Properties of the figure consisting of a triangle, and the squares described on its sides.

By J. S. MACKAY, LL.D.

[Of the following properties, some must have been long known, but I do not remember any early statement of them. The 1st, 10th, 15th, 16th, 19th, 20th, and 22nd are given, mostly without proof, in an article by Vecten in Gergonne's *Annales de Mathématiques*, vol. VII., p. 321 (1817); the first parts of the 3rd and 4th are proposed for proof by William Godward in *The Gentleman's Diary* for 1837, p. 48, and proved by him and others in the *Diary* for 1838, pp. 40–41; the first parts of the 5th, 6th, 12th, and 14th are given in M'Dowell's *Exercises on Euclid and in Modern Geometry*, §§ 27, 28, 29 (1863); the 9th in Milne's *Weekly Problem Papers*, p. 135 (1885); the first part of the 8th in Vuibert's *Journal de Mathématiques Elémentaires*, 12e Année, p. 18 (1887); the 39th in the *Journal de Mathématiques Elémentaires*, edited by De Longchamps, 3e série, tome I., p. 234 (1887). The others are believed to be new.

For the sake of brevity, I have given only one of the varieties of the figure—that, namely, where the squares are all described outwardly on the sides of the triangle, and I have not mentioned any of the numerous properties that may be derived from the several sets of concurrent straight lines.]

Figure 1.

Let ABC be a triangle; BDEC, CFGA, AHIB the squares described outwardly on its sides. GH, ID, EF are joined, and on GH is described the square GHKL; IK and FL are joined. Then

§ 1. The triangles GAH, IBD, ECF are each equivalent to the triangle ABC.

Since angles BAH, CAG are right, therefore angle GAH is supplementary to angle CAB; and GA, AH are respectively equal to CA, AB; therefore triangle GAH is equal to triangle CAB.

§ 2. The triangles FGL, IHK are each equivalent to the triangle ABC.

For these triangles are each equivalent to the triangle GAH, and therefore to the triangle ABC.

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Figure 2.

§ 3. The perpendicular from A to GH is the median from A to BC, and the perpendicular from A to BC is the median from A to GH.

Let AY which is perpendicular to GH meet BC at M, and from B and C draw BP, CQ perpendicular to AM.

Then the right-angled triangles AYH, BPA, having angles HAY, BAP complementary, and AH = BA, are congruent;
therefore AY = BP.

Similarly from the right-angled triangles AYG, CQA there results AY = CQ; therefore BP = CQ.

Hence the right-angled triangles BMP, CMQ are congruent, and BM = CM.

From the relation in which the triangles ABC, AHG stand to each other, it follows from the preceding reasoning that if AX, which is perpendicular to BC, meet HG at N, then HN = GN.

Figure 1.

§ 4. The perpendiculars from A to GH, from B to ID, from C to EF intersect at the centroid of the triangle ABC; and the medians from A to GH, from B to ID, from C to EF intersect at the orthocentre of the triangle ABC.

This follows at once from the preceding section.

Figure 2.

§ 5. GH is double of the median from A to BC, and BC is double of the median from A to GH.

The congruent triangles AYG, CQA give GY = AQ;
and the congruent triangles AYH, BPA give HY = AP;
therefore GH = GY + HY = AQ + AP.
But the congruent triangles BMP, CMQ give MP = MQ;
therefore AQ + AP = 2AM; therefore GH = 2AM.

Hence also, as in § 3, BC = 2AN.

Figure 1.

§ 6. The triangle whose sides are GH, ID, EF is three times the triangle ABC;
and the triangle whose sides are BC, FL, KI is three times the triangle ABC.

Slide the triangle IBD, without rotating it, in such a manner that B may move along BA to A ; then BI will coincide with AH. Similarly slide the triangle FCE so that C may move along CA to A ; then CF will coincide with AG. If it be not clear that BD and CE will coincide after these transferences, it may be noted that the sum of the angles GAH, IBD, ECF is four right angles.

Hence the three triangles GAH, IBD, ECF, each of which is equal to the triangle ABC, form a single triangle whose sides are GH, ID, EF.

From the relation in which the triangles ABC, AHG stand to each other, it follows that BC, FL, KI will form sides of a triangle which is three times the triangle AHG, and consequently three times the triangle ABC.

Figure 3.

§ 7. The sides of the triangle ABC are two-thirds of the medians of the triangle whose sides are GH, ID, EF.

Let GHD represent the triangle formed by the junction round the point A of the three equal triangles GAH, IBD, ECF. Then AH, AD, AG will represent AB, BC, CA.

Since the triangles GAH, HAD, DAG are all equal, therefore A is the centroid of the triangle GHD, and HA, DA, GA produced are the medians. Now HA, DA, GA, and consequently AB, BC, CA, are two-thirds of the medians from H, D, G.

§ 8. From the results established in the three preceding sections may be derived some of the relations which exist between a triangle and the triangle whose sides are the medians of the former.

(*a*) Let ABC, A′B′C′, ... DEF, D′E′F′ ... be two sets of triangles, such that the sides of DEF are the medians of ABC, the sides of A′B′C′ are the medians of DEF, the sides of D′E′F′ are the medians of A′B′C′, and so on ; the set of triangles ABC, A′B′C′, ... will be similar to each other, and the set DEF, D′E′F′ ... similar to each other.

To prove this, consider the triangle whose sides are GH, ID, EF, and the triangle ABC.

Figure 1.

The triangle ABC has its sides two-thirds of the medians of the triangle whose sides are GH, ID, EF ; and therefore it is similar to the triangle whose sides are the medians.

But the medians of the triangle ABC are respectively halves of GH, ID, EF; therefore the triangle whose sides are the medians of ABC is similar to the triangle whose sides are GH, ID, EF.

(*b*) Let ABC be a triangle whose centroid is G, DEF the triangle whose sides are the medians of ABC, the angles of DEF are such that

$$D = GCA + GAC, \ E = GAB + GBA, \ F = GBC + GCB.$$

For (*Figure* 1) the angle $CAB = AGH + AHG$, angle $ABC = BID + BDI$, angle $BCA = CEF + CFE$. But AGH, AHG, BID, BDI, CEF, CFE are the six angles formed by the medians at the vertices of the triangle whose sides are GH, ID, EF, and the triangle ABC is similar to the triangle formed by these medians.

(*c*) Let ABC be a triangle, DEF the triangle whose sides are the medians of ABC, then $DEF = \frac{3}{4}ABC$.

For (*Figure* 1) the sides of triangle ABC are two-thirds of the medians of the triangle whose sides are GH, ID, EF; therefore the area of triangle ABC is four-ninths of the area of the triangle formed by these medians. But the triangle whose sides are GH, ID, EF is three times the triangle ABC; therefore the triangle whose sides are GH, ID, EF is $(4/9) \times 3$ times the triangle formed by its medians.

§ 9. If in the same way that the triangle whose sides are GH, ID, EF is formed from ABC, there be formed a third triangle from that whose sides are GH, ID, EF, the sides of this third triangle will be three times the sides of ABC.

For this third triangle will be three times the triangle whose sides are GH, ID, EF, and therefore nine times the triangle ABC; and it is similar to ABC. Hence its sides must be three times the sides of ABC.

§ 10. $GH^2 + ID^2 + EF^2 = 3(AB^2 + BC^2 + CA^2)$.

Rotate the triangle GAH round A counter-clockwise through a right angle; then AH will coincide with AB, and AG will be CA produced. Do the same with the triangles IBD and ECF round B and C; and there will be obtained *Figure* 4 in which $AG = CA$, $BI = AB$, $CE = BC$, and where GB, IC, EA represent GH, ID, EF in *Figure* 1.

Now $$GB^2 + BC^2 = 2CA^2 + 2AB^2,$$
$$IC^2 + CA^2 = 2AB^2 + 2BC^2,$$
$$EA^2 + AB^2 = 2BC^2 + 2CA^2;$$
therefore $$GB^2 + IC^2 + EA^2 = 3(AB^2 + BC^2 + CA^2).$$

[Perhaps the following proof may be thought simpler :—

Since BA and AC are respectively equal to HA and AG, and since the defect of angle BAC from a right angle equals the excess of angle HAG above a right angle, therefore BC^2 is as much less than $BA^2 + AC^2$ as HG^2 is greater than $HA^2 + AG^2$;

$$\text{therefore } BC^2 + GH^2 = 2(AB^2 + CA^2), \text{ and so on.}]$$

§ 11. If as the square GHKL is described on GH, squares be described on ID and EF, and the two pairs of straight lines that correspond to FK, IL be drawn, the sum of the squares on these six straight lines is equal to eight times the sum of the squares on the sides of ABC.

For
$$\begin{aligned} BC^2 + FL^2 + KI^2 &= 3(AG^2 + GH^2 + HA^2) \\ &= 3(CA^2 + GH^2 + AB^2), \end{aligned}$$
therefore
$$FL^2 + KI^2 = 3(CA^2 + AB^2 + GH^2) - BC^2.$$
Hence the sum of the squares on the six straight lines
$$\begin{aligned} &= 3(CA^2 + AB^2 + GH^2) - BC^2 \\ &+ 3(AB^2 + BC^2 + ID^2) - CA^2 \\ &+ 3(BC^2 + CA^2 + EF^2) - AB^2 \\ &= 5(AB^2 + BC^2 + CA^2) + (GH^2 + ID^2 + EF^2), \\ &= 8(AB^2 + BC^2 + CA^2). \end{aligned}$$

Figure 1.

§ 12. The projections of BI and CF on BC are each equal to the perpendicular from A on BC; and the projections of HI and GF on GH are each equal to the perpendicular from A on GH.

Let I′, F′ be the projections of I, F on BC.

From the congruency of the triangles BII′, ABX it follows that BI′ = AX;
and from the congruency of the triangles CFF′, ACX it follows that CF′ = AX.

Similarly if I″, F″ be the projections of I, F on GH, it follows that HI″ = AY, and GF″ = AY.

§ 13. M is the middle point of I′F′, and N of I″F″.

This follows at once from the preceding section.

§ 14. The sum of the perpendiculars from I and F on BC is equal to BC; and the sum of the perpendiculars from I and F on GH is equal to GH.

From the congruency of the triangles BII′, ABX it follows that II′ = BX;

and from the congruency of the triangles CFF′, ACX
it follows that $FF' = CX$;

therefore $II' + FF' = BX + CX = BC$.

If I and F be on opposite sides of BC, the difference of II′ and FF′ must be taken instead of the sum.

Similarly $II'' + FF'' = GH$.

§ 15. BG, CH are equal and mutually perpendicular; so are CI, AD; and AE, BF.

For in the triangles BAG, HAC, $BA = HA$, $AG = AC$, and angle BAG = angle HAC; therefore $BG = HC$.

Again if the triangle BAG be rotated clockwise through a right angle, BA will coincide with HA, AG with AC, and consequently BG with HC.
BG therefore in its first position is perpendicular to HC.

The point of intersection of BG, CH is P_1; of CI, AD is P_2; of AE, BF is P_3.

§ 16. BF, CI intersect on the perpendicular from A to BC; CH, AE on the perpendicular from B to CA; and AD, BG on the perpendicular from C to AB.

Slide the triangle ADE, without rotating it, in such a manner that A may move along XA produced to a distance equal to BC; and let T_1 be the new position of A. Then D and E will coincide with B and C; and since T_1A is equal and parallel to BD and CE, therefore T_1B is parallel to AD, and T_1C parallel to AE. Now since BF is perpendicular to AE, and CI to AD, therefore BF is perpendicular to T_1C, and CI to T_1B. Hence T_1X, BF, CI are the three perpendiculars of the triangle T_1BC, and therefore intersect at one point.

The point of intersection of BF, CI is Q_1; of CH, AE is Q_2; of AD, BG is Q_3.

§ 17. The distance of Q_1 from BC is a fourth proportional to $BC + AX$, CX, BX.

For the similar triangles BFF′, BQ_1X give

$$BF' : FF' = BX : Q_1X,$$

that is $$BC + AX : CX = BX : Q_1X.$$

Similar statements hold for Q_2 and Q_3.

§ 18. AT_1 is bisected by GH,

Because AT_1 is perpendicular to BC, therefore it coincides with the median from A to GH, that is, it passes through N. But BC was proved in § 5 to be double of AN; therefore AT_1 is double of AN.

§ 19. The points P_1, P_2 are on the circle circumscribed about AHIB; P_2, P_3 on the circle BDEC; and P_3, P_1 on the circle CFGA.

Because the angle BP_1H is right, therefore P_1 lies on the circle whose diameter is BH, that is on the circle AHIB; and because the angle AP_2I is right, therefore P_2 lies on the circle whose diameter is AI, that is on the circle AHIB.

§ 20. FI, HE, DG pass respectively through the points P_1, P_2, P_3, and bisect the angles formed by BG and CH, by CI and AD, by AE and BF.

Join FP_1, IP_1.

Because P_1 is a point on the circumference of the circle CFGA, and CF is the chord of a quadrant of that circle, therefore the angle CP_1F is half a right angle. For a similar reason the angle HP_1I is half a right angle; therefore, since CP_1H is a straight line, FP_1I is also a straight line.

Again, the angles CP_1F, GP_1F are each half a right angle; therefore FI bisects the angles formed by BG and CH.

§ 21. AP_1 is perpendicular to FI, BP_2 to HE, and CP_3 to DG.

For the angles AP_1H, HP_1I at the circumference of the circle AHIB stand on arcs each equal to a quadrant.

§ 22. AP_1, BP_2, CP_3 are concurrent at R.

For AP_1 is the common chord or radical axis of the circles AHIB, CFGA; BP_2 that of the circles AHIB, BDEC; CP_3 that of the circles BDEC, CFGA. Hence AP_1, BP_2, CP_3 are concurrent, and R is the radical centre of the three circles.

§ 23. If U, V, W are the centres of the squares described on BC, CA, AB, then VW is parallel to FI and equal to half of it; and similar relations hold between WU and HE, and UV and DG.

For in the triangle AFI, VW joins the middle points of the sides AF, AI; therefore VW is parallel to FI and equal to half of it.

§ 24. AP_1, BP_2, CP_3 pass respectively through U, V, W, and R is the orthocentre of the triangle UVW.

For the angle BP_1C is right, and AP_1 bisects it; therefore AP_1 passes through the centre of the square described on BC.*

Hence UA, VB, WC are the three perpendiculars of the triangle UVW, and R is its orthocentre.

§ 25. *If U′ be the centre of the square described on GH, then* U, A, U′ *are collinear.*

For the point P_1 stands in the same relation to the triangle AHG as it does to the triangle ABC, and consequently AP_1 passes also through U′.

§ 26. WU passes through the intersection of CI and BT_1; and UV through the intersection of BF and CT_1.

Since the angles BP_2D, BP_2I are each half a right angle, and since BT_1 is perpendicular to CI, therefore BP_2 is one of the diagonals of a square, three of whose sides coincide with BT_1, CI, AD. Hence the other diagonal of this square, which passes through the intersection of CI and BT_1, will bisect BP_2 perpendicularly. But BP_2 is the common chord of the two circles AHIB, BDEC; therefore it is bisected perpendicularly by WU, the straight line joining the centres. Hence WU passes through the intersection of CI and BT_1.

§ 27. If VW be the side of a square, BG or CH is its diagonal; and a similar relation holds between WU and CI or AD, and between UV and AE or BF.

For $\quad BA : WA = \sqrt{2} : 1$, and $AG : AV = \sqrt{2} : 1$;

therefore in the triangles BAG, WAV, $BA : AG = WA : AV$, and the angle BAG = the angle WAV, because each is equal to the angle BAC increased by a right angle. Hence these triangles are similar, and $BG : WV = \sqrt{2} : 1$.

§ 28. If BG or CH be the side of a square, FI is its diagonal; and a similar relation holds between CI or AD and HE, and between AE or BF and DG.

For $\quad FI : VW = 2 : 1$, and $BG : VW = \sqrt{2} : 1$;

therefore $\quad FI : BG = \sqrt{2} : 1$.

§ 29. The result obtained in the preceding paragraph and several other results may be got in the following way.

Since the points P_1, P_2, P_3 are situated in pairs on the circum-

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* See (9) of Mr William Harvey's *Notes on Euclid*, I. 47, in the Proceedings of the Edinburgh Mathematical Society, vol. IV., p. 19.

ferences of the circles circumscribed about the squares, twelve encyclic quadrilaterals are formed, AP_1IH, AP_2IH, BP_1HI, BP_2HI, and two other sets of four corresponding to the sides BC, CA. From two of them BP_1HI and CP_1GF, by the application of Ptolemy's theorem, there result

$$BP_1 \cdot HI + HP_1 \cdot BI = IP_1 \cdot BH,$$

and $$CP_1 \cdot GF + GP_1 \cdot CF = FP_1 \cdot CG.$$

Now $$HI = BI = BH/\sqrt{2}, \text{ and } GF = CF = CG/\sqrt{2};$$

therefore $$BP_1 + HP_1 = IP_1\sqrt{2}, \qquad (\alpha)$$

and $$CP_1 + GP_1 = FP_1\sqrt{2}; \qquad (\beta)$$

therefore $$BG + CH = FI\sqrt{2}, \text{ by addition};$$

therefore $$2BG = 2CH = FI\sqrt{2}$$

therefore $$FI : BG = \sqrt{2} : 1.$$

§ 30. From the equations (α) and (β), by subtraction, there results

$$(BP_1 - CP_1) + (HP_1 - GP_1) = (IP_1 - FP_1)\sqrt{2}.$$

Now $$BP_1 - CP_1 = HP_1 - GP_1, \text{ since } BP_1 + GP_1 = CP_1 + HP_1;$$

therefore $$2(BP_1 - CP_1) = 2(HP_1 - GP_1) = (IP_1 - FP_1)\sqrt{2},$$

or $$IP_1 - FP_1 = (BP_1 - CP_1)\sqrt{2} = (HP_1 - GP_1)\sqrt{2}.$$

Similar relations hold for $EP_2 - HP_2$ and $GP_3 - DP_3$.

§ 31. From the encyclic quadrilaterals AP_1IH and AP_1FG there result

$$AP_1 + IP_1 = HP_1\sqrt{2},$$

and $$AP_1 + FP_1 = GP_1\sqrt{2};$$

therefore $$2AP_1 + FI = (HP_1 + GP_1)\sqrt{2}, \text{ by addition.}$$

Similar relations hold for $2BP_2 + HE$, and $2CP_3 + DG$.

§ 32. $AU = VW$, $BV = WU$, $CW = UV$.

Because $$(GP_1 + HP_1)\sqrt{2} = 2AP_1 + FI,$$

therefore $$GP_1 + HP_1 = AP_1\sqrt{2} + FI/\sqrt{2},$$

$$= AP_1\sqrt{2} + BG.$$

Now $$BP_1 + CP_1 = UP_1\sqrt{2}\,^*;$$

therefore $$BG + CH = AU\sqrt{2} + BG, \text{ by addition};$$

therefore $$CH = AU\sqrt{2}.$$

But $$CH = VW\sqrt{2}; \text{ therefore } AU = VW.$$

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* See (5) of Mr Harvey's paper, before referred to.

This result will be established more simply later on.

§ 33. The perpendiculars from M and N to BC and GH intersect each other at the middle point of FI.

Because M is the middle point of F′I′, and FF′, II′ are perpendicular to BC, therefore the perpendicular to BC at M will bisect FI. For a similar reason the perpendicular to GH at N will bisect FI.

§ 34. If Z be the middle point of FI, then ZBUC and ZHU′G are squares.

For $ZM = \frac{1}{2}(FF' + II') = \frac{1}{2}BC$; and UM is perpendicular to BC and equal to $\frac{1}{2}BC$; therefore ZBUC is a square.

For a similar reason ZHU′G is a square.

§ 35. AMZN is a parallelogram whose diagonals intersect at the middle point of VW.

For $ZM = \frac{1}{2}BC = AN$, and $ZN = \frac{1}{2}GH = AM$; therefore AMZN is a parallelogram. Hence MN bisects AZ. But VW joins the middle points of two sides of the triangle AFI; therefore VW bisects, and is itself bisected by, the median AZ.

§ 36. If O_1 is the middle point of AZ, then O_1M is parallel to AU and equal to half of it.

For in the triangle ZAU, O_1 is the middle point of ZA and M is the middle point of ZU.

Similarly O_1N is parallel to AU′ and equal to half of it. Hence $UU' = 2MN$.

§ 37. The middle point of any side of a triangle is equidistant from the centres of the squares described in the same manner on the other two sides.

For O_1 is the middle point of VW and O_1M, which is parallel to AU, is perpendicular to VW; therefore $MV = MW$.

§ 38. If V′, W′ be the projections of V, W on BC, then $VV' = XV'$ and $WW' = XW'$.

For the quadrilateral AXCV is encyclic, since the angles AXC, AVC are each right; therefore the angle VXC = the angle VAC = half a right angle; therefore $VV' = XV'$. Similarly $WW' = XW'$.

§ 39. The centroid of the triangle UVW is the same as the centroid of the triangle ABC, and that of U′VW the same as that of AHG.

The distance of the centroid of the triangle UVW from BC is $\frac{1}{3}(VV' + WW' - UM)$, and the distance of the centroid of the

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triangle ABC from BC is $\frac{1}{3}$ AX. It remains, therefore, to prove that $VV' + WW' - UM = AX$.

From the encyclic quadrilateral AXCV, by application of Ptolemy's theorem, there results

$$AX \cdot CV + CX \cdot AV = VX \cdot AC.$$

Now $\quad CV = \quad AV = AC/\sqrt{2}$;

therefore $\quad AX + CX \quad = VX\sqrt{2} = 2VV'$.

Similarly $\quad AX + BX \quad = 2WW'$.

But $\quad BC \quad = 2UM$;

therefore $\quad 2(VV' + WW' - UM) = 2AX + BX + CX - BC$

$$= 2AX.$$

Hence also the distance of the centroid of UVW from either AB or CA is the same as the distance of the centroid of ABC from AB or CA. The two triangles consequently have the same centroid.

§ 40. *O_1 is the centre of a circle which passes through the following ten points*:—V, W, M, X, N, Y, the feet of the perpendiculars from V on WU, WU′, and from W on VU, VU′.

The circle with O_1 as centre and OV or OW as radius is readily seen to pass through the feet of the four perpendiculars from V and W. Also this circle will pass through N and Y, if it can be shown to pass through M and X.

Now O_1M is half of AU, and $AU = VW$; therefore O_1M is half of VW; therefore the circle passes through M.

But since AX and ZM are perpendicular to BC, and $O_1A = O_1Z$, therefore $O_1X = O_1M$; and therefore the circle passes through X.

[The circle on VW as diameter can be proved to pass through X thus: the angle VXC is half a right angle, by § 38, and so is the angle WXB; therefore the angle VXW is a right angle.]

If O_2, O_3 be the middle points of WU, UV then, O_2, O_3 will be the centres of two other ten point circles.

The Potential of a Spherical Magnetic Shell deduced from the Potential of a Coincident Layer of Attracting Matter.

By A. C. Elliott, B.Sc., C.E.

This is the problem of § 670 in Clerk Maxwell's *Electricity and Magnetism*. The author proposes to proceed by another method and to obtain the result in a different form. Let O be the centre of the spherical surface on which the shell lies and Z the point where the