



On the Bernstein Problem in the Three-dimensional Heisenberg Group

Josef F. Dorfmeister, Jun-ichi Inoguchi, and Shimpei Kobayashi

Abstract. In this note we present a simple alternative proof for the Bernstein problem in the three-dimensional Heisenberg group Nil_3 by using the loop group technique. We clarify the geometric meaning of the two-parameter ambiguity of entire minimal graphs with prescribed Abresch-Rosenberg differential.

Introduction

The *Bernstein problem* is one of the traditional problems of global differential geometry. The original result due to Bernstein asserts that every entire minimal graph in Euclidean three-space $\mathbb{R}^3(x_1, x_2, x_3)$ is a plane. In other words, Bernstein's result shows that the only global solution on the (x_1, x_2) -plane to the so-called *minimal surface equation*

$$\{1 + (f_{x_1})^2\}f_{x_2x_2} - 2f_{x_1}f_{x_2}f_{x_1x_2} + \{1 + (f_{x_2})^2\}f_{x_1x_1} = 0$$

is a linear function of x_1 and x_2 .

The Bernstein problem has been generalized to a problem basically asking for a classification of all entire minimal graphs. On the other hand, when the ambient space is not the Euclidean three-space, the Bernstein problem often needs to be amended. For instance, in Minkowski three-space \mathbb{L}_3 equipped with the natural Lorentz metric $dx_1^2 + dx_2^2 - dx_3^2$, there are many entire (timelike) minimal graphs over the timelike plane $\mathbb{L}_2 = \mathbb{R}^2(x_2, x_3)$; see, for example, [7]. Next, we focus on the three-dimensional *Heisenberg group* Nil_3 , which is one of the model spaces of Thurston geometries [8]. The space Nil_3 is realized as Cartesian three-space $\mathbb{R}^3(x_1, x_2, x_3)$ equipped with the Riemannian metric

$$dx_1^2 + dx_2^2 + \left\{ dx_3 + \frac{1}{2}(x_2 dx_1 - x_1 dx_2) \right\}^2$$

and a nilpotent Lie group structure; see, for example, [4]. The Riemannian metric is invariant under the nilpotent Lie group structure and has a 4-dimensional isometry group. The identity component of the isometry group is a semi-direct product $\text{Nil}_3 \ltimes \text{SO}_2$.

It has been known for a long time that nontrivial entire minimal graphs exist in Nil_3 ; see, for example, [5]. Therefore, in Nil_3 the Bernstein problem has been phrased

Received by the editors April 28, 2015; revised June 11, 2015.

Published electronically October 21, 2015.

The second named author is partially supported by Kakenhi 24540063, 15K04834.

The third named author is partially supported by Kakenhi 26400059.

AMS subject classification: 53A10, 53C42.

Keywords: Bernstein problem, minimal graphs, Heisenberg group, loop groups, spinors.

more specifically as the problem to construct entire minimal graphs over the natural (x_1, x_2) -plane with a prescribed holomorphic quadratic differential.

Under this formulation, Fernández and Mira [6] studied the Bernstein problem in Nil_3 . They proved in a fairly indirect way that for a prescribed holomorphic quadratic differential $Q dz^2$ over the complex plane \mathbb{C} with $Q \neq 0$ or the unit disc \mathbb{D} , there exists a two-parameter family of entire minimal vertical graphs whose Abresch–Rosenberg differential is $Q dz^2$. Their proof relies first on the Lawson-type correspondence (often called *sister correspondence*) between minimal surfaces in Nil_3 and surfaces of constant mean curvature (CMC in short) with mean curvature $H = 1/2$ in the product space $\mathbb{H}^2 \times \mathbb{R}$, where \mathbb{H}^2 denotes the hyperbolic two-space. Secondly, they use the correspondence between harmonic maps into \mathbb{H}^2 and CMC surfaces with mean curvature $H = 1/2$ in the product space $\mathbb{H}^2 \times \mathbb{R}$. Finally, they use a result of Wan and Au [9, 10] solving the Bernstein problem for spacelike CMC surfaces in \mathbb{L}_3 and use that the Gauss map of those surfaces is also harmonic into \mathbb{H}^2 .

In this paper we give a much simpler proof of the solution to the Bernstein problem in Nil_3 by virtue of the generalized Weierstrass type representation established in our previous work [4]. The advantage of our approach is that we can give a direct relation between minimal graphs in Nil_3 and spacelike CMC surface with mean curvature $H = 1/2$ graphs in \mathbb{L}_3 (Theorem 1.7). This relation enables us to give a simple alternative proof of Fernández and Mira’s theorem (Theorem 1.8).

Our new proof also provides new insights. In fact it clarifies the geometric meaning of the two-parameter ambiguity of entire minimal graphs with prescribed Abresch–Rosenberg differential. While it is quite clear that the two-parameter family is related to the boosts in $\text{SU}_{1,1}$, our argument also shows how the corresponding family of surfaces varies in Nil_3 .

1 Bernstein Problem

We discuss the Bernstein problem in Nil_3 , that is, the classification of entire minimal vertical graphs in Nil_3 . We only consider vertical graphs; therefore, we will sometimes omit the word “vertical”. From now on, we denote the coordinates of Nil_3 or \mathbb{L}_3 by (x_1, x_2, x_3) .

1.1 Completeness

The basic result used in this paper is Theorem A.3. It provides the direct relation between minimal surfaces in Nil_3 and spacelike CMC surfaces in \mathbb{L}_3 . This close relationship is also underlined by a simple relation between the corresponding metrics.

Lemma 1.1 *Let f^λ and $f_{\mathbb{L}_3}^\lambda$ be an associated family of minimal surfaces in Nil_3 and an associated family of spacelike CMC surfaces with mean curvature $H = 1/2$ in \mathbb{L}_3 correlated and defined as in Theorem A.3, respectively. Denote the metric of f^λ by $e^u dzd\bar{z}$ and the metric of $f_{\mathbb{L}_3}^\lambda$ by $e^{u_{\mathbb{L}_3}} dzd\bar{z}$. Moreover, let $\phi_3^\lambda dz$ be the coefficient of e_3 in*

$$(f^\lambda)^{-1} f_{\mathbb{L}_3}^\lambda dz = \sum_{i=1}^3 (\phi_i^\lambda dz) e_i.$$

Then the following relation holds:

$$e^{u_{\mathbb{L}_3}} + 4|\phi_3^\lambda|^2 = e^u.$$

Proof It is known that the conformal factors e^u and $e^{u_{\mathbb{L}_3}}$ can be computed explicitly in terms of spinors (see [4, Section 3.1], Remark A.4, and [2]):

$$e^u = 4(|\psi_1^\lambda|^2 + |\psi_2^\lambda|^2)^2, \quad e^{u_{\mathbb{L}_3}} = 4(|\psi_1^\lambda|^2 - |\psi_2^\lambda|^2)^2,$$

where ψ_j^λ ($j = 1, 2$) is a family of spinors for the associated family f^λ . Since $\phi_3^\lambda = 2\psi_1^\lambda \overline{\psi_2^\lambda}$, the claim follows. ■

Remark 1.2 It is known that the metrics $e^{u_{\mathbb{L}_3}} dzd\bar{z}$ of an associated family of space-like CMC surfaces $f_{\mathbb{L}_3}^\lambda$ are independent of λ ; that is, on a simply connected domain, any two members of the associated family $\{f_{\mathbb{L}_3}^\lambda\}_{\lambda \in \mathbb{S}^1}$ are isometric. In fact the metric can be computed by the support $h(dz)^{1/2}(d\bar{z})^{1/2}$ (see Appendix A for the definition), as

$$(1.1) \quad h^2 dzd\bar{z} = e^{u_{\mathbb{L}_3}} dzd\bar{z}.$$

However, the metrics $e^u dzd\bar{z}$ of an associated family of minimal surface f^λ depend on λ ; that is, any two members of the associated family $\{f^\lambda\}_{\lambda \in \mathbb{S}^1}$ are, in general, non-isometric.

Using the relation above, we have the following theorem.

Theorem 1.3 *Let f^λ and $f_{\mathbb{L}_3}^\lambda$ be an associated family of minimal surfaces in Nil_3 and an associated family of spacelike CMC surfaces with mean curvature $H = 1/2$ in \mathbb{L}_3 correlated and defined as in Theorem A.3, respectively. Assume that one member of the associated family $\{f_{\mathbb{L}_3}^\lambda\}_{\lambda \in \mathbb{S}^1}$ is closed with respect to the Euclidean topology. Then each member of the associated family $\{f^\lambda\}_{\lambda \in \mathbb{S}^1}$ is a complete, entire graph.*

Proof We denote the spacelike CMC surface in \mathbb{L}_3 , which is closed with respect to the Euclidean topology, by $f_{\mathbb{L}_3}^* = f_{\mathbb{L}_3}^\lambda|_{\lambda^* \in \mathbb{S}^1}$. From the assumption and by [3, p. 415], we conclude that $f_{\mathbb{L}_3}^*$ is complete. Moreover, from [9, Proposition 2], $f_{\mathbb{L}_3}^*$ is also an entire graph. Since the metric is invariant within the associated family $\{f_{\mathbb{L}_3}^\lambda\}_{\lambda \in \mathbb{S}^1}$ (see Remark 1.2), each member of the associated family of spacelike CMC surfaces $\{f_{\mathbb{L}_3}^\lambda\}_{\lambda \in \mathbb{S}^1}$ is also complete. Then from Lemma 1.1, we have that each member of the associated family of minimal surfaces $\{f^\lambda\}_{\lambda \in \mathbb{S}^1}$ is complete.

Let us look more closely at the correspondence between $f_{\mathbb{L}_3}^\lambda$ and f^λ . From formulas (A.4) and (A.5) we infer by inspection that f^λ and $f_{\mathbb{L}_3}^\lambda$ share the same x_1 -, x_2 -components. ■

1.2 Rigid Motions

It is known that the isometry group of \mathbb{L}_3 is the six-dimensional Lie group that is generated by a one-parameter family of rotations around the x_3 -axis (the timelike axis), a two-parameter family of boosts, and three families of translations. In contrast, the

isometry group of Nil_3 is only four-dimensional and is generated by a one-parameter family of rotations around x_3 -axis and three families of translations in Nil_3 .

A comparison of the two Sym formulas in Theorem A.3 indicates that isometries of Minkowski space will not necessarily become isometries of Nil_3 . The precise relation will be made clear in the next lemma.

Lemma 1.4 *Let $f_{\mathbb{L}_3}^\lambda$ and $\tilde{f}_{\mathbb{L}_3}^\lambda$ be two associated family of spacelike CMC surfaces with mean curvature $H = 1/2$ in \mathbb{L}_3 defined by the Sym-formula in Theorem A.3 for some extended frames F^λ and \tilde{F}^λ , respectively and set $f_{\mathbb{L}_3} = f_{\mathbb{L}_3}^\lambda|_{\lambda=1}$ and $\tilde{f}_{\mathbb{L}_3} = \tilde{f}_{\mathbb{L}_3}^\lambda|_{\lambda=1}$. Moreover, let f^λ and \tilde{f}^λ denote the two associated families of minimal surfaces in Nil_3 defined from the same extended frames F^λ and \tilde{F}^λ , respectively and set $f = f^\lambda|_{\lambda=1}$ and $\tilde{f} = \tilde{f}^\lambda|_{\lambda=1}$. Assume that $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric by some rigid motion in \mathbb{L}_3 . Then the following statements hold:*

- (i) *If $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric by a rotation around the x_3 -axis (the timelike axis), then f and \tilde{f} are isometric by the rotation around the x_3 -axis (the same angle) and some translation.*
- (ii) *If $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric by a translation, then f and \tilde{f} are isometric by some translation (not necessarily the same translation).*
- (iii) *If $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric by a boost, then f and \tilde{f} are, in general, not isometric.*

Proof Since $f_{\mathbb{L}_3} (= f_{\mathbb{L}_3}^\lambda|_{\lambda=1})$ and $\tilde{f}_{\mathbb{L}_3} (= \tilde{f}_{\mathbb{L}_3}^\lambda|_{\lambda=1})$ are isometric by a rigid motion in \mathbb{L}_3 , the isometry between these two surfaces lifts to the level of frames $F = F^\lambda|_{\lambda=1}$ and $\tilde{F} = \tilde{F}^\lambda|_{\lambda=1}$ as $\tilde{F} = MFk$, where M is a z -independent $\text{SU}_{1,1}$ -valued matrix and k is a U_1 -valued matrix. After introducing the loop parameter, we obtain the relation

$$\tilde{F}^\lambda = M^\lambda F^\lambda k.$$

Note that M^λ is a $(\Lambda\text{SU}_{1,1})_\sigma$ -valued matrix and satisfies $M^\lambda|_{\lambda=1} = M$. Then it is easy to see that $\tilde{f}_{\mathbb{L}_3}^\lambda$ and $f_{\mathbb{L}_3}^\lambda$ satisfy the relation

$$(1.2) \quad \tilde{f}_{\mathbb{L}_3}^\lambda = M^\lambda f_{\mathbb{L}_3}^\lambda (M^\lambda)^{-1} - i\lambda(\partial_\lambda M^\lambda)(M^\lambda)^{-1}.$$

Now a straightforward computation shows that the corresponding two minimal surfaces f^λ and \tilde{f}^λ have the following relation:

$$(1.3) \quad \tilde{f}^\lambda = (\text{Ad}(M^\lambda)(f_{\mathbb{L}_3}^\lambda) - X^\lambda)^o - \frac{1}{2} \{ \text{Ad}(M^\lambda)(i\lambda\partial_\lambda f_{\mathbb{L}_3}^\lambda) + [X^\lambda, \text{Ad}(M^\lambda)(f_{\mathbb{L}_3}^\lambda)] - Y^\lambda \}^d,$$

where we set

$$(1.4) \quad X^\lambda = i\lambda(\partial_\lambda M^\lambda)(M^\lambda)^{-1}, \quad Y^\lambda = i\lambda(\partial_\lambda X^\lambda),$$

and the superscripts “ o ” and “ d ” denote the off-diagonal and diagonal part, respectively. For simplicity of notation we do not distinguish here f^λ in Nil_3 and \tilde{f}^λ in $\mathfrak{su}_{1,1}$.

We note that for each fixed $\lambda \in \mathbb{S}^1$, the first part of the right-hand side in (1.2) describes a Lorentz transformation and the second part of the right-hand side in (1.2) describes a translation. We now consider each of the three types of generators separately.

(1) First, suppose that $\tilde{f}_{\mathbb{L}_3}$ and $f_{\mathbb{L}_3}$ are isometric by a rotation around the x_3 -axis (the timelike axis). Since the original transformation M was a rotation, it follows that

$$M(= M^\lambda|_{\lambda=1}) = \text{diag}(e^{i\theta}, e^{-i\theta}), \quad \partial_\lambda M^\lambda|_{\lambda=1} = \mathbf{0}.$$

Here 2θ is the angle of rotation. A straightforward computation shows that f and \tilde{f} satisfy the equation $\tilde{f} = \text{Ad}(M)(f) + \frac{1}{2}Y^d$, where the translation term Y can be computed as

$$Y = Y^\lambda|_{\lambda=1} = -\lambda^2(\partial_\lambda^2 M^\lambda)(M^\lambda)^{-1}|_{\lambda=1}.$$

Therefore, this one-parameter family consists of isometric minimal surfaces in Nil_3 .

(2) Next suppose that $\tilde{f}_{\mathbb{L}_3}$ and $f_{\mathbb{L}_3}$ are isometric by some translation. Since the original transformation M was a translation, it follows

$$M(= M^\lambda|_{\lambda=1}) = \text{id}, \quad (\partial_\lambda M^\lambda)|_{\lambda=1} \neq \mathbf{0}.$$

Substituting $\lambda = 1$ into (1.3) we see immediately that f and \tilde{f} satisfy the relation $\tilde{f} = f + A$, where $A = A(x_1, x_2)$ is given by

$$A = -X^o - \frac{1}{2}([X, f_{\mathbb{L}_3}] - Y)^d,$$

where $X = X^\lambda|_{\lambda=1}$ and $Y = Y^\lambda|_{\lambda=1}$ for X^λ and Y^λ in (1.4). It is clear that Y is independent of x_1 and x_2 , the coordinates for f , but x_1 and x_2 enter the commutator. An explicit computation, using the basis $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ given in [4, (6.1)] and the transformation formula stated in [4, Appendix B], now shows that \tilde{f} can be obtained from f by a translation in Nil_3 (with a constant vector, whose coefficients basically are the components of X^o and of Y^d).

(3) Let us finally consider the transformations M given by boosts in \mathbb{L}_3 . These transformations form a two-parameter family. Since the original transformation was a boost, it follows that

$$(1.5) \quad M = M^\lambda|_{\lambda=1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix}, \quad \partial_\lambda M^\lambda|_{\lambda=1} = \mathbf{0}.$$

Here, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $\alpha^2 - |\beta|^2 = 1$ and we obtain

$$\tilde{f} = (\text{Ad}(M)(f_{\mathbb{L}_3}))^o - \frac{1}{2}(\text{Ad}(M)(i\lambda\partial_\lambda f_{\mathbb{L}_3}^\lambda|_{\lambda=1}) - Y)^d,$$

where $Y = Y^\lambda|_{\lambda=1}$ for Y^λ in (1.4). Now it follows by a straightforward computation that

$$(1.6) \quad \tilde{f} = \begin{pmatrix} (\alpha^2 + |\beta|^2)r + \alpha\beta\bar{s} - \alpha\bar{\beta}s & -2\alpha\beta p + \alpha^2 q - \beta^2\bar{q} \\ 2\alpha\bar{\beta}p + \alpha^2\bar{q} - \bar{\beta}^2 q & -(\alpha^2 + |\beta|^2)r - \alpha\beta\bar{s} + \alpha\bar{\beta}s \end{pmatrix} + \frac{1}{2}Y^d,$$

where $p, r \in i\mathbb{R}$ and $q, s \in \mathbb{C}$ are functions defined by

$$(1.7) \quad f_{\mathbb{L}_3}^\lambda|_{\lambda=1} = \begin{pmatrix} p & q \\ \bar{q} & -p \end{pmatrix}, \quad -\frac{1}{2}i\lambda(\partial_\lambda f_{\mathbb{L}_3}^\lambda)|_{\lambda=1} = \begin{pmatrix} r & s \\ \bar{s} & -r \end{pmatrix}.$$

Note that the components of the minimal surface f in the basis $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ are given by

$$(x_1, x_2, x_3) = (2 \text{Im } q, -2 \text{Re } q, -2 \text{Im } r).$$

Thus, from (1.6) and the action of the isometry group of Nil_3 as described in (A.1), it is easy to see that f and \tilde{f} are in general not isometric; see Remark 1.5 in detail. ■

Remark 1.5 In Lemma 1.4(iii), from (1.6) and the action of the isometry group of Nil_3 , we see that the f and \tilde{f} are isometric in Nil_3 if and only if there exist some $\theta \in \mathbb{R}$, $(a_1, a_2, a_3) \in \mathbb{R}^3$ such that the following two equations hold:

$$\begin{aligned} (\alpha^2 + |\beta|^2)x_3 - 4\alpha \operatorname{Im}(\beta\bar{s}) - Y^{11} &= ax_1 + bx_2 + x_3 + a_3, \\ -4\alpha\beta p + i(\beta^2 + \alpha^2)x_1 + (\beta^2 - \alpha^2)x_2 &= e^{i\theta}(ix_1 - x_2) + ia_1 - a_2, \end{aligned}$$

where $Y^d = \operatorname{diag}(iY^{11}, -iY^{11})$, $a = \frac{1}{2}(a_1 \sin \theta - a_2 \cos \theta)$, $b = \frac{1}{2}(a_1 \sin \theta + a_2 \cos \theta)$, and p and s are purely imaginary and complex valued functions, respectively, defined in (1.7). From these two equations, it is easy to see that they are satisfied for very special minimal surfaces f only.

Remark 1.6 After fixing base points, the Sym-formula establishes a one-to-one relation between spacelike CMC with mean curvature $H = 1/2$ surfaces in \mathbb{L}_3 and minimal surfaces in Nil_3 . Clearly, the Poincaré group $\text{SU}_{1,1} \times \mathbb{L}_3$ acts on the family of spacelike CMC surfaces in \mathbb{L}_3 . If we fix base points, we eliminate the action of the translation part of the Poincaré group, reducing the action to the Lorentz group $\text{SU}_{1,1}$.

Via the Sym formula, the Poincaré group also acts on the family of minimal surfaces in Nil_3 . Since we fix base points, we can also eliminate the translation part of the isometry group of Nil_3 . So generically, the dimension of the family of minimal surfaces should be three. But from Lemma 1.4, identifying minimal surfaces that are isometric by rotations, we see that two is the highest dimension of any orbit. These orbits are realized by the action of boosts. From (1.5), the set of boosts \mathcal{B} can be computed as

$$\mathcal{B} = \{X \bar{X} \mid X \in \text{SU}_{1,1}\}.$$

From this, it is clear that \mathcal{B} is the symmetric space $\text{SU}_{1,1}/\text{U}_1$.

1.3 Bernstein Problem

We will finally present a short alternative proof of the Bernstein problem in Nil_3 using the loop group method. The heart of the proof is the following simple relation between spacelike CMC graphs in \mathbb{L}_3 and minimal graphs in Nil_3 .

Theorem 1.7 *Every entire, complete, spacelike CMC graph in \mathbb{L}_3 with mean curvature $H = 1/2$ and the Hopf differential $Q_{\mathbb{L}_3} dz^2$ induces, via the Sym-formula (applied to its associated family), an entire, complete, minimal graph in Nil_3 with Abresch–Rosenberg differential $-Q_{\mathbb{L}_3} dz^2$.*

Conversely, every entire, complete, minimal graph in Nil_3 is obtained in this way.

Proof Let $g_{\mathbb{L}_3}$ be an entire complete spacelike CMC graph with mean curvature $H = 1/2$ over the (x_1, x_2) -plane in \mathbb{L}_3 whose Hopf differential is $Q_{\mathbb{L}_3} dz^2$. Let F^λ be the extended frame of $g_{\mathbb{L}_3}$, defined by (A.3), and apply the Sym-formulas of Theorem A.3 to obtain $f_{\mathbb{L}_3}^\lambda$ and f^λ from the same extended frame F^λ . Note that $f_{\mathbb{L}_3}^\lambda$ and

f^λ define an associated family of spacelike CMC surfaces in \mathbb{L}_3 and minimal surfaces in Nil_3 , respectively. Moreover, $f_{\mathbb{L}_3} = f_{\mathbb{L}_3}^\lambda|_{\lambda=1}$ and $g_{\mathbb{L}_3}$ are isometric by some rigid motion in \mathbb{L}_3 , by the fundamental theorem of surface theory (the mean curvature H , the Hopf differential $Q_{\mathbb{L}_3} dz^2$ and the metric $e^{u_{\mathbb{L}_3}} dzd\bar{z}$ are the same), thus $f_{\mathbb{L}_3}$ is also an entire, complete, spacelike CMC graph, [9, Proposition 1]. From formulas (A.4) and (A.5) we infer by inspection that f^λ and $f_{\mathbb{L}_3}^\lambda$ share the same x_1 -, x_2 -components. Thus $f = f^\lambda|_{\lambda=1}$ is an entire minimal graph as well. Moreover, from Theorem 1.3 we obtain that $f (= f^\lambda|_{\lambda=1})$ is complete, and by Remark A.4 we know that the Abresch–Rosenberg differential is $-Q_{\mathbb{L}_3} dz^2$.

To verify the second statement, let f be an entire, complete, minimal graph in Nil_3 whose Abresch–Rosenberg differential is $Q dz^2$ and let F^λ be the extended frame of f and f^λ its associated family from the extended frame F^λ . Then we have $f = f^\lambda|_{\lambda=1}$ up to translation in Nil_3 . Moreover, let $f_{\mathbb{L}_3}^\lambda$ be the spacelike CMC surface in \mathbb{L}_3 defined by the same extended frame F^λ in (A.4). Note that the Hopf differential of $f_{\mathbb{L}_3}^\lambda$ is $Q_{\mathbb{L}_3}^\lambda dz^2 = -\lambda^{-2} Q dz^2$. Since f and $f_{\mathbb{L}_3}^\lambda|_{\lambda=1}$ have the same x_1 -, x_2 -components, the latter surface is an entire CMC graph in \mathbb{L}_3 , and thus by [3, p. 415], the proof is complete. ■

Using Theorem 1.7, it is easy to give the proof of the solution to the Bernstein problem.

Theorem 1.8 *Let $Q dz^2$ be a holomorphic quadratic differential on \mathbb{D} or $M = \mathbb{C}$ with $Q \neq 0$. Then the following statements hold:*

- (i) *There exists a two-parameter family of entire, complete, minimal graphs in Nil_3 , whose Abresch–Rosenberg differential is $Q dz^2$.*
- (ii) *Any two members of this two-parameter family are generically non-congruent.*
- (iii) *Each member of this two-parameter family is induced via the Sym-formula by (the associated family of) an entire, complete, spacelike CMC graph in \mathbb{L}_3 with the Hopf differential $-Q dz^2$.*

Proof First we note that it is known that for a given holomorphic quadratic differential $Q_{\mathbb{L}_3} dz^2$ on \mathbb{D} or \mathbb{C} , there exists a unique entire complete spacelike CMC graph $g_{\mathbb{L}_3}$ over the (x_1, x_2) -plane in \mathbb{L}_3 whose Hopf differential is $Q_{\mathbb{L}_3} dz^2$ [9, 10]. Here “unique” means that any other such spacelike CMC graph whose Hopf differential is $Q_{\mathbb{L}_3} dz^2$ is isometric to $g_{\mathbb{L}_3}$ by an isometry of \mathbb{L}_3 . We normalize the mean curvature of $g_{\mathbb{L}_3}$ as $H = 1/2$ and set $Q_{\mathbb{L}_3} dz^2 = -Q dz^2$, where $Q dz^2$ is the quadratic differential satisfying the condition in the theorem. Let F^λ be the extended frame of $g_{\mathbb{L}_3}$ and apply the Sym formulas of Theorem A.3 to obtain $f_{\mathbb{L}_3}^\lambda$ and f^λ from the same extended frame F^λ . From Theorem 1.7 we know that $f_{\mathbb{L}_3} = f_{\mathbb{L}_3}^\lambda|_{\lambda=1}$ and $f = f^\lambda|_{\lambda=1}$ are complete entire graphs. From the construction it is clear that f is a minimal surface. Moreover, the Abresch–Rosenberg differential of f is $Q dz^2$.

We now consider a spacelike CMC surface $\tilde{g}_{\mathbb{L}_3}$ isometric to $g_{\mathbb{L}_3}$ in \mathbb{L}_3 . Then as explained in the proof of Lemma 1.4, the extended frame \tilde{F}^λ of $\tilde{g}_{\mathbb{L}_3}$ satisfies $\tilde{F}^\lambda = M^\lambda F^\lambda k$ for some z -independent $(\text{ASU}_{1,1})_\sigma$ -valued matrix M^λ and a U_1 -valued matrix k , in particular independent of λ . For the associated family $\tilde{f}_{\mathbb{L}_3}^\lambda$ of $\tilde{g}_{\mathbb{L}_3}$ which is defined by the Sym formula (A.4) from the extended frame \tilde{F}^λ , we see that $\tilde{f}_{\mathbb{L}_3} = \tilde{f}_{\mathbb{L}_3}^\lambda|_{\lambda=1}$ and

$\tilde{g}_{\mathbb{L}_3}$ are isometric. Thus $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric, and again from [9, Proposition 1] we obtain that $\tilde{f}_{\mathbb{L}_3}$ is an entire, complete, spacelike CMC graph. Let \tilde{f}^λ be the corresponding associated family of minimal surfaces in Nil_3 which is defined by the Sym formula (A.5) from the extended frame \tilde{F}^λ . Then using the argument in Theorem 1.7, we see that $\tilde{f} = \tilde{f}^\lambda|_{\lambda=1}$ is an entire, complete minimal graph in Nil_3 . Note that the Abresch–Rosenberg differential of \tilde{f} is also $Q dz^2$.

We now apply Lemma 1.4. If the isometry $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ is of case (i) or (ii), then $\tilde{f}(= \tilde{f}^\lambda|_{\lambda=1})$ is congruent to $f(= f^\lambda|_{\lambda=1})$. However, if the isometry of $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ is of case (iii), then \tilde{f} is in general non-congruent to f . In particular, in Lemma 1.4(iii) corresponds to a two-parameter family of boosts in \mathbb{L}_3 . Therefore, for an entire, complete, spacelike CMC surface, there exists a two-parameter family of non-congruent complete minimal graphs in Nil_3 which have the same Abresch–Rosenberg differential $Q dz^2$. ■

1.4 The Two-parameter of Ambiguity and the Associated Family

From Theorem 1.8, we have the geometric meaning of the two-parameter ambiguity of entire minimal graphs. We also have the associated family of entire minimal graphs.

Remark 1.9 In [6], the two-parameter family of an entire, complete, minimal graphs in Nil_3 was obtained by the choice of the initial condition for a nonlinear partial differential equation. The solution corresponds to ϕ_3 ; that is, the e_3 -component of $f^{-1}f_z$ and the initial condition is the initial value $\phi_3(z_*)$ for some base point z_* in \mathbb{C} or \mathbb{D} . In our setting, this freedom naturally appears as the two-parameter family of boosts in \mathbb{L}_3 . As we see from the proof of Theorem 1.8, two minimal surfaces f^λ and \tilde{f}^λ satisfy the relation (1.3). Set

$$M^\lambda|_{\lambda=1} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \in \text{SU}_{1,1}, \quad (f^\lambda)^{-1}f_z|_{\lambda=1} = \sum_{k=1}^3 \phi_k e_k, \quad (\tilde{f}^\lambda)^{-1}\tilde{f}_z|_{\lambda=1} = \sum_{k=1}^3 \tilde{\phi}_k e_k,$$

where α is real. Then a straightforward computation (using the proof of [4, Theorem 6.1]) shows

$$\tilde{\phi}_3 = (\alpha^2 + |\beta|^2)\phi_3 + 2i \text{Re}(\alpha\beta)\phi_1 + 2i \text{Im}(\alpha\beta)\phi_2.$$

From this expression, it is clear that our two-parameter family of boosts induces a freedom of the initial condition of ϕ_3 that is naturally parametrized by \mathbb{C} .

Corollary 1.10 *The associated family of every entire, complete, minimal graph in Nil_3 with a given Abresch–Rosenberg differential $Q dz^2$ is a family of entire, complete, minimal graphs in Nil_3 with the Abresch–Rosenberg differential $\lambda^{-2}Q dz^2$ ($\lambda \in \mathbb{S}^1$). Moreover, within a given associated family, complete minimal graphs have the same support $h(dz)^{1/2}(d\bar{z})^{1/2}$.*

Proof From the proof of Theorem 1.8, it is clear that for an entire minimal graph $f^\lambda|_{\lambda=1}$ all members of its associated family of minimal surfaces have the Abresch–Rosenberg differential $\lambda^{-2}Q dz^2$ ($\lambda \in \mathbb{S}^1$) and the same support $h(dz)^{1/2}(d\bar{z})^{1/2}$. To prove that the minimal surfaces in the associated family are graphs, we consider the

spacelike CMC surfaces $f_{\mathbb{L}_3}^\lambda$ given in the proof of Theorem 1.8. Then, since $f_{\mathbb{L}_3}^\lambda|_{\lambda=1}$ is entire, it is complete [3, p. 415] and thus the spacelike CMC surfaces $f_{\mathbb{L}_3}^\lambda$ in the associated family are also complete. They are in fact isometric to $f_{\mathbb{L}_3}^\lambda|_{\lambda=1}$. Note that the complete metric is given by $h^2 dzd\bar{z}$; see (1.1). Therefore, by [9, Proposition 1], all $f_{\mathbb{L}_3}^\lambda$ are entire graphs, and thus the corresponding minimal surfaces f^λ in Nil_3 are also entire graphs. The completeness of the associated family follows from Lemma 1.1. ■

Remark 1.11 (Canonical examples) In [6], all entire, complete, minimal vertical graphs are called the *canonical examples*.

Appendix A Basic Results

A.1 Isometry Group

The identity component $\text{Iso}_o(\text{Nil}_3)$ of the isometry group of Nil_3 is the semi-direct product $\text{Nil}_3 \times \text{SO}_2$. If we identify Nil_3 with $\mathbb{C} \times \mathbb{R}$ and SO_2 with U_1 , respectively, then the action of $\text{Nil}_3 \times \text{SO}_2 (\cong (\mathbb{C} \times \mathbb{R}) \times U_1)$ is given by

$$(A.1) \quad ((\alpha = a_1 + ia_2, a_3), e^{i\theta}) \cdot (z = x_1 + ix_2, x_3) = (e^{i\theta}z + \alpha, x_3 + \frac{1}{2} \text{Im}(\bar{\alpha}e^{i\theta}z) + a_3),$$

where $\theta, a_3, x_3 \in \mathbb{R}$ and $\alpha, z \in \mathbb{C}$. Here (x_1, x_2, x_3) is a coordinate system of Nil_3 , θ is a rotation angle, and (a_1, a_2, a_3) is a translation vector.

A.2 Basic Notation

Let $f: M \rightarrow \text{Nil}_3$ be a conformal immersion of a Riemann surface M into Nil_3 . Denote the natural orthonormal basis of the Lie algebra of Nil_3 by $\{e_1, e_2, e_3\}$; see [4, (2.1)]. Then the Maurer-Cartan form $f^{-1}df = (f^{-1}f_z)dz + (f^{-1}f_{\bar{z}})d\bar{z}$ can be expanded as

$$f^{-1}f_z = \sum_{k=1}^3 \phi_k e_k \quad \text{and} \quad f^{-1}f_{\bar{z}} = \overline{f^{-1}f_z} = \sum_{k=1}^3 \overline{\phi_k} e_k.$$

Here, $z = x + iy$ are conformal coordinates, $\bar{z} = x - iy$ is its complex conjugate. The subscripts z and \bar{z} denote the partial differentiations with respect to z and \bar{z} , respectively. Moreover, ϕ_k is a complex-valued function and $\overline{\phi_k}$ is its complex conjugate function. Since f is a conformal, $\phi_k (k = 1, 2, 3)$ satisfy $\sum_{k=1}^3 \phi_k^2 = 0$ and $\sum_{k=1}^3 |\phi_k|^2 = \frac{1}{2}e^u \neq 0$. We note that the induced metric of f is given by $ds^2 = e^u dzd\bar{z}$. Then using the *generating spinors* ψ_1 and ψ_2 , the first equation can be solved by

$$\phi_1 = (\overline{\psi_2})^2 - \psi_1^2, \quad \phi_2 = i((\overline{\psi_2})^2 + \psi_1^2), \quad \phi_3 = 2\psi_1\overline{\psi_2}.$$

Then the condition $\sum_{k=1}^3 |\phi_k|^2 = e^u/2$ is equivalent to $e^u = 4(|\psi_1|^2 + |\psi_2|^2)^2$. Let N be the positively oriented unit normal vector field along f . We define the *support* $h(dz)^{1/2}(d\bar{z})^{1/2}$ by $h = \langle f^{-1}L, e_3 \rangle$, where $L = e^{u/2}N$. Then it is easy to compute h by the generating spinors ψ_1 and ψ_2 : $h = 2(|\psi_1|^2 - |\psi_2|^2)$. Moreover, let $e^{w/2}$ and $Q dz^2 = 4B dz^2$ be the Dirac potential and the Abresch–Rosenberg differential [1],

given by

$$e^{w/2} = \mathcal{U} = \mathcal{V} = -\frac{H}{2}e^{u/2} + \frac{i}{4}h \quad \text{and} \quad B = \frac{2H+i}{4} \left(\langle f_{zz}, N \rangle + \frac{\phi_3^2}{2H+i} \right),$$

respectively. It is known that the vector of generating spinors $\tilde{\psi} = (\psi_1, \psi_2)$ satisfies the so-called “linear spinor system” [4]:

$$\tilde{\psi}_z = \tilde{\psi}\tilde{U}, \quad \tilde{\psi}_{\bar{z}} = \tilde{\psi}\tilde{V},$$

where

$$\tilde{U} = \begin{pmatrix} \frac{1}{2}w_z + \frac{1}{2}H_z e^{-w/2+u/2} & -e^{w/2} \\ B e^{-w/2} & 0 \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} 0 & -\bar{B}e^{-w/2} \\ e^{w/2} & \frac{1}{2}w_{\bar{z}} + \frac{1}{2}H_{\bar{z}} e^{-w/2+u/2} \end{pmatrix}.$$

We note that the second column of the first equation and the first column of the second equation together are the nonlinear Dirac equations; that is,

$$\partial_z \psi_2 = -\mathcal{U}\psi_1, \quad \partial_{\bar{z}} \psi_1 = \mathcal{V}\psi_2,$$

where $\mathcal{U} = \mathcal{V} = e^{w/2}$.

A.3 Flat Connections

From now on we assume that the unit normal $f^{-1}N$ is upward; that is, the e_3 -component of $f^{-1}N$ is positive. Since $f^{-1}N$ is upward, there is a stereographic projection π of the unit normal $f^{-1}N$ from the south pole to the unit disk in \mathbb{C} . We denote the map $\pi \circ f^{-1}N$ by g and call g the *normal Gauss map*. Then it is easy to see that g can be represented by the generating spinors as $g = \frac{\psi_2}{\psi_1}$. We now define the family of Maurer–Cartan forms α^λ as $\alpha^\lambda = \tilde{U}^\lambda dz + \tilde{V}^\lambda d\bar{z}$ with

$$(A.2) \quad \tilde{U}^\lambda = \begin{pmatrix} \frac{1}{4}w_z + \frac{1}{2}H_z e^{-w/2+u/2} & -\lambda^{-1}e^{w/2} \\ \lambda^{-1}B e^{-w/2} & -\frac{1}{4}w_z \end{pmatrix},$$

$$\tilde{V}^\lambda = \begin{pmatrix} -\frac{1}{4}w_{\bar{z}} & -\lambda\bar{B}e^{-w/2} \\ \lambda e^{w/2} & \frac{1}{4}w_{\bar{z}} + \frac{1}{2}H_{\bar{z}} e^{-w/2+u/2} \end{pmatrix}.$$

Then minimal surfaces in Nil_3 are characterized in terms of the normal Gauss map as follows.

Theorem A.1 ([4, Theorem 5.3]) *Let $f: \mathbb{D} \rightarrow \text{Nil}_3$ be a conformal immersion of a simply connected domain \mathbb{D} that is nowhere vertical and let α^λ be the 1-form defined in (A.2). Moreover, assume that the unit normal $f^{-1}N$ is upward. Then the following statements are equivalent:*

- (i) f is a minimal surface.
- (ii) $d + \alpha^\lambda$ is a family of flat connections on $\mathbb{D} \times \text{SU}_{1,1}$.
- (iii) The normal Gauss map g for f is a non-conformal harmonic map into the hyperbolic two-space \mathbb{H}^2 .

Denote by σ the involution corresponding to the Riemmanian symmetric space $\mathbb{H}^2 = \text{SU}_{1,1}/\text{U}_1$ and $(\Lambda\text{SU}_{1,1})_\sigma$ the twisted loop group of $\text{SU}_{1,1}$ twisted by σ .

Definition A.2 ([4, Definition 1]) Let f be a minimal surface in Nil_3 and F^λ a $(\Lambda\text{SU}_{1,1})_\sigma$ -valued solution to the equation $(F^\lambda)^{-1}dF^\lambda = \alpha^\lambda$ such that

$$(A.3) \quad F^\lambda|_{\lambda=1} = \frac{1}{\sqrt{|\psi_1|^2 - |\psi_2|^2}} \begin{pmatrix} \sqrt{i}^{-1} \psi_1 & \sqrt{i}^{-1} \psi_2 \\ \sqrt{i} \psi_2 & \sqrt{i} \psi_1 \end{pmatrix}.$$

Then F is called an *extended frame* of the minimal surface f .

A.4 Sym-formula

Identifying the basis $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ of $\mathfrak{su}_{1,1}$ in [4, (6.1)] with the basis $\{e_1, e_2, e_3\}$ of \mathfrak{nil}_3 in [4, (2.1)], we obtain the Sym-formula, [4, Section 6.1]. In what follows we will take derivatives for functions of λ . Note that for $\lambda = e^{i\theta} \in \mathbb{S}^1$, we have $\partial_\theta = i\lambda\partial_\lambda$.

Theorem A.3 ([4, Theorem 6.1]) For the extended frame F^λ of some minimal surface f , define maps $f_{\mathbb{L}_3}^\lambda$ and $N_{\mathbb{L}_3}^\lambda$ respectively by

$$(A.4) \quad f_{\mathbb{L}_3}^\lambda = -i\lambda(\partial_\lambda F^\lambda)(F^\lambda)^{-1} - N_{\mathbb{L}_3}^\lambda \quad \text{and} \quad N_{\mathbb{L}_3}^\lambda = \frac{i}{2} \text{Ad}(F^\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, define a map $f^\lambda: \mathbb{D} \rightarrow \text{Nil}_3$ by $f^\lambda := \Xi_{\text{nil}} \circ \widehat{f}^\lambda$ with

$$(A.5) \quad \widehat{f}^\lambda = (f_{\mathbb{L}_3}^\lambda)^o - \frac{i}{2}\lambda(\partial_\lambda f_{\mathbb{L}_3}^\lambda)^d,$$

where the superscripts “o” and “d” denote the off-diagonal and diagonal parts, respectively. Then, for each $\lambda \in \mathbb{S}^1$, the following statements hold:

- (i) The map $f_{\mathbb{L}_3}^\lambda$ is a spacelike CMC surface with mean curvature $H = 1/2$ in \mathbb{L}_3 and $N_{\mathbb{L}_3}^\lambda$ is the timelike unit normal vector of $f_{\mathbb{L}_3}^\lambda$.
- (ii) The map f^λ is a minimal surface in Nil_3 and $N_{\mathbb{L}_3}^\lambda$ is the normal Gauss map of f^λ . In particular, $f^\lambda|_{\lambda=1}$ gives the original minimal surface f up to translation.

Remark A.4

(i) It is known that the Maurer-Cartan form $\alpha^\lambda = \widetilde{U}^\lambda dz + \widetilde{V}^\lambda d\bar{z}$ in (A.2) with $H = 0$ and $\lambda = 1$ is the Maurer-Cartan form of a spacelike CMC surface with mean curvature $H = 1/2$, the Hopf differential $Q_{\mathbb{L}_3} dz^2 = -4B dz^2$ and the metric $h^2 dz d\bar{z}$, see [2, Lemma 3.1].¹ Any $(\Lambda\text{SU}_{1,1})_\sigma$ -valued solution F^λ of $(F^\lambda)^{-1}dF^\lambda = \alpha^\lambda$ is called the *extended frame* of a spacelike CMC surface in \mathbb{L}_3 .

(ii) The Hopf differential of the spacelike CMC surface $f_{\mathbb{L}_3}^\lambda$ in Theorem A.3 can be computed as $Q_{\mathbb{L}_3}^\lambda dz^2 = -4\lambda^{-2}B dz^2$, where $Q^\lambda dz^2 = 4\lambda^{-2}B dz^2$ is the Abresch-Rosenberg differential of the minimal surface f^λ in Nil_3 .

(iii) Note that in Theorem A.3 the choice of coordinates is free. We will therefore apply this result to graph coordinates as well as to conformal coordinates without further mentioning.

¹ The mean curvature and the Hopf differential for spacelike surfaces in Minkowski space could be defined differently from the definition in Euclidean space; The second fundamental form II and the mean curvature H could be changed to $-\text{II}$ and $-H$, respectively, that is, in the above case the mean curvature and the Hopf differential would then become $H = -1/2$ and $Q_{\mathbb{L}_3} dz^2 = 4B dz^2$, respectively.

In the following corollary, we compute the Abresch–Rosenberg differential Bdz^2 for the 1-parameter family f^λ in Theorem A.3, and it implies that the family f^λ actually defines the associated family.

Corollary A.5 *Let f^λ be the family of minimal surfaces in Nil_3 defined by (A.5). Then f^λ preserves the mean curvature ($= 0$) and the support. Moreover, the Abresch–Rosenberg differential $Q^\lambda dz^2$ for f^λ is given by $Q^\lambda dz^2 = 4\lambda^{-2}Bdz^2$, where $Q dz^2 = 4B dz^2$ is the Abresch–Rosenberg differential for $f^\lambda|_{\lambda=1}$. Therefore, $\{f^\lambda\}_{\lambda \in \mathbb{S}^1}$ is the associated family of the minimal surface $f^\lambda|_{\lambda=1}$.*

References

- [1] U. Abresch and H. Rosenberg, *Generalized Hopf differentials*. Mat. Contemp. 28(2005), 1–28.
- [2] D. Brander, W. Rossmann, and N. Schmitt, *Holomorphic representation of constant mean curvature surfaces in Minkowski space: consequences of non-compactness in loop group methods*. Adv. in Math. 223(2010), 949–986. <http://dx.doi.org/10.1016/j.aim.2009.09.006>
- [3] S.-Y. Cheng and S.-T. Yau, *Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces*. Ann. Math. (2) 104(1976), no. 3, 407–419. <http://dx.doi.org/10.2307/1970963>
- [4] J. F. Dorfmeister, J. Inoguchi, and S.-P. Kobayashi, *A loop group method for minimal surfaces in the three-dimensional Heisenberg group*. Asian J. Math., to appear. [arxiv:1210.7300v3](https://arxiv.org/abs/1210.7300v3)
- [5] J. Inoguchi, T. Kumamoto, N. Ohsugi, and Y. Suyama, *Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces I, II*. Fukuoka Univ. Sci. Rep. 29(2000), 155–182; 30(2000), 17–47.
- [6] I. Fernández and P. Mira, *Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space*. Trans. Amer. Math. Soc. 361(2009), no. 11, 5737–5752. <http://dx.doi.org/10.1090/S0002-9947-09-04645-5>
- [7] T. K. Milnor, *A conformal analog of Bernstein's theorem in Minkowski 3-space*. In: The legacy of Sonya Kovalevskaya, Contemp. Math., 64, American Mathematical Society, Providence, RI, 1987, pp. 123–132.
- [8] W. M. Thurston, *Three-dimensional geometry and topology, Vol. I*. Princeton Mathematical Series, 35, Princeton University Press, Princeton, NJ, 1997.
- [9] T. Y.-H. Wan, *Constant mean curvature surface, harmonic maps, and universal Teichmüller space*. J. Differential Geom. 35(1992), no. 3, 643–657.
- [10] T. Y.-H. Wan and T. K.-K. Au, *Parabolic constant mean curvature spacelike surfaces*. Proc. Amer. Math. Soc. 120(1994), no. 2, 559–564. <http://dx.doi.org/10.1090/S0002-9939-1994-1169052-5>

Fakultät für Mathematik, TU-München, Boltzmann str. 3, D-85747, Garching, Germany
e-mail: dorf@ma.tum.de

Institute of Mathematics, University of Tsukuba, Tsukuba 305-8571, Japan
e-mail: inoguchi@math.tsukuba.ac.jp

Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan
e-mail: shimpei@math.sci.hokudai.ac.jp