a}{c}(x_2 + \xi_t) \right\rangle,$$

for any complete set $\{\xi_t\}_{t=1}^{|c|}$ of representatives modulo c. $\square$

REMARK 4. The combination of Theorem 3.1 with Example 1 gives an explicit form of Theorem 2.2, showing how to express any $\mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s})$ in terms of the Dedekind Sums $S_{m,n}$. The properties of the Shintani functions proved in Section 2 will be applied to give essentially algebraic proofs of some important results on Dedekind sums. Before doing so we list for future reference a few of the more elementary relations that they satisfy.

PROPOSITION 3.1. *Let a and c be integers with $c \neq 0$ and let* $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mathbb{Z}^2$ *lie in* $(\mathbb{R}/\mathbb{Z})^2$. *We write* $\mathbf{x}'$ *for* $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} + \mathbb{Z}^2, \mathbf{x}''$ *for* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + \mathbb{Z}^2, a'$ *for* a/d *and* c' *for* c/d *where* $d = \pm\mathrm{h.c.f.}(a, c)$. *Then*

(i) $S_{m,n}(ra, rc, \mathbf{x}) = \mathrm{sgn}(r) r^{1-n} S_{m,n}(a, c, \mathbf{x}) \quad \forall m, n \in \mathbb{N}, \quad \forall r \in \mathbb{Z}\backslash\{0\}$.

(ii) $S_{m,n}(a, c, -\mathbf{x}) = (-1)^{m+n} S_{m,n}(a, c, \mathbf{x}) \quad \forall m, n \in \mathbb{N}$.

(iii) $S_{m,n}(a, c, \mathbf{x}') = (-1)^{m+n} S_{m,n}(a, -c, \mathbf{x})$ *and* $S_{m,n}(a, c, \mathbf{x}'') = S_{m,n}(a, -c, \mathbf{x}) \forall m, n \in \mathbb{N}$.

(iv) $$\frac{1}{z} \sum_{n \in \mathbb{N}} S_{0,n}(a, c, \mathbf{x}) z^n = \mathrm{sgn}(c)\, p(x_2 + \mathbb{Z}; z/c) \quad \textit{and}$$

$$S_{0,n}(a, c, \mathbf{x}) = \mathrm{sgn}(c) c^{1-n} \frac{1}{n!} \hat{B}_n(\langle x_2 \rangle)$$

(v) $$\frac{1}{z} \sum_{m \in \mathbb{N}} S_{m,0}(a, c, \mathbf{x}) z^m = \mathrm{sgn}(c')\, |d|\, p(c'x_1 - a'x_2 + \mathbb{Z}; z/c') \quad \textit{and}$$

$$S_{m,0}(a, c, \mathbf{x}) = \mathrm{sgn}(c') c'^{1-m}\, |d|\, \frac{1}{m!} \hat{B}_m(\langle c'x_1 - a'x_2 \rangle)$$

Proof. Parts (i) and (ii) are easily deduced either from the definition of $S_{m,n}(a, c, \mathbf{x})$ together with the parity/distribution relations (17) for the $\hat{B}_m$, or by using Theorem 3.1 and properties of the Shintani functions. We skip the details. The second formula of part (iii) follows on sending ξ_t to $-\xi_t$ in the definition and the first is then a consequence of part (ii). For (iv), $\hat{B}_0(T) \equiv 1$ implies that $S_{0,n}(a, c, \mathbf{x})$ is independent of a, so equals $S_{0,n}(0, c, \mathbf{x}) = \mathrm{sgn}(c) c^{1-n} S_{0,n}(0, 1, \mathbf{x})$ by part (i). This proves the second equation and the first is simply a reformulation. For the first equation in (v) we reduce to the case $\mathrm{h.c.f.}(a, c) = |d| = 1$ (and replace a by a' and c by c') by using part (i) to transform the left-hand side. We can then write the latter as $\Sigma_{t=1}^{|c'|} p((1/c')(c'x_1 - a'x_2) + (a'/c')\xi_t + \mathbb{Z}; z)$ and since $(a'/c')\xi_t$ runs exactly once through $\{i/c'\}_{i=1}^{|c'|}$ *modulo* $\mathbb{Z}$, the first equation follows from the relations (17). The second is a reformulation. □

Before passing to the applications of Theorem 3.1 we introduce three pieces of notation: The abbreviation $\mathfrak{r}_\infty$ stands for the ray $\mathbb{Q}_+^\times\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{P}_+(\mathbb{Q}^2)$, (the positive rational x-axis). For any $N \in \mathbb{N}$, $N > 0$ we define a finite set of matrices $\mathfrak{A}_N \subset \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{R})$ by

$$\mathfrak{A}_N := \left\{\begin{pmatrix} d & b \\ 0 & a\end{pmatrix}: d, a, b \in \mathbb{N}, \qquad ad = N, 0 \leqslant b < d\right\}$$

and for any $l \in \mathbb{Z}$ we denote by σ_l the arithmetic function

$$\sigma_l(N) := \sum_{\substack{q\in\mathbb{N}\\ q|N}} q^l.$$

Thus we have $|\mathfrak{A}_N| = \sigma_1(N)$ for all $N > 0$. The first application is

THEOREM 3.2. *For all $N \in \mathbb{N}$, $N > 0$ and every $\mathfrak{r} \in \mathbb{P}_+(\mathbb{Q}^2)$ we have*

$$\sum_{\mathbf{A}\in\mathfrak{A}_N} \mathcal{P}(\mathbf{A}\mathbb{Z}^2, \mathbf{0}, \mathfrak{r}_\infty, \mathfrak{r}) = \sum_{\substack{q\in\mathbb{N}\\ q|N}} \mathbf{q} \star q\mathcal{P}(\mathbb{Z}^2, \mathbf{0}, \mathfrak{r}_\infty, \mathfrak{r}), \tag{21}$$

from which we deduce the (essentially equivalent)

COROLLARY 3.1 (The Generalised Petersson–Knopp Identities). *Fix m, n and $N \in \mathbb{N}$, $N > 0$. Then, for all $h, k \in \mathbb{Z}$, $k \neq 0$, we have*

$$\begin{aligned} &N^{n-1} \sum_{\substack{ad=N\\ 0\leqslant b<d}} d^{m-1} S_{m,n}(-(ah+bk), dk, \mathbf{0}) \\ &\quad = \sigma_{m+n-1}(N) S_{m,n}(-h, k, \mathbf{0}). \end{aligned} \tag{22}$$

REMARK 5. A brief history of these identities is as follows (see e.g. [A-V] for more details). For $m = n = 1$, Dedekind himself treated the case where N is prime, and in [Kn], M. Knopp, stimulated by a conjecture of H. Petersson, removed this condition on N. The proofs of these results depended on an analysis of the behaviour of log $\eta(z)$ under the action of Hecke operators. The identities of Corollary 3.1 were first proven for general m and n by L. A. Parson and K. Rosen in [P-R]. In place of log $\eta(z)$, they used the transformation properties of certain Lambert series studied by Apostol. Other, essentially elementary, proofs were given in e.g. [Pa] and [A-V], the latter being totally analysis-free and applicable to more general sums 'of Dedekind type'. Subsequently, C. Nagasaka [N] proved even more general versions for sums which also involve Dirichlet characters.

DEDUCTION OF THE COROLLARY. $\mathfrak{A}_N$ fixes $\mathfrak{r}_\infty$ so, by Proposition 2.2, the left-hand side of (21) can be written as $\Sigma_{\mathfrak{A}_N}\mathbf{A}\star\mathcal{P}(\mathbb{Z}^2,\mathbf{0},\mathfrak{r}_\infty,\mathbf{A}^{-1}\mathfrak{r})$. Now, setting $\mathfrak{r}=\mathbb{Q}_+^\times\begin{pmatrix}-h\\k\end{pmatrix}$, $\left(\text{so that } \mathbf{A}^{-1}\mathfrak{r}=(N\mathbf{A}^{-1})\mathfrak{r}=\mathbb{Q}_+^\times\begin{pmatrix}-(ah+bk)\\dk\end{pmatrix}\right)$, we use (20) to substitute for the $\mathcal{P}(\mathbb{Z}^2,\mathbf{0},\mathfrak{r}_\infty,\cdot)$'s on both sides. Applying $\begin{pmatrix}1&-h\\0&k\end{pmatrix}^{-1}$ to the resulting equation, we find

$$\sum_{\substack{ad=N\\0\leqslant b<d}}\begin{pmatrix}d&0\\0&N\end{pmatrix}\star\left[\frac{1}{z_1z_2}\sum_{m,n\in\mathbb{N}}S_{m,n}(-(ah+bk),dk,\mathbf{0})z_1^mz_2^n-\tfrac{1}{4}\right]$$

$$=\sum_{\substack{q\in\mathbb{N}\\q|N}}\mathbf{q}\star q\left[\frac{1}{z_1z_2}\sum_{m,n\in\mathbb{N}}S_{m,n}(-h,k,\mathbf{0})z_1^mz_2^n-\tfrac{1}{4}\right]$$

and the Corollary follows on multiplying by z_1z_2 and equating coefficients of $z_1^mz_2^n$. □

We now prove Theorem 3.2 by means of two lemmas. The first is fundamental in the theory of Hecke operators and shows that 'that's really what's going on here'.

LEMMA 3.1. *The set* $\mathfrak{A}_N$ *is in bijective correspondence with the set of lattices* $\{\Lambda\subset\mathbb{Z}^2:[\mathbb{Z}^2:\Lambda]=N\}$ (*respectively, with the set of subgroups* $\{L<(\mathbb{Z}/N\mathbb{Z})^2:[(\mathbb{Z}/N\mathbb{Z})^2:L]=N\}$) *via the mapping*

$$\begin{pmatrix}d&b\\0&a\end{pmatrix}\mapsto\begin{pmatrix}d&b\\0&a\end{pmatrix}\mathbb{Z}^2=\mathbb{Z}\begin{pmatrix}d\\0\end{pmatrix}\oplus\begin{pmatrix}b\\a\end{pmatrix},$$

(*respectively, via the mapping* $\begin{pmatrix}d&b\\0&a\end{pmatrix}\mapsto\begin{pmatrix}d&b\\0&a\end{pmatrix}(\mathbb{Z}/N\mathbb{Z})^2=\mathbb{Z}\begin{pmatrix}\bar d\\\bar 0\end{pmatrix}\oplus\mathbb{Z}\begin{pmatrix}\bar b\\\bar a\end{pmatrix}$).

Proof. The fact that the first mapping is a bijection is the contents of Lemma 2, [Se1, p. 99]. The second bijection is a consequence of the natural one between the two specified sets (namely $\Lambda\leftrightarrow L=\bar\Lambda$, since $[\mathbb{Z}^2:\Lambda]=N$ implies $\Lambda\supset N\mathbb{Z}^2$). □

LEMMA 3.2. *Let* $\mathbf{y}$ *be an element of* $(\mathbb{Z}/N\mathbb{Z})^2$ $(N\in\mathbb{N},N>0)$, *whose additive order we denote by* $o(\mathbf{y})$. *Then*

$$|\{L<(\mathbb{Z}/N\mathbb{Z})^2:[(\mathbb{Z}/N\mathbb{Z})^2:L]=N\quad\text{and}\quad\mathbf{y}\in L\}|$$
$$=\sigma_1(N/o(\mathbf{y}))=\sum_{\substack{q|N\\\mathbf{y}\in(q\mathbb{Z}/N\mathbb{Z})^2}}q. \tag{23}$$

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Proof. (Sketch). If N has the prime decomposition $N = p_1^{\mathrm{e}_1} \dots p_s^{\mathrm{e}_s}$ then the isomorphism $(\mathbb{Z}/N\mathbb{Z})^2 \cong \bigoplus_{i=1}^{s} (\mathbb{Z}/p_i^{\mathrm{e}_i}\mathbb{Z})^2$ allows us to reduce to the case $N = p^{\mathrm{e}}$ by a simple multiplicativity argument. Furthermore, for any $\mathbf{y} \in (\mathbb{Z}/p^{\mathrm{e}}\mathbb{Z})^2$ with $o(\mathbf{y}) = p^{\mathrm{e}'}$, $0 \leqslant \mathrm{e}' \leqslant \mathrm{e}$ there exists an automorphism of $(\mathbb{Z}/p^{\mathrm{e}}\mathbb{Z})^2$ taking $\mathbf{y}$ to $\begin{pmatrix} \overline{p^{\mathrm{e}-\mathrm{e}'}} \\ \overline{0} \end{pmatrix}$. To see this, set $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, so that, without loss of generality, we have $p^{\mathrm{e}'} = o(y_1) \geqslant o(y_2)$ (orders in $\mathbb{Z}/p^{\mathrm{e}}\mathbb{Z}$) and $\mathbf{y} = p^{\mathrm{e}-\mathrm{e}'} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, with $x_1 \in (\mathbb{Z}/p^{\mathrm{e}}\mathbb{Z})^{\times}$. Thus $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} \overline{0} \\ \overline{1} \end{pmatrix} \right\}$ is a base of $(\mathbb{Z}/p^{\mathrm{e}}\mathbb{Z})^2$ and there exists an automorphism taking it to the base $\left\{ \begin{pmatrix} \overline{1} \\ \overline{0} \end{pmatrix}, \begin{pmatrix} \overline{0} \\ \overline{1} \end{pmatrix} \right\}$. Now for $\mathbf{y} = \begin{pmatrix} \overline{p^{\mathrm{e}-\mathrm{e}'}} \\ \overline{0} \end{pmatrix}$, the second two quantities in (23) are clearly both equal to $1 + p + p^2 + \dots + p^{\mathrm{e}-\mathrm{e}'}$. That this is also equal to the first quantity follows from the previous lemma, since for $\mathbf{A} = \begin{pmatrix} p^u & b \\ 0 & p^t \end{pmatrix} \in \mathfrak{A}_{p^{\mathrm{e}}}$, it is clear that $\mathbf{y}$ lies in $\mathbf{A}(\mathbb{Z}/p^{\mathrm{e}}\mathbb{Z})^2$ if and only if $0 \leqslant u \leqslant \mathrm{e} - \mathrm{e}'$, and for each such u there are p^u possibilities for b. $\square$

Proof of Theorem 3.2. Using successively Lemma 3.1, Proposition 2.3, Lemma 3.2, Proposition 2.3 again and Proposition 2.2 we get

$$\begin{aligned}
\sum_{\mathbf{A} \in \mathfrak{A}_N} \mathcal{P}(\mathbf{A}\mathbb{Z}^2, \mathbf{0}, \mathfrak{r}_\infty, \mathfrak{r}) &= \sum_{\substack{\Lambda \subset \mathbb{Z}^2 \\ [\mathbb{Z}^2:\Lambda]=N}} \mathcal{P}(\Lambda, \mathbf{0}, \mathfrak{r}_\infty, \mathfrak{r}) \\
&= \sum_{\substack{\Lambda \subset \mathbb{Z}^2 \\ [\mathbb{Z}^2:\Lambda]=N}} \sum_{\mathbf{y} \in (\Lambda/N\mathbb{Z})^2} \mathcal{P}((N\mathbb{Z})^2, \mathbf{y}, \mathfrak{r}_\infty, \mathfrak{r}) \\
&= \sum_{\mathbf{y} \in (\mathbb{Z}/N\mathbb{Z})^2} \sigma_1(N/o(\mathbf{y})) \mathcal{P}((N\mathbb{Z})^2, \mathbf{y}, \mathfrak{r}_\infty, \mathfrak{r}) \\
&= \sum_{\substack{q \in \mathbb{N} \\ q|N}} q \left(\sum_{\mathbf{y} \in (q\mathbb{Z}/N\mathbb{Z})^2} \mathcal{P}((N\mathbb{Z})^2, \mathbf{y}, \mathfrak{r}_\infty, \mathfrak{r}) \right) \\
&= \sum_{\substack{q \in \mathbb{N} \\ q|N}} q \mathcal{P}((q\mathbb{Z})^2, \mathbf{0}, \mathfrak{r}_\infty, \mathfrak{r}) \\
&= \sum_{\substack{q \in \mathbb{N} \\ q|N}} \mathbf{q} \star q \mathcal{P}(\mathbb{Z}^2, \mathbf{0}, \mathfrak{r}_\infty, \mathfrak{r})
\end{aligned}$$

as required. $\square$

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Dedekind proved the following 'reciprocity law' for his sums: *'For all* $h, k \in \mathbb{Z}$*, with* $h, k > 0$ *and* $(h, k) = 1$*, one has* $s(h, k) + s(k, h) = \frac{1}{12}(k/h + h/k + 1/hk) - \frac{1}{4}$*'*. Various other authors (Apostol, Berndt, Carlitz, Mikolás ...) subsequently devised versions and variants of this law which apply to more and more generalised Dedekind sums and whose proofs, especially in the more complicated cases, tended to rely on complex- or real-analytic arguments. One of the most recent versions is due to Halbritter [Hal, Thm 2]. It includes many of the previous versions as specialisations and can be seen as a 'transformation law' for his generalised sums under the action of $\mathrm{GL}_2(\mathbb{Z})$. We shall now give an essentially algebraic proof of (a reformulation of) Halbritter's law by means of Theorem 3.1 and the properties (especially the 'Formal Cauchy Theorem') of Shintani functions.

THEOREM 3.3 (Reciprocity Law for Generalised Dedekind Sums). *Let* $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a matrix in* $\mathrm{GL}_2(\mathbb{Z})$ *with* $c \neq 0$. *Let* a' *and* $c' \neq 0$ *be integers,* $\mathbf{x}$ *an element of* $(\mathbb{R}/\mathbb{Z})^2$ *and* $m, n \in \mathbb{N}$, $m, n \geqslant 1$. *We set*

$$\eta = \det(\mathbf{A}), \qquad \begin{pmatrix} a'' \\ c'' \end{pmatrix} = \mathbf{A} \begin{pmatrix} a' \\ c' \end{pmatrix},$$

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \qquad \delta = \delta_{\mathbb{Z}^2}(\mathbf{x}) = \delta_{\mathbb{Z}^2}(\mathbf{y}), \qquad l = m + n \geqslant 2$$

and we suppose that $c'' \neq 0$. *Then*

$$\begin{aligned} &S_{m,n}(a'', c'', \mathbf{y}) \\ &\quad = \left|\frac{c''}{c}\right| \left(\sum_{\substack{m' \geqslant 0,\, n' \geqslant n \\ m'+n'=l}} \binom{n'-1}{n-1} \frac{c^n (\eta c')^{n'-n}}{c''^{n'}} S_{m',n'}(a, c, \mathbf{y}) \right. \\ &\qquad\qquad \left. - \left(\frac{-c}{\eta c'}\right)^n S_{l,0}(a, c, \mathbf{y}) \right) \\ &\qquad + \left|\frac{c''}{c'}\right| \sum_{\substack{m' \geqslant m,\ n' \geqslant 0 \\ m'+n'=l}} \binom{m'-1}{m-1} \frac{c^{m'-m} (\eta c')^m}{c''^{m'}} S_{m',n'}(a', c', \mathbf{x}) + G_{m,n}\delta, \end{aligned} \tag{24}$$

where $G_{m,n}$ is defined to be $-\frac{1}{4}\mathrm{sgn}(\eta c c')$ if $m = n = 1$ and is otherwise 0.

Proof. We set

$$\mathfrak{r}' = \mathbb{Q}_+^\times \begin{pmatrix} a' \\ c' \end{pmatrix}, \qquad \mathfrak{r}'' = \mathbb{Q}_+^\times \begin{pmatrix} a'' \\ c'' \end{pmatrix} = \mathbf{A}\mathfrak{r}'$$

and apply Theorem 2.1 to the triple of rays $(\mathfrak{r}_\infty, \mathfrak{r}'', \mathbf{A}\mathfrak{r}_\infty)$, taking the lattice Λ to be $\mathbb{Z}^2$ (and dropping it henceforth from the notation). We obtain

$$\mathcal{P}(\mathbf{y}, \mathfrak{r}_\infty, \mathfrak{r}'') + \mathcal{P}(\mathbf{y}, \mathfrak{r}'', \mathbf{A}\mathfrak{r}_\infty) + \mathcal{P}(\mathbf{y}, \mathbf{A}\mathfrak{r}_\infty, \mathfrak{r}_\infty) = -\delta W(\mathfrak{r}_\infty, r'', \mathbf{A}\mathfrak{r}_\infty)$$

and hence

$$\mathcal{P}(\mathbf{y}, \mathfrak{r}_\infty, \mathfrak{r}'') = \mathcal{P}(\mathbf{y}, \mathfrak{r}_\infty, \mathbf{A}\mathfrak{r}_\infty) + \mathbf{A} \star \mathcal{P}(\mathbf{x}, \mathfrak{r}_\infty, \mathfrak{r}') - \delta W(\mathfrak{r}_\infty, \mathfrak{r}'', \mathbf{A}\mathfrak{r}_\infty).$$

Now substitute for the $\mathcal{P}$'s using Theorem 3.1 and apply $\begin{pmatrix} 1 & a'' \\ 0 & c'' \end{pmatrix}^{-1} \star$ to the resulting equation, noting that

$$\frac{1}{c''} \begin{pmatrix} c'' & -a'' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & f' \\ 0 & f \end{pmatrix}$$

and

$$\frac{1}{c''} \begin{pmatrix} c'' & -a'' \\ 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & a' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} f' & 0 \\ f & 1 \end{pmatrix},$$

where we have written f and f' for the nonzero rationals c/c'' and $(c''a - a''c)/c'' = \eta c'/c''$ respectively. This gives

$$\begin{aligned} &\frac{1}{z_1 z_2} \sum_{m',n' \in \mathbb{N}} S_{m',n'}(a'', c'', \mathbf{y}) z_1^{m'} z_2^{n'} - \tfrac{1}{4}\delta \\ &\quad = \begin{pmatrix} 1 & f' \\ 0 & f \end{pmatrix} \star \left(\frac{1}{z_1 z_2} \sum_{m',n' \in \mathbb{N}m} S_{m',n'}(a, c, \mathbf{y}) z_1 m' z_2^{n'} - \tfrac{1}{4}\delta \right) \\ &\qquad + \begin{pmatrix} f' & 0 \\ f & 1 \end{pmatrix} \star \left(\frac{1}{z_1 z_2} \sum_{m',n' \in \mathbb{N}} S_{m',n'}(a', c', \mathbf{x}) z_1 m' z_2^{n'} - \tfrac{1}{4}\delta \right) \\ &\qquad - \delta \,\mathrm{sgn}(c'') W(\mathfrak{r}_\infty, \mathfrak{r}'', \mathbf{A}\mathfrak{r}_\infty). \end{aligned}$$

Now 'do the $\star$'s', multiply by $z_1 z_2$ and then equate the lth homogeneous parts (see the Introduction) to obtain

$$\begin{aligned}
&\sum_{m'+n'=l} S_{m',n'}(a'',c'',\mathbf{y}) z_1^{m'} z_2^{n'}\\
&= \mathrm{sgn}(f) z_2 \sum_{\substack{m'\geqslant 0,\ n'\geqslant 1\\ m'+n'=l}} S_{m',n'}(a,c,\mathbf{y}) z_1^{m'} (f'z_1 + fz_2)^{n'-1}\\
&\quad + \mathrm{sgn}(f') z_1 \sum_{\substack{m'\geqslant 1,\ n'\geqslant 0\\ m'+n'=l}} S_{m',n'}(a',c',\mathbf{x}) (f'z_1 + fz_2)^{m'-1} z_2^{n'}\\
&\quad + \frac{z_1 z_2}{(f'z_1 + fz_2)} [\mathrm{sgn}(f) S_{l,0}(a,c,\mathbf{y}) z_1^{l-1}\\
&\qquad\qquad - \mathrm{sgn}(f') S_{0,l}(a',c',\mathbf{x}) z_2^{l-1}] + \tilde{G}_{m,n} \delta z_1 z_2, \qquad (25)
\end{aligned}$$

where $\tilde{G}_{m,n}$ is 0 unless $l = 2$ ($\Leftrightarrow m = n = 1$) in which case it equals $\frac{1}{4}(1 - \mathrm{sgn}(f) - \mathrm{sgn}(f')) - \mathrm{sgn}(c'') W(\mathfrak{r}_\infty, \mathfrak{r}'', \mathbf{A}\mathfrak{r}_\infty)$. This equation takes place *a priori* in $\mathbb{R}(\mathbf{z})_l \subset \mathbb{R}(\mathbf{z})$ but since all the terms are evidently polynomials except possibly the third on the R.H.S., this one must be as well. In other words $(f'z_1 + fz_2)$ must divide the quantity in square brackets (this also follows from Proposition 3.1 and (17)) and the quotient must be

$$\frac{\mathrm{sgn}(f)}{f'} S_{l,0}(a,c,\mathbf{y}) \left(z_1^{l-2} + z_1^{l-3} \left(\frac{-fz_2}{f'} \right) + \cdots + \left(\frac{-fz_2}{f'} \right)^{l-2} \right).$$

Substituting this into (25), expanding and equating coefficients of $z_1^m z_2^n$ we obtain Equation (24) with $\tilde{G}_{m,n}$ in place of $G_{m,n}$ (and, of course, with $\mathrm{sgn}(f)/f = 1/|f|$ standing in for $|c''/c|$ etc.). It therefore only remains to show that $\tilde{G}_{1,1} = G_{1,1}$, an equation which may be written

$$\mathrm{sgn}(c'') W(\mathfrak{r}_\infty, \mathfrak{r}'', \mathbf{A}\mathfrak{r}_\infty) = \tfrac{1}{4}(1 - \mathrm{sgn}(f))(1 - \mathrm{sgn}(f')). \qquad (26)$$

Now the conditions $c, c', c'' \neq 0$ are easily seen to imply that the triple $(\mathfrak{r}_\infty, \mathfrak{r}'', \mathbf{A}\mathfrak{r}_\infty) = (\mathfrak{r}_\infty, \mathbf{A}\mathfrak{r}', \mathbf{A}\mathfrak{r}_\infty)$ is non-degenerate and non-critical. Therefore, by Lemma 2.1, it is splayed if and only if

$$\begin{aligned}
&\mathfrak{S}(\mathfrak{r}_\infty, \mathfrak{r}'') = \mathfrak{S}(\mathbf{A}\mathfrak{r}', \mathbf{A}\mathfrak{r}_\infty) = \mathfrak{S}(\mathbf{A}\mathfrak{r}_\infty, \mathfrak{r}_\infty),\\
&\text{i.e.}\quad \mathrm{sgn}(c'') = -\mathrm{sgn}(\eta c') = -\mathrm{sgn}(c),\\
&\text{i.e.}\quad -1 = \mathrm{sgn}(f') = \mathrm{sgn}(f),
\end{aligned}$$

in which case $W(\mathfrak{r}_\infty, \mathfrak{r}'', \mathbf{A}\mathfrak{r}_\infty) = \mathfrak{S}(\mathfrak{r}_\infty, \mathfrak{r}'') = \mathrm{sgn}(c'')$ and both sides of (26) are equal to 1. Otherwise, the triple must be folded so that $W = 0$, either $\mathrm{sgn}(f)$ or $\mathrm{sgn}(f')$ must be equal to 1 and consequently both sides of (26) vanish. □

Dedekind's law is obtained from Theorem 3.3 by specialising

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad a' = -h,\ \ c' = k,\ \ \mathbf{x} = 0 \quad \text{and} \quad n = m = 1.$$

The proof of Theorem 2 of [Hal] relies on the manipulation of Fourier series such as (16) and occupies fifteen journal pages. The result itself is, however, equivalent to Theorem 3.3. This can be seen as follows: Replace our matrix $\mathbf{A}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ (i.e. our 'a' by Halbritter's 'δd' etc.) and our $a, c', \mathbf{x}, m, n, \eta, a'', c'', \mathbf{y}$ etc. by the quantities which would be denoted $ah + bk, ch + dk, \begin{pmatrix} av - bu \\ cv - du \end{pmatrix} + \mathbb{Z}^2, q, p, \delta, h, k, \begin{pmatrix} v \\ -u \end{pmatrix} + \mathbb{Z}^2$ etc. using the notation of [Hal]. Taking into account the conversion between our sums 'S' and Halbritter's 'C's', the equivalence results from Proposition 3.1 and a certain amount of rearranging. Notice that we have excluded consideration of the three simplest of the seven special cases in [Hal] (namely when one or both of m and n is zero) since these are easily treated by means of Proposition 3.1. For more details and a comparison with previous reciprocity laws, the reader should consult Halbritter's paper.

(*Note added June 1997*: In addition to reciprocity laws of this type, there are also various generalisations springing from Rademacher's so-called 'three-term relation'. One very general such law has appeared since the writing of the present article, in [H-W-Z]. While its formulation does not involve matrices, nor is a logical connection with Theorem 3.3 immediately apparent (except, perhaps, when $x = y = z = 0$ in the notation of loc. cit.), we nevertheless note that the proof of this law uses quotients of certain 2-variable formal power-series which resemble our $\mathcal{P}$'s).

4. Cocycles

The complicated formulae of the previous chapter resulted from explicitly writing out certain simple functional relations obtained in Chapter 2 for the Shintani functions in terms of the latter's coefficients. Drawing the appropriate conclusion we now proceed in the opposite direction, showing that these relations have a neat reformulation in a certain abstract framework of 1-cocycles on $\mathrm{PGL}_2(\mathbb{Q})$ similar to that appearing in [St].

4.1. Distributions and Matrix Actions

Given a positive integer d, a (rank-d) lattice $\Lambda \subset \mathbb{R}^d$ and any Abelian group $\mathcal{A}$ we shall write $\mathcal{C}(\mathbb{R}^d/\Lambda, \mathcal{A})$ for $\mathcal{A}^{(\mathbb{R}^d/\Lambda)}$, the Abelian group of all functions $g\colon \mathbb{R}^d/\Lambda \to \mathcal{A}$ under pointwise addition. Given any linear left action '$*$' of the multiplicative monoid of $\mathbb{Z}\backslash\{0\}$ on $\mathcal{A}$ (for example $n * a =: n^t a$ for some fixed $t \in \mathbb{N}$), we define a subgroup $\mathcal{D}(\mathbb{R}^d/\Lambda, \mathcal{A})$ of $\mathcal{C}(\mathbb{R}^d/\Lambda, \mathcal{A})$ by setting

$$\mathcal{D}(\mathbb{R}^d/\Lambda, \mathcal{A}) = \left\{ g \in \mathcal{C}(\mathbb{R}^d/\Lambda, \mathcal{A}) : g(\mathbf{x}) = n * \sum_{\substack{\mathbf{y} \in \mathbb{R}^d/\Lambda \\ \mathbf{y} = \mathbf{x}}} g(\mathbf{y}),\ \forall n \in \mathbb{Z}\backslash\{0\},\ \forall \mathbf{x} \in \mathbb{R}^d/\Lambda \right\}.$$

We shall refer to the elements of $\mathcal{D}(\mathbb{R}^d/\Lambda, \mathcal{A})$ as '*$\mathcal{A}$-valued distributions on* $\mathbb{R}^d/\Lambda$' (*with respect to* $*$) although the term 'distribution' in this general context has been defined by various authors in various ways that are not always precisely equivalent to ours.

EXAMPLE 4. (Dirac Distributions). Let δ_Λ be the Dirac function on $\mathbb{R}^d/\Lambda$, defined just as it was in Chapter 2 in the case $d = 2$. Any element $a \in \mathcal{A}$ on which $\mathbb{Z}\backslash\{0\}$ acts trivially gives rise to an $\mathcal{A}$-valued distribution $a\delta_\Lambda\colon \mathbf{x} \mapsto \delta_\Lambda(\mathbf{x})a$.

EXAMPLE 5. For $d = 2$ and $\mathcal{A} = \mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$, we set $n * F = \mathbf{n} \star F = F(nz_1, nz_2)$ for any $n \in \mathbb{Z}\backslash\{0\}$ and any $F \in \mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$. Equation (5) then amounts to the statement that for any rank-2 lattice $\Lambda \subset \mathbb{R}^2$ and any two fixed rays $\mathfrak{r}$ and $\mathfrak{s}$ in $\mathbb{P}_+(\mathbb{Q}\Lambda)$, the map

$$\mathcal{P}(\Lambda, \mathfrak{r}, \mathfrak{s})\colon \mathbb{R}^2/\Lambda \to \mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$$
$$\mathbf{x} \mapsto \mathcal{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z})$$

lies in $\mathcal{D}(\mathbb{R}^2/\Lambda, \mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$. Similarly, for $d = 1$, formula (17) implies that the map $\mathbf{x} \mapsto p(\mathbf{x}; z)$ is an $\mathbb{R}((z))$-valued distribution on $\mathbb{R}/\mathbb{Z}$, where $n \in \mathbb{Z}\backslash\{0\}$ acts on $\mathbb{R}((z))$ by sending $F(z)$ to $n * F(z) := \mathrm{sgn}(n)F(nz)$.

In the second example above we can identify Λ with $\mathbb{Z}^2$ by means of a choice of $\mathbb{Z}$-basis as in Example 1. The distribution property for $\mathcal{P}(\Lambda, \mathfrak{r}, \mathfrak{s})$ is then a consequence of the same property in the special case $\Lambda = \mathbb{Z}^2$. From now on we shall consider *only* the case $\Lambda = \mathbb{Z}^d$. The quotient group $\mathbb{R}^d/\mathbb{Z}^d$ then comes equipped with a natural action of the multiplicative monoid of $\mathrm{GL}_d(\mathbb{Q}) \cap \mathrm{M}_d(\mathbb{Z})$ which contains $\mathbb{Z}\backslash\{\mathbf{0}\}$ identified with the set of scalar matrices. Suppose that the $*$-action of the latter extends to a linear, left action of $\mathrm{GL}_d(\mathbb{Q}) \cap \mathrm{M}_d(\mathbb{Z})$ on $\mathcal{A}$, also denoted $*$. Then, for any $\mathbf{A} \in \mathrm{GL}_d(\mathbb{Q}) \cap \mathrm{M}_d(\mathbb{Z})$ and $g \in \mathcal{C}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A})$ we write $\mathbf{A} \cdot g \in \mathcal{C}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A})$

for the function which sends $\mathbf{x}$ to $\mathbf{A} * \Sigma_{\substack{\mathbf{y}\in\mathbb{R}^d/\mathbb{Z}^d \\ \mathbf{Ay}=\mathbf{x}}} g(\mathbf{y})$, (a finite sum).

PROPOSITION 4.1. (i) *The mapping* $(\mathbf{A}, g) \mapsto \mathbf{A}\cdot g$ *defines a linear left action of the monoid* $\mathrm{GL}_d(\mathbb{Q}) \cap \mathrm{M}_d(\mathbb{Z})$ *on* $\mathcal{C}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A})$.

(ii) $\mathcal{D}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A}) = \{g \in \mathcal{C}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A}) : \mathbf{n}\cdot g = g\ \forall n \in \mathbb{Z}\backslash\{\mathbf{0}\}\}$.

(iii) $\mathcal{D}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A})$ *is stable under this 'dot'-action of* $\mathrm{GL}_d(\mathbb{Q}) \cap \mathrm{M}_d(\mathbb{Z})$ *on* $\mathcal{C}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A})$.

Proof. The verification of (i) is left to the reader and part (ii) is a tautology. Part (iii) follows from (i) and (ii) since $\mathbf{n}$ commutes with each $\mathbf{A} \in \mathrm{GL}_d(\mathbb{Q}) \cap \mathrm{M}_d(\mathbb{Z})$. □

To obtain a *group* action on $\mathcal{C}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A})$ one can simply restrict '$\cdot$' to $\mathrm{GL}_d(\mathbb{Z})$. Explicitly, this gives: $(\mathbf{M}\cdot g)(\mathbf{x}) = \mathbf{M} * g(\mathbf{M}^{-1}\mathbf{x})$ for all $\mathbf{M} \in \mathrm{GL}_d(\mathbb{Z})$, $g \in \mathcal{C}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A})$ and $\mathbf{x} \in \mathbb{R}^d/\mathbb{Z}^d$. This in turn restricts to a $\mathrm{GL}_d(\mathbb{Z})$-action on the distributions, but we can do better by performing this restriction first of all: Proposition 4.1 has the easily deduced

COROLLARY 4.1. *There is a unique left action of* $\mathrm{GL}_d(\mathbb{Q})$ *on* $\mathcal{D}(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{A})$ *extending the 'dot' action of* $\mathrm{GL}_d(\mathbb{Q}) \cap \mathrm{M}_d(\mathbb{Z})$. *Explicitly, a matrix M sends a distribution g to* $(n\mathbf{M})\cdot g$ *where n is any nonzero integer chosen so that $n\mathbf{M}$ lies in* $\mathrm{GL}_d(\mathbb{Q}) \cap \mathrm{M}_d(\mathbb{Z})$. □

The point is, of course, that $(n\mathbf{M})\cdot g$ doesn't depend on n. This extended action of will be of principal interest in this section. It clearly factors through the quotient group $\mathrm{PGL}_d(\mathbb{Q})$ and there will be no ambiguity in denoting it $\mathbf{M}\cdot g$ for any distribution g and any $\mathbf{M}$ considered either as an element of $\mathrm{GL}_d(\mathbb{Q})$ or of $\mathrm{PGL}_d(\mathbb{Q})$.

4.2. The Shintani Cocycle (and Variations)

Henceforth we shall work in dimension $d = 2$ so that the lattice Λ will always be $\mathbb{Z}^2 \subset \mathbb{R}^2$ and will usually be suppressed from the notation. Let's consider the distribution $\mathcal{P}(\mathfrak{r}, \mathfrak{s}) := \mathcal{P}(\mathbb{Z}^2, \mathfrak{r}, \mathfrak{s})$ lying in $\mathcal{D}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}) := \mathcal{D}(\mathbb{R}^2/\mathbb{Z}^2, \mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$, as defined in Example 5 for each pair of rays $\mathfrak{r}$ and $\mathfrak{s}$ in $\mathbb{P}_+(\mathbb{Q}^2)$. Since the $\star$-action on $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ is defined for all $\mathbf{A} \in \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$, it gives rise as above to a $(\mathrm{P})\mathrm{GL}_2(\mathbb{Q})$-action on $\mathcal{D}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$ which we denote '$\cdot$' and which is clearly $\mathbb{R}$-linear. We can reformulate Equation (6) of Example 3 as

$$\mathcal{P}(\mathbf{M}\mathfrak{r}, \mathbf{M}\mathfrak{s}) = \mathbf{M}\cdot\mathcal{P}(\mathfrak{r}, \mathfrak{s}) \quad \text{in } \mathcal{D}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$$
$$\text{for all } \mathbf{M} \in \mathrm{GL}_2(\mathbb{Q}) \quad \text{and} \quad \mathfrak{r}, \mathfrak{s} \in \mathbb{P}_+(\mathbb{Q}^2). \tag{27}$$

The action of $\mathbb{Z}\backslash\{0\}$ on constant power-series in $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ is trivial so that the set $\frac{1}{2}\mathbb{Z}\delta := \{\frac{1}{2}m\delta_{\mathbb{Z}^2} : m \in \mathbb{Z}\}$ is a subgroup of $\mathcal{D}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$ on which any $\mathbf{M} \in \mathrm{GL}_2(\mathbb{Q})$

acts by multiplication by $\mathrm{sgn}(\det(\mathbf{M}))$. We shall denote the quotient $(\mathrm{P})\mathrm{GL}_2(\mathbb{Q})$-module $\mathcal{D}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})/(\frac{1}{2}\delta\mathbb{Z})$ by $\bar{\mathcal{D}}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$. The point of all this definition-making is to formulate the

THEOREM 4.1. *For each rational ray* $\mathfrak{r} \in \mathbb{P}_+(\mathbb{Q}^2)$ *we define a map* $\Psi_\mathfrak{r}$ *from* $\mathrm{GL}_2(\mathbb{Q})^2$ *to* $\mathcal{D}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$ *by setting* $\Psi_\mathfrak{r}(\mathbf{M}_0, \mathbf{M}_1) = \mathcal{P}(\mathbf{M}_0\mathfrak{r}, \mathbf{M}_1\mathfrak{r})$ *and write* $\bar{\Psi}_\mathfrak{r}\colon \mathrm{GL}_2(\mathbb{Q})^2 \to \bar{\mathcal{D}}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$ *for the composite of* $\Psi_\mathfrak{r}$ *with the quotient map. Then*

(i) $\Psi_\mathfrak{r} = \Psi_{-\mathfrak{r}}$, *i.e.* $\Psi_\mathfrak{r}$ *depends only on the image* $\mathfrak{r} \cup -\mathfrak{r}$ *of* $\mathfrak{r}$ *in the rational projective line* $\mathbb{P}^1(\mathbb{Q})$.

(ii) *The same is true of the map* $\bar{\Psi}_\mathfrak{r}$ *which in addition factors through* $\mathrm{PGL}_2(\mathbb{Q})^2$. *It defines a homogeneous* 1*-cocycle on* $\mathrm{PGL}_2(\mathbb{Q})$ *(or on* $\mathrm{GL}_2(\mathbb{Q})$*) with values in* $\bar{\mathcal{D}}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$.

(iii) *The cohomology class represented by* $\bar{\Psi}_\mathfrak{r}$ *in* $H^1(\mathrm{PGL}_2(\mathbb{Q}), \bar{\mathcal{D}}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}))$ *is independent of the ray* $\mathfrak{r}$.

Proof. Part (i) and the first statement in Part (ii) follow from parts (iii) and (iv) of Proposition 2.1. The rest of part (ii) is a consequence of the following two equations in $\mathcal{D}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$, valid for all $\mathbf{M}, \mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2 \in \mathrm{GL}_2(\mathbb{Q})$

$$\Psi_\mathfrak{r}(\mathbf{M}\mathbf{M}_0, \mathbf{M}\mathbf{M}_1) = \mathbf{M} \cdot \Psi_\mathfrak{r}(\mathbf{M}_0, \mathbf{M}_1),$$

(from (27)) and, by Theorem 2.1 and part (i) of Proposition 2.1

$$\Psi_\mathfrak{r}(\mathbf{M}_1, \mathbf{M}_2) - \Psi_\mathfrak{r}(\mathbf{M}_0, \mathbf{M}_2) + \Psi_\mathfrak{r}(\mathbf{M}_0, \mathbf{M}_1) = -W(\mathbf{M}_0\mathfrak{r}, \mathbf{M}_1\mathfrak{r}, \mathbf{M}_2\mathfrak{r})\delta.$$

The images of these two equations in $\bar{\mathcal{D}}$ constitute respectively the homogeneous 1-cochain and 1-cocyle conditions on the map $\bar{\Psi}_\mathfrak{r}$ (see [Se2, p. 112]). Finally, if $\mathfrak{s} \in \mathbb{P}_+(\mathbb{Q}^2)$ is any other rational ray, then applying Theorem 2.1 to the two triples $(\mathbf{M}_0\mathfrak{r}, \mathbf{M}_1\mathfrak{r}, \mathbf{M}_1\mathfrak{s})$ and $(\mathbf{M}_0\mathfrak{s}, \mathbf{M}_1\mathfrak{s}, \mathbf{M}_0\mathfrak{r})$ and subtracting gives on the one hand

$$\begin{aligned} &\Psi_\mathfrak{r}(\mathbf{M}_0, \mathbf{M}_1) - \Psi_\mathfrak{s}(\mathbf{M}_0, \mathbf{M}_1) \\ &\quad = \mathcal{P}(\mathbf{M}_0\mathfrak{r}, \mathbf{M}_0\mathfrak{s}) - \mathcal{P}(\mathbf{M}_1\mathfrak{r}, \mathbf{M}_1\mathfrak{s}) \pmod{\tfrac{1}{2}\mathbb{Z}\delta}, \end{aligned} \tag{28}$$

for all $\mathbf{M}_0, \mathbf{M}_1 \in \mathrm{GL}_2(\mathbb{Q})$, while on the other hand Equation (27) tells us that the map $\mathbf{M} \mapsto \mathcal{P}(\mathbf{M}\mathfrak{r}, \mathbf{M}\mathfrak{s})$ is a homogeneous 0-cochain with values in $\mathcal{D}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$. The right-hand side of (28) is its corresponding 1-coboundary so that $\bar{\Psi}_\mathfrak{r} - \bar{\Psi}_\mathfrak{s}$ lies in the group $B^1(\mathrm{PGL}_2(\mathbb{Q}), \bar{\mathcal{D}}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}))$, as required for part (iii). $\square$

We call $\bar{\Psi}_\mathfrak{r}$ the *homogeneous Shintani cocycle* (associated to the ray $\mathfrak{r}$).

To round off this article we introduce a number of variants of $\bar{\Psi}_\mathfrak{r}$. This involves no substantial new mathematics but the new notations and the ideas they represent will be of use in the sequel to the present paper. For each $l \in \mathbb{Z}$ the lth homogeneous component $\mathbb{R}(\mathbf{z})_l$ of $\mathbb{R}((\mathbf{z}))^{\mathrm{hd}}$ (see the Introduction) is stable for

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the $\star$-action of $\mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$ and in particular each $n \in \mathbb{Z}\setminus\{0\}$ acts on it by multiplication by n^l. We write $\mathcal{D}(\mathbb{R}(\mathbf{z})_l)$ for the associated space of distributions on $\mathbb{R}^2/\mathbb{Z}^2$. Since $\frac{1}{2}\mathbb{Z}\delta$ is entirely contained in $\mathcal{D}(\mathbb{R}(\mathbf{z})_0)$, the composition of distributions with the lth-homogeneous-parts map π_l gives rise to a well-defined $\mathbb{R}[\mathrm{GL}_2(\mathbb{Q})]$-projection from $\bar{\mathcal{D}}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$ onto $\mathcal{D}(\mathbb{R}(\mathbf{z})_l)$ for each $l \neq 0$ and onto $\bar{\mathcal{D}}(\mathbb{R}(\mathbf{z})_0) =: \mathcal{D}(\mathbb{R}(\mathbf{z})_0)/(\frac{1}{2}\mathbb{Z}\delta)$ in the case $l = 0$. (The product of these projections defines an embedding of $\bar{\mathcal{D}}(\mathbb{R}((\mathbf{z}))^{\mathrm{hd}})$ into $(\prod_{l\neq 0}\mathcal{D}(\mathbb{R}(\mathbf{z})_l))\times\bar{\mathcal{D}}(\mathbb{R}(\mathbf{z})_0)$). The composite of the lth projection with $\Psi_{\mathfrak{r}}$ will be denoted $\Psi_{\mathfrak{r},l}$ for $l \neq 0$, (respectively, $\bar{\Psi}_{\mathfrak{r},0}$ for $l = 0$). Concretely, it sends $(\mathbf{M}_0, \mathbf{M}_1) \in \mathrm{GL}_2(\mathbb{Q})^2$ to the $\mathbb{R}(\mathbf{z})_l$-valued distribution $\mathcal{P}(\mathbf{M}_0\mathfrak{r}, \mathbf{M}_1\mathfrak{r})_l : \mathbf{x} \mapsto \mathcal{P}(\mathbf{x}, \mathbf{M}_0\mathfrak{r}, \mathbf{M}_1\mathfrak{r})_l$ (respectively, to $\mathcal{P}(\mathbf{M}_0\mathfrak{r}, \mathbf{M}_1\mathfrak{r})_0$ modulo $\frac{1}{2}\mathbb{Z}\delta$, as an element of $\bar{\mathcal{D}}(\mathbb{R}(\mathbf{z})_0)$). It depends only on the image of $\mathfrak{r}$ in $\mathbb{P}^1(\mathbb{Q})$ and is zero for $l < -2$. The following is an easy consequence of parts (ii) and (iii) of Theorem 4.1.

COROLLARY 4.2. *For each $l \in \mathbb{Z}$, $l \geqslant -2$ and each $\mathfrak{r} \in \mathbb{P}_+(\mathbb{Q}^2)$, the map $\Psi_{\mathfrak{r},l}$ (respectively, the map $\bar{\Psi}_{\mathfrak{r},0}$ if $l = 0$) defines a homogeneous 1-cocycle on $\mathrm{GL}_2(\mathbb{Q})$ with values in $\mathcal{D}(\mathbb{R}(\mathbf{z})_l)$ (respectively, in $\bar{\mathcal{D}}(\mathbb{R}(\mathbf{z})_0)$, if $l = 0$) which factors through $\mathrm{PGL}_2(\mathbb{Q})$. The corresponding $(\mathrm{P})\mathrm{GL}_2(\mathbb{Q})$ cohomology classes do not depend on the ray $\mathfrak{r}$.* □

The infinite sequence of cocycles $\Psi_{\mathfrak{r},l}$ for $l \neq 0$ and the cocycle $\bar{\Psi}_{\mathfrak{r},0}$ recall those defined in [St] and [Sc1]. Since these papers deal largely in terms of *inhomogeneous* 1-cocycles (meaning now that they are functions of a single group element instead of two), we hereby introduce the notations $\bar{\Phi}_{\mathfrak{r}}$, $\Phi_{\mathfrak{r},l}$ and $\bar{\Phi}_{\mathfrak{r},0}$ for the versions of our cocycles $\bar{\Psi}_{\mathfrak{r}}$, $\Psi_{\mathfrak{r},l}$ and $\bar{\Psi}_{\mathfrak{r},0}$ which are 'inhomogeneous' in this sense. Thus, generically, $\Phi(\mathbf{M}) =: \Psi(\mathbf{1}, \mathbf{M})$ for any $\mathbf{M} \in (\mathrm{P})\mathrm{GL}_2(\mathbb{Q})$, so that $\Psi(\mathbf{M}_0, \mathbf{M}) = \mathbf{M}_0 \cdot \Phi(\mathbf{M}_0^{-1}\mathbf{M}_1)$. The homogeneous 1-cocycle condition on the Ψ's is equivalent to the familiar 'crossed homomorphism' property of the Φ's

$$\Phi(\mathbf{M}\mathbf{M}') = \Phi(\mathbf{M}) + \mathbf{M}\cdot\Phi(\mathbf{M}') \quad \text{for all } \mathbf{M}, \mathbf{M}' \text{ in } (\mathrm{P})\mathrm{GL}_2(\mathbb{Q}).$$

Notice that, if $\mathbf{M}\mathfrak{r} = \pm\mathfrak{r}$ then for each $l \in \mathbb{Z}$, $l \neq 0$, $\mathcal{P}(\mathbf{x}, \mathfrak{r}, \mathbf{M}\mathfrak{r})_l$ is zero for all $\mathbf{x}$ so that the cocycle $\Phi_{\mathfrak{r},l}$ vanishes on such $\mathbf{M}$, as does the cocycle $\bar{\Phi}_{\mathfrak{r},0}$. In particular, the cocycles $\Phi_{\mathfrak{r}_\infty,l}$ ($l \neq 0$) and $\bar{\Phi}_{\mathfrak{r}_\infty,0}$ are *parabolic* (see the Introduction). Their values therefore depend only on the first column of $\mathbf{M}$ and are given explicitly by Theorem 3.1 in terms of Dedekind sums.

It is natural to ask whether any of the cocycles that we have constructed here are actually coboundaries, i.e. whether or not they represent the trivial class in the appropriate cohomology group H^1. The answer to this question clearly does not depend on the choice of the ray $\mathfrak{r}$. One can prove the following statement (which certainly implies the non-triviality of $\Phi_{\mathfrak{r}}$): '*For every even $l \geqslant 2$ (respectively, for $l = 0$) the cocycle $\Phi_{\mathfrak{r},l}$ (respectively, the cocycle $\bar{\Phi}_{\mathfrak{r},0}$) does not lie in the group $B^1(\mathrm{PGL}_2(\mathbb{Q}), \mathcal{D}(\mathbb{R}(\mathbf{z})_l))$ (respectively, in the group $B^1(\mathrm{PGL}_2(\mathbb{Q}), \bar{\mathcal{D}}(\mathbb{R}(\mathbf{z})_0)))$ of* *coboundaries*.' The idea, as mentioned in the Introduction, is to use the non-vanishing of certain L-values over real quadratic fields, calculated by means of the Shintani cocycle. In fact, this method gives much stronger non-triviality statements concerning certain *restrictions* of the cocycle, for example to principal congruence subgroups $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$.

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