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SOLVING SINGULAR INTEGRAL EQUATIONS VIA (0,2,3) LACUNARY INTERPOLATION

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A modified product rule method for solving Cauchy-type singular integral equations is proposed. It is based on interpolating the unknown and some of its higher derivatives at any prespecified points. At these nodes the value of the unknown can be calculated directly by solving the discretized linear system. No need of further interpolatory formulae arises, as is the case with other quadrature methods.

1. Introduction

Modified product rules for the solution of Fredholm integral equations are well known. Atkinson [1] gives a detailed treatment of some of these methods, considering for instance equations having kernels which possess logarithmic singularities. These methods proceed by splitting the kernel in two parts, one part which is well-behaved and another one which contains weak singularities or worse.

In connection with singular integral equations also, such methods

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have been recently introduced. Gerasoulis and Srivastav [8] considered the replacement of the unknown function by a piecewise linear interpolant. After quadrature and collocation a linear algebraic system is obtained, which is the discretized version of the original equation. Gerasoulis then [7] analyzed the case of a piecewise quadratic polynomial being the approximation to the sought solution. Later, Jen and Srivastav [11] considered the use of cubic splines. There are at least two major grounds for seeking the solution in this form. First of all, the ordinary direct methods, such as Gauss-Chebyshev or Gauss-Jacobi quadrature, [5], [6], [12], require the collocation nodes to be well-determined points, usually the zeros of a polynomial related to the one whose zeros provide the quadrature nodes [2], [3]. The disadvantage in this case is the fact that if the solution is sought at points other than the collocation nodes, an interpolatory formula is needed. The second and perhaps more important advantage is the possibility of avoiding computation and use of polynomials of high degree. Also, the methods discussed here could be applied, with due modification, to singular integro-differential equations.

Three methods are described in the next section. They are slight modifications of each other, in the sense that the satisfaction of only one condition makes the difference while the general quadrature-collocation scheme remains unaltered. The corresponding matrices of the linear algebraic system which is obtained by discretization of the singular integral equation, are given in the following section. In section 4, error analysis is provided, based upon the results of Jen [10] and our estimates given in [14]. Finally some numerical examples are presented as support of the theoretical investigation.

2. Description of the algorithms

Consider the following singular integral equation

$$(2.1)\quad (1/\pi)\int_{-1}^{1} g(t)/(t-x)\,dt + \int_{-1}^{1} K(x,t)g(t)\,dt = f(x) \qquad -1 < x < 1$$

together with the additional normalization condition

$$(2.2)\quad \int_{-1}^{1} g(t)\,dt = C\ .$$

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The solution is then known to possess square root singularities at the endpoints and a new unknown function $y(t)$ can be introduced by the following definition

$$g(t) = y(t)/\sqrt{1-t^2} \; . \tag{2.3}$$

Suppose moreover that the partition $\{x_i\}$ of the interval $(-1,1)$ is given, for $i = 0,\dots,n+1$, with $x_0 = -1$ and $x_{n+1} = 1$. Also a second set of nodes is assumed to be known, $\{t_j\}$, $i = 0,\dots,n$. The following relations hold

$$\begin{aligned} h_i &= x_{i+1} - x_i \\ t_i &= x_i + zh_i \; . \end{aligned} \tag{2.4}$$

We can replace the unknown function by a spline, a deficient spline or a piecewise polynomial. We want to consider lacunary data, and they are assumed as follows. The function and its second derivative are sought at the nodes of the first set, while at the nodes of the second we assign the third derivative. In this way the situation, except for the fact that all the above are unknowns, becomes a $(0,2,3)$-type lacunary interpolation problem. The function

$$\begin{aligned} s_i(x) = {} & s(x_i)A((x-x_i)/h_i) \\ & + s(x_{i+1})B((x-x_i)/h_i + s''(x_i)C(x-x_i)/h_i) \\ & + s''(x_{i+1})D((x-x_i)/h_i) + s'''(t_i)E((x-x_i)/h_i) \end{aligned} \tag{2.5}$$

replaces the unknown function $y(t)$ in the singular integral equation, where the quartic polynomials $A(x)$, $B(x)$, $C(x)$, $D(x)$, $E(x)$ are defined as follows

$$\begin{aligned} A(x) &= 1 - x \\ B(x) &= x \\ C(x) &= [(8z-5)x + 6(1-2z)x^2 + 4zx^3 - x^4]/(12(1-2z)) \\ D(x) &= [(4z-1)x - 4zx^3 + x^4]/(12(1-2z)) \\ E(x) &= [-x + 2x^3 - x^4]/(12(1-2z)) \; . \end{aligned} \tag{2.6}$$

The kernel $K(x,t)$ is instead replaced in each subinterval by a quadratic interpolatory polynomial, $K_i(x,t)$. In this way we obtain for the singular integral equation the expression

$$(2.7)\qquad (1/\pi)\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} s_i(t)/[\sqrt{1-t^2}\,(t-x)]\,dt + \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} K_i(x,t)s_i(t)/\sqrt{1-t^2}\,dt = f(x)\ .$$

By collocation at a set of nodes lying in $(-1,1)$, this functional relation will be reduced to a discrete algebraic system. The nice feature of such replacements now comes into the picture, since the integrals in the above formula can all be evaluated in closed form. The use of some of the formulae in Gradshteyn and Ryzhik [9] and integration by parts when necessary provides us with the analytical expressions for the coefficients of the matrix in the above mentioned system. Three different algorithms can be proposed based on the results of the preceding section. We may collocate at $n+1$ points and seek the quartic spline interpolating the unknown function. This operation needs another $2n$ equations which come from the continuity of the first and third derivative at the nodes $\{x_i\}$, $i = 1,\dots,n$. Moreover, there is the normalization condition, and the difference between the $3n + 5$ unknowns and the $3n + 3$ equations can be filled in by two extra collocation conditions, or we may want to use the natural splines by simply prescribing

$$(2.8)\qquad s''(-1) = s''(1) = 0\ .$$

The second approach consists in seeking the solution as a deficient quartic spline. In this case, we would impose the continuity conditions only for the first derivative and then collocate at $2n+2$ distince nodes. One extra condition comes from the normalization equation and the remaining two equations can be obtained from the natural conditions (2.8) or by the two extra collocation nodes. The third approach makes use of a piecewise polynomial scheme. Here, instead of the continuity of the first derivative, we require the continuity of the third one, and then proceed as in the case of the deficient spline. The form of the matrix of the linear algebraic system in each case is the same for the latter ones, while it is more "sparse" for the first one. In each case, block structure of some form is obtained. The first part has a full structure, corresponding to the collocation equations and the normalization conditions. The continuity conditions instead provide a submatrix. The structure of this is given by three blocks, each of which possesses a tridiagonal form.

3. The coefficient matrix

We turn now to the constriction of the matrix of the system. A few results from Gradshteyn and Ryzhik [9] are needed, namely the formulae 2.27, 2.272.3, 2.273.3, 2.274.3, 2.264.4 and 2.551.3. If we define

$$I(k,i) = \int_{x_i}^{x_{i+1}} (t^k / \sqrt{1-t^2})\, dt$$

$$[J(k,i)](x) = \int_{x_i}^{x_{i+1}} \{t^k/[(t-x)\sqrt{1-t^2}]\}\, dt$$

there is an obvious relationship between the two operators, given by

$$I(0,i) = \arcsin(x_{i+1}) - \arcsin(x_i)$$

$$I(1,i) = \sqrt{1-x_i^2} - \sqrt{1-x_{i+1}^2}$$

$$I(2,i) = (x_i/2)\sqrt{1-x_i^2} - (x_{i+1}/2)\sqrt{1-x_{i+1}^2} + (\tfrac{1}{2})\, I(0,i)$$

$$I(3,i) = x_i^2 \sqrt{1-x_i^2} + (2/3)(1-x_i)^{3/2} - x_{i+1}^2\sqrt{1-x_{i+1}^2} - (2/3)(1-x_{i+1}^2)^{3/2}$$

$$I(4,i) = (x_i^3/4)\sqrt{1-x_i^2} + (3x_i/8)\sqrt{1-x_i^2} - (x_{i+1}^3/4)\sqrt{1-x_{i+1}^2} - (3x_{i+1}/8)\sqrt{1-x_{i+1}^2} + (3/8)\, I(0,i)$$

$$I(5,i) = x_i^4\sqrt{1-x_i^2} + 4[(x_i^2/5) + (2/15)](1-x_i^2)^{3/2} - x_{i+1}^4\sqrt{1-x_{i+1}^2} - 4[(x_{i+1}^2/5) + (2/15)](1-x_{i+1}^2)^{3/2}$$

$$I(6,i) = (x_i^5/6)\sqrt{1-x_i^2} + (5x_i^3/24)\sqrt{1-x_i^2} + (5x_i/16)\sqrt{1-x_i^2} - (x_{i+1}^5/6)\sqrt{1-x_{i+1}^2} - (5x_{i+1}^3/24)\sqrt{1-x_i^2}$$

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$$- (5x_{i+1}/16)\sqrt{1-x_{i+1}^2} + (5/16)\ I(0,i)$$

and also

$$[J(0,i)](x) = (1-x^2)^{-1/2}\left\{ \ln\left|\tan(u_i)\left(\sqrt{1-x^2}-1+x\tan(u_{i+1})\right) + x - (1+\sqrt{1-x^2})\tan(u_{i+1})\right| - \ln\left|x - (1-\sqrt{1-x^2})\tan(u_{i+1}) + \tan(u_i)\left(-1-\sqrt{1-x^2} + x\tan(u_{i+1})\right)\right|\right\}$$

where

$$u_i = \arcsin(x_i)/2 .$$

The kernel $K(x,t)$ of the equation is replaced in each subinterval (x_i,x_{i+1}) by a quadratic polynomial, by evaluating the values $K(x,x_i)$, $K(x,t_i)$, $K(x,x_{i+1})$. Hence in the equation $K(x,t)$ gets replaced by $K_i(x,t)$, with

$$K_i(x,t) = a(i,x)t^2 + b(i,x)t + c(i,x) \qquad i = 0,\dots,n$$

where

$$a(i,x) = \Big[K(x,x_i)t_i - K(x,t_i)x_i - K(x,x_i)x_{i+1} + K(x,x_{i+1})x_i - K(x,x_{i+1})t_i + K(x,t_i)x_{i+1}\Big]/H(i)$$

$$b(i,x) = \Big[x_i^2K(x,t_i) - t_i^2K(x,x_i) - K(x,x_{i+1})x_i^2 + x_{i+1}^2K(x,x_i) + K(x,x_{i+1})t_i^2 - x_{i+1}^2K(x,t_i)\Big]/H(i)$$

$$c(i,x) = \Big[K(x,x_i)x_{i+1}t_i(z-1)h_i + K(x,t_i)x_ix_{i+1}h_i - K(x,x_{i+1})x_it_izh_i\Big]/H(i)$$

$$H(i) = x_{i+1}t_i(z-1)h_i + x_ix_{i+1}h_i - zx_it_ih_i .$$

The above formulae allow the explicit construction of the matrices of the system. Given their complexity however we refer the reader to [15], in which the entries are given in a suitable form for implementation. The following equations then provide the continuity conditions for the first derivative of the function $s(x)$ at the nodes $\{x_i\}$, $i = 1,\dots,n$

$$12h_i(1-2z)s(x_{i-1}) - 12(1-2z(h_{i-1}+h_i)s(x_i)$$
$$+ 12(1-2z)h_{i-1}s(x_{i+1}) + h_i h_{i-1}^2(4z-3)s''(x_{i-1})$$
$$+ (h_{i-1}(8z-3) + h_i(8z-5))h_i h_{i-1}s''(x_i)$$
$$+ h_i^2 h_{i-1}(4z-1)s''(x_{i+1}) - h_i^3 s'''(t_i)$$
$$- h_{i-1}^3 s'''(t_{i-1}) = 0 \ .$$

The next ones instead provide the continuity of the third derivative

$$2h_i(1-z)s''(x_{i-1}) + 2((x-1)h_i+zh_{i-1})s''(x_i)$$
$$- 2zh_{i-1}s''(x_{i+1}) + h_i h_{i-1}s'''(t_{i-1})$$
$$+ h_i h_{i-1}s'''(t_i) = 0 \ .$$

4. Error analysis

In this section we sketch an error analysis for the proposed algorithm. However nothing really new is contained in what follows since the the basic results are the ones in [14], as far as the convergence of the deficient spline and the piecewise polynomial to the solution are concerned. Moreover the estimates for the singular integral equation are basically the same as provided by Jen [10] and thus will not be repeated here. For further details the reader is referred to the above mentioned dissertation [10].

Let the discretized equation be represented by the system

$$Ax = f \ .$$

The vector x^* providing the correct solution satisfies the equation

$$Ax^* = f^* = f + E$$

where E is the error vector induced in the numerical scheme by the use of the piecewise polynomials or the deficient splines. Let K^* be the piecewise quadratic interpolant to the kernel K. By applying to the

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Fredholm term in the definiton of each component of E the triangle inequality we obtain the bound

$$\{\max |y^*| \max |K^*-K| + \max |y^*-y| \max |K|\}$$

where y^* is the deficient spline or the piecewise polynomial interpolating on the true values $y(x_i)$, $i = 0,\ldots,n+1$. From the analysis of [14] it follows that the second term in the sum is of order $O(h^3)$. The first one is also, while the error term corresponding to the dominant part can be shown to be $O(h^{5/2-\delta})$ for some $\delta > 0$. This last expression thus gives the order of E . We should remark however that the expression for the error is

$$\|x^*-x\| < \|A^{-1}\| \quad \|E\|$$

and A^{-1} depends on h , so that the above result should be used with caution.

5. Numerical examples

In this section we describe some numerical implementations of the algorithms.

Example 1. Here we considered the equation

$$(1/\pi) \int_{-1}^{1} g(t)/(t-x)dt = 0$$

subject to the normalization condition

$$(1/\pi) \int_{-1}^{1} g(t)dt = 1 .$$

The analytic solution is

$$g(t) = 1/\sqrt{1-t^2}$$

For the spline algorithm the maximum error at the nodes is 4×10^{-5} . It is obtained by dividing the interval $(-1,1)$ into six subintervals, that is for $n = 4$. For $n = 6$ it is 2×10^{-6} for a value of z close to

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$1/2$ and it is not significantly affected by varying z .

The deficient spline algorithm gives for $n = 5$ an error of about 3×10^{-5} or better. Here z has been chosen close to the endpoints 0 and 1 .

For the piecewise continuous polynomial case with $n = 4$ the situation seems also to be well behaved. Choosing $z = .8$ the maximum error turns out to be 4×10^{-6} .

Example 2. Consider the equation

$$(1/\pi) \int_{-1}^{1} g(t)/(t-x)\, dt = U_{m-1}(x)$$

subject to

$$(1/\pi) \int_{-1}^{1} g(t)\, dt = 0$$

whose analytical solution is $g(t) = T_m(t)/\sqrt{1-t^2}$

The spline case for $n = 4$ and $m = 1$ gives a maximum error of 1×10^{-4} for $z = .7$ and 2×10^{-5} for $z = .9$.

The deficient spline behaves similarly, giving an error of 9×10^{-6} for $z = .2$ and $n = 4$.

The algorithm for the piecewise continuous polynomials yields instead an error of 6×10^{-6} already for $n = 4$ and $z = .1$.

Example 3. Here we considered the equation

$$(1/\pi) \int_{-1}^{1} g(t)/(t-x)\, dt + (1/\pi) \int_{-1}^{1} tg(t)(t^2-x^2)/(t^2+x^2)^2 dt = 1$$

subject to

$$\int_{-1}^{1} g(t)\, dt = 0 .$$

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This equation arises in the problem of a cruciform crack in an infinite medium under constant load. The value of the unknown $\phi(t)$,

$$g(t) = \phi(t)/\sqrt{1-t^2}$$

at $+1$ has been analytically calculated by Rooke and Sneddon [13] to four correct decimal digits to be $.8636$. By applying the above algorithms we obtained the following results.

We examine first the spline method. In the case $n = 5$ and $z = .854$ we have $\phi(1) = .86358321$. In the case $n = 9$ and $z = .80731$, then $\phi(1) = .86365350$. Also for $n = 5$ and $z = .9$, $\phi(1) = .86360225$.

For the deficient spline solution the following results hold. In the case $n = 4$ and $z = .4413$, we have $\phi(1) = .86363869$. For $n = 5$ and $z = .90485$, then $\phi(1) = .86365817$.

Finally for the piecewise continuous polynomial solution we have the following result. For $n = 4$ and $z = .7$ the solution is $\phi(1) = .86367947$.

As a general comment we can say that the differences between the results provided by the three different methods are minor. The last one however requires fewer computations due to the special structure of the matrix. We present these algorithms as an alternative to methods currently employed in the solution of SIE's. We do not claim any superiority of the codes presented in this paper over other methods in common use.

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